Hyperbolic Property of Earthquake Networks

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Abstract

Advanced statistical research on robust quantification of seismicity patterns is required for mitigating the devastating impacts of earthquakes. The need for such research is highlighted by increasing population density in large urban areas near major active faults (e.g., Tokyo, Istanbul, Los Angeles, San Francisco), the catastrophic earthquakes in Japan, Haiti and Indonesia (life loss over 500,000, economical damage over $US 100 billion), and recent earthquakes in the Midwestern US and other areas with hydrocarbon and geothermal production.

A particularly important aspect of seismicity that might benefit from better statistical and physical understanding is earthquake clustering in time and space. A nearest-neighbor analysis has been shown recently to be an effective tool for identifying earthquake clusters, quantifying regional cluster styles in relation to the physical properties of lithosphere, discriminating natural and human-induced seismicity, and developing earthquake declustering techniques. The key technical component of this analysis is a network (graph) of earthquakes connected according to their space-time-energy proximity. Large-scale geometric analysis and, particularly, hyperbolic embedding of the examined networks has facilitated network analysis in various applied fields during the recent decade. This thesis is focused on exploring hyperbolic properties of the earthquake networks.

We examine the geometry of earthquakes in time-space-magnitude domain using the Gromov hyperbolic property of metric spaces. Gromov $\delta$-hyperbolicity quantifies the curvature of a metric space via four point condition, which is a computationally convenient analog of the famous slim triangle property. We estimate the $\delta$-hyperbolicity for observed events from several different earthquake catalogs. A set of earthquakes is represented by a point field in space-time-magnitude domain $D$. The separation between earthquakes is quantified by the Baiesi-Paczuski
proximity $\eta$ that has been shown efficient in applied cluster analyses of natural and human-induced seismicity and acoustic emission experiments. The Gromov $\delta$ is estimated in the earthquake space $(D, \eta)$ and in the proximity graphs $G_{\eta_0}$ obtained by connecting pairs of earthquakes within proximity $\eta_0$. All experiments result in the values of $\delta$ that are bounded from above and do not tend to increase as the examined region expands. This suggests that the earthquake field has hyperbolic geometry. We discuss the properties naturally associated with hyperbolicity in terms of the examined earthquake field. The results improve the understanding of the dynamics of seismicity and further expand the list of natural processes characterized by underlying hyperbolic geometry.

**Keywords:** Large scale geometry, Gromov $\delta$-hyperbolicity, earthquake networks.
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1 Introduction

1.1 Hyperbolic geometry

Large scale geometry, which studies objects and spaces as if viewed from afar, is a well established area of mathematics, with specific ramifications in random walk theory, differential geometry, geometric topology, and more [37, 43]. During the recent decades large scale geometry has been recognized as an efficient tool for applied analysis of complex networks [30, 31]. In particular, the graphs that describe such diverse phenomena as the Internet, biological, social, or science networks, have shown to have hyperbolic geometry at large scale [11, 15, 22, 27, 28, 30]. This means that these graphs can be naturally embedded in spaces of negative curvature, as opposed to the familiar flat Euclidean space. Krioukov et al. [31] have shown that such common properties of observed complex networks as power law degree distribution and high cluster coefficient follow naturally from hyperbolicity of the embedding space. Moreover, the converse is also true – if a network has a power law degree distribution, then its underlying geometry is effectively hyperbolic. The underlying hyperbolicity has multiple tangible implications for structure and dynamics of networks. For instance, the self-similarity (a.k.a. scale-free property) of a network may reflect particular invariances of a hyperbolic space with respect to symmetry transformations [31]. Optimal routing (sending a message between two points in a shortest time) has a natural implementation in a hyperbolic network, avoiding the intrinsic complications that make this problem unsolvable in general graphs. This novel understanding explains an interest to applied analysis of observed networks, with the goal of revealing their underlying geometry [11, 15, 16, 22, 27, 28, 30, 31]. Here we perform a large scale geometric analysis of earthquakes in time-space-energy domain.
1.2 Seismicity

Seismicity is a complex natural phenomenon that poses important challenges for science and society [8]. Quantification of seismicity patterns is needed to mitigate the devastating economic and humanitarian impact of natural and human-induced earthquakes. The need for such research is highlighted by increasing population density in large urban areas near major active faults (e.g., Tokyo, Istanbul, Los Angeles, San Francisco), the catastrophic earthquakes in Japan, Haiti and Indonesia (life loss over 500,000, economical damage over $US 100 billion), and recent earthquakes in the Midwestern US and other areas with hydrocarbon and geothermal production [21].

Graph-theoretic analyses have shown efficient for better understanding the dynamics of seismicity on the intermediate time scales, from days to hundreds of years [1, 3, 7, 18, 38, 41, 42, 44, 45]. In this work, we estimate the large scale geometry of earthquakes proximity graphs, and the general earthquake proximity space, using Gromov $\delta$-hyperbolicity that measures large scale curvature in a metric space. We find that the examined earthquake space and graphs exhibit strong $\delta$-hyperbolicity. This sheds some light at space-time organization of seismicity and further expands the list of natural processes characterized by underlying hyperbolic geometry.

1.3 Contribution of this thesis and dissemination of results

The thesis work included review of literature on large-scale geometric analysis, $\delta$-hyperbolicity, hyperbolic analysis of observed networks, and network analysis of earthquakes. The thesis includes analytical treatment of the conventional and scaled hyperbolicity statistics (Theorems 2,3), deriving distributional properties of random sets of points in 2-D and 3-D hyperbolic spaces (Theorems 4,5), and dimensional analysis of
earthquake proximity (Theorem 6).

The main contribution of this thesis is in applied statistical hyperbolic analysis of the observed earthquake networks. This thesis work resulted in the following original contributions:

- Hyperbolic analysis of earthquake networks in Southern California and North-West Pacific. (Selecting network parameters suitable for the conducted analyzes, constructing networks, estimating the \( \delta \)-hyperbolicity, establishing a strong hyperbolic behavior of the examined networks, interpreting the results.)

- Developing Matlab code for hyperbolic analysis of earthquake networks. (Constructing combinatorial and metric earthquake proximity networks with a given set of parameters and estimating \( \delta \)-hyperbolicity, estimating \( \delta \)-hyperbolicity using the earthquake proximity in original time-space-magnitude domain.)

- Developing auxiliary Matlab codes for simulating and examining random sets in hyperbolic spaces. (Generating uniform distributions in 2-D and 3-D hyperbolic spheres, estimating \( \delta \)-hyperbolicity, estimating scaled \( \delta \)-hyperbolicity.)

The preliminary thesis results have been presented at the following venues:

- The Graduate Seminar, Department of Mathematics and Statistics, University of Nevada Reno (November 22, 2018)

- 2019 Joint Statistics Meeting, Denver, CO, July 27 – August 1 (abstract #305119)

- 2019 Fall Symposium of Nevada Chapter of American Statistical Association (October 19, 2019)

The thesis results are presented in a publication:

1.4 Thesis organization

The rest of the thesis is organized as follows. We begin by reviewing several related works on $\delta$-hyperbolicity and their results in Sect. 2 to understand how hyperbolicity has been used previously. Section 3, provides necessary definitions related to metric spaces and graphs and trees. We review key facts about hyperbolic metric spaces in Sect. 4. In particular, we introduce Gromov $\delta$-hyperbolicity via the Slim Triangle Condition and the Four Point Condition in Sect. 4.1. Section 5 discusses the proposed estimation approach and illustrates it in the Euclidean and hyperbolic planes. The earthquake data and necessary phenomenological background is described in Sect. 6. The Baiesi-Paczuski earthquake proximity that is used to measure separation between earthquakes in space-time-energy domain is described in Sect. 6.2. The main results are presented in Sect. 7 that performs hyperbolicity analysis in the proximity space of earthquakes, and in earthquake proximity graphs (combinatorial and metric). Section 8 concludes.

2 Literature Review

This work uses the concept of a hyperbolic space that combines the properties of classical hyperbolic geometry and that of tree graphs. This concept was introduced by Michail Gromov [24] in the framework of studying hyperbolic groups. The modern textbook account of the theory can be found in [13], [5].

Practical importance of hyperbolic embedding of graphs (networks) has been ele-
gantly emphasized in the paper by Krioukov et al. [31] titled “Hyperbolic Geometry of Complex Networks”. This work shows that if a complex network is assumed to have an underlying hyperbolic structure, then some common properties of complex networks arise naturally. Conversely, if a complex network has these common properties, then the network has an underlying hyperbolic geometry. They begin with the assumption that a complex network has hyperbolic structure. They construct a model in 2-dimensional hyperbolic space in which \( N \gg 1 \) nodes are distributed quasi-uniformly distributed in a disk of radius \( R \gg 1 \). Nodes are assigned an angular coordinate \( \theta \in [0, 2\pi] \) with density \( \rho(\theta) = \frac{1}{2\pi} \) and a radial coordinate with density

\[
\rho(r) = \alpha \frac{\sinh \alpha r}{\cosh \alpha R - 1} \approx \alpha e^{\alpha (r-R)} \sim e^{\alpha r},
\]

(1)

where \( \alpha > 0 \). They ultimately found that this lead to a heterogenous degree distribution in the network.

They then demonstrate that the converse is also true. A scale-free network with some metric structure can be rescaled to create a metric space that is hyperbolic. To show this, they use an underlying metric structure of the circle \( S_1 \). Then \( N \) uniformly distributed nodes are placed in a circle with radius \( N/(2\pi) \). Each node is assigned a degree \( k \) according to the power-law distribution

\[
\rho(k) = k_0^{\gamma-1}(\gamma - 1)k^{-\gamma}, k \geq k_0,
\]

(2)

where \( \gamma > 2 \) is the exponent of the target degree distribution and \( k_0 \) is the minimum expected degree. Each pair of nodes with expected degrees \( (k, k') \) and angular coordinates \( (\theta, \theta') \) at distance \( d = N \Delta \theta/(2\pi) \), \( \Delta \theta = \pi - \pi ||\theta - \theta'|| \), is connected with probability
\( \hat{p}(\chi) \), which is any integrable function of

\[ \chi = \frac{d}{\mu kk'}, \]  

where \( \mu > 0 \) is the parameter controlling the average degree in the network. After a change of variables mapping expected degree of a node \( k \) to its radial coordinate \( r \) on a disk of radius \( R \), nodes become distributed on the disk as in the 2-dimensional hyperbolic space model used previously. Thus their findings show that hyperbolic geometry arises naturally from network heterogeneity, just as network heterogeneity arises from hyperbolic geometry.

With increasing popularity of network representation of various physical and social phenomena [34], the idea of a hidden hyperbolic embedding of a network has been thoroughly explored in the literature. Below we review the recent works that evaluate \( \delta \)-hyperbolicity in complex networks observed in various applied fields.

In “Efficient Navigation in Scale-Free Networks Embedded in Hyperbolic Metric Spaces”, Krioukov et al. [29] examine the navigability of hyperbolic networks. They begin with several different types of networks including Autonomous Systems (AS) of the Internet, film actor collaboration, metabolic reactions, protein interactions, etc. For each network they embed a “hidden metric space,” which is an underlying geometry that determines connections between nodes. Nodes are connected depending on their distance in the so-called hidden metric space. The hidden space in this paper is chosen to be hyperbolic space. They found that this leads to scale-free topologies over the hidden spaces and efficient greedy routing on these topologies, which is the focus of this paper. Greedy routing is when information to be transferred from one node to a destination is first transferred to the node closest to the destination. Each network is described by
the following specifications: the hidden hyperbolic space, the node density in it taken from some probability distribution, and the connection probability obtained from the hyperbolic distance between nodes. In this paper, they have chosen their hyperbolic space to be the hyperbolic plane and the node density to be a uniform distribution. Two nodes are connected if their hyperbolic distance is below some threshold. With this set up, the network is found to have a power-law degree distribution which arises naturally. Ultimately, they have found that because routing relies on hidden geometries to find paths, greedy routing is exceptionally efficient, more so than any other compact routing proposal.

De Montgolfier, Soto, and Viennot [20] explore in their paper “Treewidth and Hyperbolicity of the Internet” the $\delta$-hyperbolicity of the Internet in regards to two different aspects: the Autonomous Systems (AS) and router levels. The hyperbolicity parameter for the Internet, if found to be low, could offer more efficient algorithms for routing issues such as compact routing and diameter estimation. Three different types of graphs are analyzed - the AS level graph, where each AS is a node and ASes are connected if they interchange data; the router level graph, where each Internet router is a node and edges correspond to IP connections; and the router level graph within individual ASes. The $\delta$-hyperbolicity is computed for several graphs of these types. They found that in most graphs the value of $\delta$ never exceeds 2. They also consider several generated graphs which are models of the AS graphs. For all generated graphs $\delta$ was found not to exceed 2.5.

In the paper “Metric tree-like structures in real-life networks: an empirical study” [4], Abu-Ata and Dragan studied several different graph parameters, including $\delta$ - hyperbolicity, of biological networks, social and collaboration networks, web graphs, Internet measurement networks, and one information network. They aimed to show that these
networks have an underlying tree-like structure. Along with other graph parameters, \( \delta \)-hyperbolicity is a measure of tree-likeness, where the smaller the value of \( \delta \), the closer the graph is to a tree. In all cases where the graphs were small enough (less than 10,000 nodes), exact values of \( \delta \) were computed and found to be small enough to consider the networks as hyperbolic, or tree-like, see Table 1. They go on to describe some implications of tree-likeness (small hyperbolicity), including approximations of diameters and radii, distances, and optimal routes.

In their paper, “On the Hyperbolicity of Large-Scale Networks”, Kennedy, Narayan, and Saniee [28] analyzed dozens of social and communication networks to show that they have bounded values of \( \delta \)-hyperbolicity and hence the metric on such networks is characterized by a negative curvature. These authors have shown, at the same time, that the network of roads in California does not have this (hyperbolic) property and is better embedded in a flat space. Specifically, they have converted each network to a simple, undirected, unweighted graph and computed the maximal and expected value of \( \delta \)-hyperbolicity via the Four Point Condition. (We will discuss below in Sect. 4.1 that the individual value of \( \delta \) used to estimate \( \delta \)-hyperbolicity of a graph is associated with four vertices on that graph. We refer to each such group of points as a quadruple.) They found that maximal values of \( \delta \) did not exceed 3 for social and communication networks, while road networks had maxima in the hundreds, see Table 1. In order to determine if a network is considered hyperbolic or not, they created curvature plots which compare the expected value of \( \delta \) for each graph to the particular measure of the quadruple diameter, \( S_{\min} \). Networks whose curvature plots increase similarly to a square grid (linearly) are considered nonhyperbolic, but networks whose curvature plots tend to flatten as \( S_{\min} \) increases are considered hyperbolic. Thus they conclude that the communication and social networks are hyperbolic, but the road networks are not. They
go on to describe the method of renormalization as an easier and more efficient way to compute the value of \( \delta \)-hyperbolicity for a large network. They found, in particular, that if neighboring nodes are merged into “supernodes” to reduce the size of the graph, then the new graph is hyperbolic only if the original network is hyperbolic and the negative curvature increases.

*Narayan and Saniee* [35] explore the effect of negative curvature on core congestion in their paper “Large-scale curvature of networks”. They calculate the load at each node in a network, where the load is the traffic flowing through each node if a unit of traffic flows between each pair of nodes and geodesic routing is used. They define the network core as the group of nodes which sees the most traffic. In short, they found that negative curvature in the network causes the load at the network core to scale with the number of nodes \( N \) as \( \sim N^2 \), whereas in flat networks it scales as \( \sim N^{3/2} \).

*Borassi, Chessa, and Caldarelli* [11] have studied the \( \delta \)-hyperbolicity as an interpretation of the so-called “democracy” of a network. In this paper titled “Hyperbolicity Measures “Democracy” in Real-World Networks”, they have formally shown that a network that has a small value of \( \delta \)-hyperbolicity is more “aristocratic” in that a small set of nodes in the network controls the shortest paths and therefore are the connectors of the entire network. On the other hand, a large value of \( \delta \)-hyperbolicity has more nodes that are crucial to creating paths and is thus more “democratic.” They analyzed several different real-world networks including social, biological, and technological autonomous system networks and computed both the maximum and average value of \( \delta \). They considered the value \( \frac{2\delta_{\text{max}}}{D} \in [0, 1] \), where \( D \) is the diameter of the network, as a worst case scenario for the \( \delta \)-hyperbolicity and found that the largest value of the ratio was 0.8 among all networks. As a more robust estimate, they examined the behavior of
the ratio \( \frac{2\delta_{\text{avg}}}{d_{\text{avg}}} \in [0, 1] \), where \( d_{\text{avg}} \) is the average distance in the network. Their results show that in most cases, the average \( \delta \)-hyperbolicity was an order of magnitude smaller than the average distance, which they consider to be hyperbolic. As to the “democracy” of the networks, they have concluded that social and biological networks are more “democratic,” while technological networks are more “aristocratic.”

*Cohen, Coudert, and Lancin* [16] proposed exact and approximate algorithms to compute \( \delta \)-hyperbolicity for large graphs. Computing the \( \delta \)-hyperbolicity for a graph of order \( n \) using the Four Point Condition can be accomplished in time \( O(n^4) \). For example, a graph with 25,815 vertices would have roughly \( 1.85 \times 10^{16} \) quadruples, which would take several weeks to complete. Their algorithm for computing \( \delta \)-hyperbolicity iterates over quadruples such that it tests those that are more likely to result in a large value of \( \delta \). A quadruple \( a, b, c, d \) is tested before \( a', b', c', d' \) if \( \min(d(a, b), d(c, d)) > \min(d(a', b'), d(c', d')) \). They apply the algorithm to some CAIDA Autonomous Systems maps and found that the computation of \( \delta \) was over 100 times faster, and found that \( \delta \) did not exceed 3, with most values around 2 or 2.5.

*Jonckheere, Lohsoonthorn, and Ariaei* [27] have developed several different scaled versions of \( \delta \)-hyperbolicity in their paper “Scaled Gromov Four-Point Condition for Network Graph Curvature Computation.” The purpose for these alternative scalings is because there is a quandary as to how large \( \delta \) can be for a graph to still be considered hyperbolic. They point out that for the Slim Triangle Condition, a graph can be considered scaled Gromov hyperbolic if \( \delta(\triangle)/\text{diam}(\triangle) < \frac{3}{2} \), \( \forall \triangle \), where the bound \( 3/2 \) is the maximum achievable in standard hyperbolic or in Euclidean space. This concept is applied to the Four Point Condition, where the following scalings are considered: \( \delta(\Box)/\text{diam}(\Box) \), \( \delta(\Box)/L(\Box) \), \( \delta(\Box)/(L + M + S)(\Box) \), and \( \delta(\Box)/(L - M)(\Box) \). For
these three scalings, upper bounds are computed for various spaces such as the Riemannian manifold $H = M_{\kappa^2}$ of constant negative curvature, the Euclidean space $E$, and the manifold $S = M_{\kappa^2}$ of constant positive curvature. In particular, they found that the upper bounds for hyperbolic spaces for the $\text{diam}$, $L$, $L + M + S$, and $L - S$ scalings were 0.1464, 0.0607, 0.2929, and 0.5, respectively. They go on to compute $\text{diam}$ scaled $\delta$-hyperbolicities for several generated graphs – a random graph, a small world graph, a scale free graph, and a growth with uniform attachment graph, and they found that the scale free graph was the most hyperbolic with a maximum scaled value of around 0.4.

*Baiesi and Paczuski* [7] proposed a measure to quantify the correlation between earthquakes in “*Scale-free Networks of Earthquakes and Aftershocks.*” The measure, which we reformulate in our analyses in Section 6.2, is essentially a product involving inter-event time, inter-event space, and the magnitude of the first event. They have taken into account the Gutenberg-Richter distribution for the number of events with magnitude $m$, $P(m) \sim 10^{-bm}$, where $b$ is usually around 1. They also include the fractal appearance of earthquake epicenters with fractal dimension $d_f$. Intuitively, the distance between two earthquakes $i$ and $j$, where $i$ occurs before $j$, is taken as the average number of events with uniform time, homogeneous space, and exponential magnitudes occurring in the space-time domain bounded by events $i$ and $j$. This distance has been shown to be useful in a multitude of applied problems, such as the classification of aftershocks. Where previously aftershock classification involved choosing a space-time window by an observer to consider if an event is an aftershock or not, the proposed distance classifies aftershocks without imposing a predetermined window. Furthermore they show that in earthquake networks, where vertices are earthquakes and edges are weighted using the proposed distance and directed according to time orientation, the cluster size distribution and the distribution of outgoing edges are both scale free.
Abe and Suzuki [1] have studied earthquake networks in their paper “Scale-free Network of Earthquakes.” They constructed earthquake networks by first dividing the geographic region into many small cubic cells. Each cell, which contains events with any magnitude, is considered to be a vertex. When two successive events occur, an edge is created between cells their respective cells. They have created earthquake networks of this type for two earthquake catalogs belonging to Southern California and Japan, respectively. They found that the distributions of connectivities for the earthquake networks obey a power law, and therefore the network is scale-free.

In the paper “Scaling and Precursor Motifs in Earthquake Networks”, Baiesi [6] constructed earthquake networks using the earthquake metric proposed by Baiesi and Paczuski in [7]. Edges are created between events $i$ and $j$ when their distance is less than $\eta_c$ and less than $\phi \eta_j^*$, where $\eta_c$ is a fixed threshold, $\eta_j^*$ is the shortest distance from event $j$ to any previous event, and $\phi$ is set to be 10. The event $i$ is called the “mainshock” and $j$ is called the “aftershock” even if the magnitude of $j$ is larger than that of $i$. An earthquake network of this type is constructed for a catalog of Southern California. The analysis is focused on network motifs, which are small pieces of the network made of a few nodes and edges. The motifs of study are triangles in which the quantity $\rho_{ij} = l_{ij}^D/10^{-b_m}$, which is the space-magnitude part of the metric of [7], is greater than a given threshold $\rho_0$. It was found that these types of motifs, which contain connections to far aftershocks, occur often before the three largest earthquakes in Southern California in the last 16 years. Thus, these constructed networks would be useful to identify common premonitory symptoms of major events.

In their paper “Small-world Structure of Earthquake Network”, Abe and Suzuki [2] study earthquake networks that were constructed as described above in [1] for a Southern California catalog. They computed degrees of separation for random samples of
vertices. They found that all degrees of separation were fairly small, typically between 2 and 3, which shows the small-world nature of the earthquake network. They also computed the clustering coefficient after removing loops and multiple edges from the network. They found that the clustering coefficient was much larger than that of a completely random graph, which also points to the small-world nature of the earthquake networks.

Zaliapin, Gabrielov, Keilis-Borok, and Wong [44] introduced a statistical methodology for seismic clustering analysis in their paper “Clustering Analysis of Seismicity and Aftershock Identification”. Using the earthquake proximity of [7] they analyzed the distribution of nearest-neighbor distances for various point-field models and also for a real earthquake catalog of Southern California. They found that the distributions were bimodal, exhibiting two distinct earthquake populations corresponding to uniform seismicity (homogeneous in time, but not necessarily in space) and to aftershock clustering. This nearest-neighbor approach is also seen in “Earthquake Clusters in Southern California I: Identification and Stability” by Zaliapin and Ben-Zion [45]. Here they found that the nearest-neighbor analysis along with the earthquake proximity of [7] provides a robust method of detection and analysis of earthquake clusters comprised of foreshocks, mainshocks, and aftershocks.
<table>
<thead>
<tr>
<th>Network</th>
<th>Ref</th>
<th>Nodes</th>
<th>Edges</th>
<th>Diameter</th>
<th>Avg dist</th>
<th>$C$</th>
<th>$\max(\delta)$</th>
<th>$&lt;\delta&gt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Social</td>
<td>[28]</td>
<td>$6 \times 10^4$–$3 \times 10^9$</td>
<td>$3 \times 10^4$–$5 \times 10^6$</td>
<td>7–15</td>
<td>3.3–5.1</td>
<td>0.1–0.2</td>
<td>1.5–3.0</td>
<td>0.2–0.3</td>
</tr>
<tr>
<td>Signed</td>
<td>[28]</td>
<td>$5 \times 10^4$–$8 \times 10^4$</td>
<td>$5 \times 10^5$–$6 \times 10^5$</td>
<td>11–14</td>
<td>3.6–4.1</td>
<td>0.1–0.3</td>
<td>1.5</td>
<td>0.2–0.4</td>
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<td>RocketFuel</td>
<td>[28]</td>
<td>$1 \times 10^2$–$1 \times 10^4$</td>
<td>$6 \times 10^3$–$1 \times 10^4$</td>
<td>6–11</td>
<td>3.2–7.0</td>
<td>0.0–0.2</td>
<td>2.0–2.5</td>
<td>0.1–0.3</td>
</tr>
<tr>
<td>Peer-to-peer</td>
<td>[28]</td>
<td>$8 \times 10^3$–$6 \times 10^4$</td>
<td>$2 \times 10^4$–$1 \times 10^5$</td>
<td>9–11</td>
<td>4.6–5.9</td>
<td>0.0–0.0</td>
<td>2.0–2.5</td>
<td>0.3–0.3</td>
</tr>
<tr>
<td>Collaboration</td>
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<td>$1 \times 10^4$–$2 \times 10^5$</td>
<td>13–18</td>
<td>4.2–6.1</td>
<td>0.5–0.6</td>
<td>2.0–3.0</td>
<td>0.2–0.4</td>
</tr>
<tr>
<td>Web</td>
<td>[28]</td>
<td>$2 \times 10^5$–$8 \times 10^5$</td>
<td>$2 \times 10^6$–$6 \times 10^6$</td>
<td>24–164</td>
<td>6.3–7.1</td>
<td>0.6–0.6</td>
<td>1.5–2.0</td>
<td>0.2–0.3</td>
</tr>
<tr>
<td>Road</td>
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<td>$1 \times 10^5$–$2 \times 10^6$</td>
<td>850–1049</td>
<td>310.5–416.4</td>
<td>0.1–0.1</td>
<td>195.5–222.0</td>
<td>41.0–54.2</td>
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<td>$2 \times 10^3$–$1 \times 10^5$</td>
<td>10–19</td>
<td>–</td>
<td>–</td>
<td>2.0–3.5</td>
<td>–</td>
</tr>
<tr>
<td>Social</td>
<td>[4]</td>
<td>$3 \times 10^3$–$3 \times 10^5$</td>
<td>$4 \times 10^3$–$1 \times 10^6$</td>
<td>4–23</td>
<td>–</td>
<td>–</td>
<td>1.0–4.0</td>
<td>–</td>
</tr>
<tr>
<td>Web</td>
<td>[4]</td>
<td>$4 \times 10^3$–$6 \times 10^3$</td>
<td>$9 \times 10^3$–$1 \times 10^4$</td>
<td>10–13</td>
<td>–</td>
<td>–</td>
<td>2.5–3.0</td>
<td>–</td>
</tr>
<tr>
<td>Internet</td>
<td>[4]</td>
<td>$1 \times 10^4$–$4 \times 10^4$</td>
<td>$2 \times 10^4$–$1 \times 10^5$</td>
<td>8–17</td>
<td>–</td>
<td>–</td>
<td>2.0–2.5</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 1: Statistics for studied networks of *Kennedy, Narayan, and Saniee* [28] and *Abu-Ata and Dragan* [4].
3 Background: Metric spaces, graphs, trees

This section reviews the key concepts related to metric spaces, graphs, and trees.

3.1 Metric Spaces

Recall that a metric on a set $X$ is a positive-definite symmetric function $d : X \times X \rightarrow [0, \infty)$ that satisfies the triangle inequality [13]. A pair $(X, d)$, where $X$ is a set and $d$ is a metric on it, is called a metric space. A geodesic is the shortest path between two points in a metric space [13]. A metric space is called geodesic if any two points are connected by a geodesic path. The familiar $n$-dimensional Euclidean space $\mathbb{R}^n$ that consists of all points $x = (x_1, \ldots, x_n)$ with Euclidean metric

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

is a geodesic metric space.

3.2 Graphs and Trees

Recall that a graph is a collection of vertices connected by edges [34]. Here we introduce various graph-related definitions.

Definition 1 (Weighted Graph, [34]). A weighted graph is a graph whose edges may be assigned a numerical value, or weight.

Note that for a tree with positively-valued weights, if we interpret the weights as lengths, then the graph becomes a geodesic metric space.
Definition 2 (Connected, [10]). A graph is connected if there is a path connecting every pair of points.

Definition 3 (Simply Connected, [10]). A graph is simply connected if it is connected and contains no loops and no two edges join the same pair of vertices.

Definition 4 (Simple Graph, [32]). A simple graph is a graph whose edges are unweighted and undirected and does not have multi-edges or self-loops.

Definition 5 (Degree, [32]). The degree of a vertex $i$, denoted $d_i$, is the number of vertices in the graph connected to vertex $i$ by an edge. The degree distribution of the graph is the $P(d_i = k)$ for a randomly chosen vertex $i$ in the network.

Definition 6 (Clustering Coefficient, [32]). Let a graph $G$ have $N$ vertices. The clustering coefficient for vertex $i$, denoted $c_i$ is equal to the number of pairs of connected neighbors of $i$, denoted $\Delta_i$, divided by the total possible number of pairs of connected neighbors $\frac{d^2(d^2-1)}{2}$.

$$c_i = \frac{2\Delta_i}{d^2(d^2-1)}$$  \hspace{1cm} (4)

The clustering coefficient for the graph $G$ is the average clustering coefficient for all vertices in the graph and is denoted $C$.

$$C = \frac{1}{N}\sum_{i=1}^{N} c_i$$  \hspace{1cm} (5)

The clustering coefficient indicates the degree of interconnectedness between vertices in a graph.

Definition 7 (Average Path Length, [32]). Let $d_{ij}$ represent the shortest path length between vertices $i$ and $j$ on a simple graph $G$, where each edge has length 1. The
average path length \( (L) \) is defined as:

\[
L = \frac{1}{2n(n + 1)} \sum_{i \geq j} d_{ij}
\]  

(6)

Average path length helps to understand the spread of the network and how quickly information travels in the network.

A graph \( G \) is said to have small-world nature if it has both a high clustering coefficient and a low average path-length.

A graph \( G \) is said to be scale-free if its degree distribution, \( P(d) \), follows a power law.

\[
P(d) \sim d^{-\alpha} \quad \alpha > 0
\]

(7)

It has been proven that scale-free networks also display small-world nature [32]

**Definition 8** (Tree, [10]). A tree is a graph that is connected and contains no cycles.

**Definition 9** (Rooted Tree, [34]). A rooted tree is a tree that has a designated root node from which the tree branches.

**Definition 10** (Metric Tree, [34]). A metric space \( (X, d) \) is called a tree if for each choice of \( u, v \in M \) there is a unique continuous path \( \sigma_{u,v}: [0, d(u, v)] \rightarrow X \) that travels from \( u \) to \( v \) at unit speed, and for any simple continuous path \( F : [0, L] \rightarrow X \) with \( F(0) = u \) and \( F(L) = v \), the ranges of \( F \) and \( \sigma_{u,v} \) coincide.

4 Background: Hyperbolic Metric Spaces

This section reviews the key concepts related to hyperbolic metric spaces. The familiar \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) that consists of all points \( x = (x_1, \ldots, x_n) \) with
Euclidean metric

\[ d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \]

is a geodesic metric space. The Euclidean metric plays a special role in the Euclidean space since this is the only metric (up to scaling) that is related to the inner product

\[ d^2(x, y) = \langle x - y, x - y \rangle = \sum_i (x_i - y_i)^2. \]

The metric space \((\mathbb{R}^3, d)\) describes many familiar properties of the physical three-dimensional world around us. For instance, the geodesics are straight lines, the sum of the angles of any triangle equals \(\pi\), the area of a circle growth polynomially with its radius \(r\), as \(\pi r^2\), and so on. Interestingly, many of these properties are tightly connected to the underlying flat geometry of the Euclidean space, which has been axiomatically described by Euclid via his five postulates [14]:

1) Each pair of points can be connected by a single straight line segment.
2) Any straight line segment can be indefinitely extended in either direction.
3) There is exactly one circle of any given radius with any given center.
4) All right angles are congruent to each other.
5) If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended indefinitely, meet on the side on which the angles are less than two right angles.

Given the first four postulates, it can be shown that the fifth postulate is equivalent to the so called parallel postulate which states [14]:
5’) Given a line and a point not on it, there is exactly one line going through the
given point that is parallel to the given line.

An interesting alternative geometry arises when the parallel postulate is dropped,
while the other four still hold. Some consequences of this omission are that geodesic
lines are no longer straight but curved, the sum of angles in a triangle is not equal to $\pi$,
and so on, resulting in curved spaces [14]. It can be shown that in an isotropic space, the
parallel postulate can be violated in exactly two ways: there might be no parallel lines
through a given point, which leads to so-called spherical (positively curved) geometry,
or there exists an infinite number of parallel lines, which leads to hyperbolic (negatively
curved) geometry [12, 14].

We are interested here in negatively curved hyperbolic spaces. An isotropic hyper-
bolic space is characterized by its constant curvature $K = -\zeta^2$ for some $\zeta > 0$. The
hyperbolic triangles become thin, with the sum of the angles being less than $\pi$ (and
equal to zero for so-called ideal triangles); see Fig. 1. The area $A_\zeta(r)$ of a hyperbolic
circle grows much faster than in Euclidean case, depending exponentially on the radius

(a) Euclidean triangle
(b) Hyperbolic (thin) triangle

Figure 1: Euclidean (a) and hyperbolic (b) triangles $ABC$. 
The two-dimensional hyperbolic distance \( d = d(u, v) \) between points with polar coordinates \( u = (r_1, \theta_1) \) and \( v = (r_2, \theta_2) \) is defined via the hyperbolic law of cosines [31]:

\[
cosh \zeta d = \cosh \zeta r_1 \cosh \zeta r_2 - \sinh \zeta r_1 \sinh \zeta r_2 \cos \Delta \theta,
\]

where \( \Delta \theta = \pi - | \pi - | \theta_1 - \theta_2 || \) is the angle between \( u \) and \( v \). It is straightforward to check that the above expressions for \( A_\zeta(r) \) and \( d \) converge to the familiar Euclidean formulas as \( \zeta \to 0 \), i.e. when the space flattens.

The thin triangle property and exponential growth of circle area are closely connected to another informal statement: *a hyperbolic space is similar to a tree*. To develop intuition, consider a combinatorial tree with branching index \( b \) (so that all internal vertices have degree \( b + 1 \)). In such a tree, the number of vertices at distance less or equal to \( r \) from a given one grows as \( b^r \) [31]. This is similar to how the area \( A_\zeta(r) \) of a circle with radius \( r \) grows in a hyperbolic space with curvature \( K = -(\ln b)^2 \), i.e. with \( \zeta = \ln b \). Furthermore, any triangle formed by three vertices in a tree is a tripod, which is the ultimate form of a thin triangle.

This tree analogy is made formal by various rigorous results that show how a hyperbolic metric can be approximated by a tree. For instance, consider a hyperbolic space \( X \) and any natural \( n \). It can be shown that there exists such \( C > 0 \) that any \( n \)-element subset of \( X \) can be mapped to the set of leaves of a finite tree so that all distances are distorted by no more than \( C \) [13, Lemma 8.4.15].

The hyperbolic geometry of a space can be more complicated than that in a space
of constant negative curvature. The degree of negative curvature of a space at large scale can be evaluated using the Gromov \( \delta \)-hyperbolicity conditions discussed in the next section.

4.1 Slim Triangle and Four Point Conditions

Gromov \( \delta \)-hyperbolicity is a measure of negative curvature in a metric space. There exist two main approaches to measuring \( \delta \)-hyperbolicity. The first approach is based on the slim triangle condition, also known as the Rips condition.

**Definition 11** (Slim Triangle Condition, [27]). Consider a geodesic metric space \((X, d)\). A triangle \(ABC\) with endpoints \(A, B, C \in X\) is called \(\delta\)-slim if there exists \(\delta > 0\) such that any side of the triangle \(ABC\) is within \(\delta\) of the union of the other two sides. Equivalently, the inscribed circle in the triangle has radius of no more than \(\delta\) (Fig. 1b). A metric space in which all possible triangles are \(\delta\)-slim is said to be \(\delta\)-hyperbolic.

This definition of \(\delta\)-hyperbolicity is quite intuitive; however, it requires one to construct the geodesics between points which can be difficult to do in practice. A computationally convenient alternative is the four point condition that only requires the pairwise distances between points.

**Definition 12** (Four Point Condition, [28]). Given any four points \(A, B, C,\) and \(D\), in a metric space \((X, d)\) denote:

\[
L := d(A, B) + d(C, D), \quad (9)
\]
\[
M := d(A, C) + d(B, D), \quad (10)
\]
\[
S := d(A, D) + d(B, C), \quad (11)
\]
such that $L \geq M \geq S$, relabeling if necessary (Fig. 2). Then the points $A, B, C, D$ satisfy the $\delta$-Four Point Condition for $\delta > 0$ if

$$\Delta := \frac{L - M}{2} \leq \delta.$$  \hspace{1cm} (12)

We say a metric space satisfies the $\delta$-Four Point Condition and is $\delta$-hyperbolic if all possible quadruples satisfy the $\delta$-four point condition.

The two definitions of $\delta$-hyperbolicity are equivalent, differing only by a constant, as described by the following theorem.

**Theorem 1** (Equivalence of the Slim Triangle and Four Point Conditions, [12]). Suppose a metric space $(X, d)$ is $\delta$-hyperbolic according to the slim triangle condition. Then $(X, d)$ satisfies the $2\delta$ four point condition. Conversely, suppose $(X, d)$ satisfies the $\delta$ four point condition. Then $(X, d)$ is at most $6\delta$-hyperbolic according to the slim triangle condition.
4.2 An upper bound on $\Delta$

The following result gives an upper limit to the value of $\Delta$ for any given quadruple of points.

**Theorem 2** (Upper bound for $\delta$). *Let $A$, $B$, $C$, and $D$ be any four points in a metric space $X$. Then,*

$$
\Delta = \frac{L - M}{2} \leq \min\{d(A, D), d(B, C)\}. \tag{13}
$$

**Proof.** By definition,

$$
\frac{L - M}{2} = \frac{d(A, B) + d(C, D) - d(A, C) - d(B, D)}{2} \tag{14}
$$

$$
= \frac{(d(A, B) - d(A, C)) + (d(C, D) - d(B, D))}{2}
$$

The triangle inequality gives:

$$
d(A, B) - d(A, C) \leq d(B, C),
$$

$$
d(C, D) - d(B, D) \leq d(B, C),
$$

$$
d(A, B) - d(B, D) \leq d(A, D),
$$

$$
d(C, D) - d(A, C) \leq d(A, D).
$$

This immediately implies

$$
\frac{L - M}{2} \leq \min\{d(A, D), d(B, C)\},
$$
which completes the proof.

4.3 Scaled Hyperbolicity

Applied studies [27] consider a scaled version of δ-hyperbolicity in which δ is normalized by some quantity that approximates the linear size of the quadruple ABCD. For instance, δ can be normalized by $L - S$. The following theorem applies to this version of δ-hyperbolicity.

**Theorem 3** (Maximal scaled hyperbolicity). The value of scaled hyperbolicity $\delta_{L-S} = \frac{\delta}{L - S}$ achieves maximum at $M = S$.

**Proof.** Since $L \geq M \geq S$, it follows that $L - S \geq L - M$. Thus, the scaled hyperbolicity,

$$\frac{\delta}{L - S} = \frac{L - M}{2(L - S)}, \quad (15)$$

has a maximum value of 1/2, which is achieved when $M = S$.

4.4 Generating uniformly distributed points in 2-dimensional hyperbolic circle

In our experiments in the following section we need uniformly distributed points in a hyperbolic circle. To generate such points, we use the following result.

**Theorem 4** (Uniform point in hyperbolic circle). Suppose the point with polar coordinates $(r, \theta)$ is uniformly distributed in a hyperbolic circle of radius $R$ centered at the origin. Then the probability density function of $r$ is given by
\[ f(r) = \frac{\zeta \sinh \zeta r}{\cosh \zeta R - 1}, \quad 0 \leq r \leq R, \quad (16) \]

and the probability density function of \( \theta \) is given by

\[ f(\theta) = \frac{1}{2\pi}, \quad 0 \leq \theta \leq 2\pi. \quad (17) \]

**Proof.** We observe that \( \theta \sim \text{Uniform}[0, 2\pi] \) by circle symmetry. The distribution of the radius is found by examining the volume of hyperbolic space available at distance \( r \) from the origin. The area of a hyperbolic circle of radius \( r \) in a space of curvature \( K = -\zeta^2 \) is given by [9]:

\[ A_\zeta(r) = \frac{2\pi}{\zeta^2} (\cosh \zeta r - 1). \quad (18) \]

Accordingly, the cumulative distribution function for the radius \( r \) is given by

\[ F(r) = \frac{A_\zeta(r)}{A_\zeta(R)} = \frac{\frac{2\pi}{\zeta^2} (\cosh \zeta r - 1)}{\frac{2\pi}{\zeta^2} (\cosh \zeta R - 1)} = \frac{\cosh \zeta r - 1}{\cosh \zeta R - 1}. \quad (19) \]

We differentiate to get the probability density function:

\[ f(r) = F'(r) = \frac{\zeta \sinh \zeta r}{\cosh \zeta R - 1}. \quad (20) \]

This completes the proof. \( \square \)

To simulate a uniform point in a hyperbolic circle, we use a uniform random variable
$u \sim \text{Uniform}[0,1]$ and apply the inverse transform:

$$u = \frac{\cosh \zeta r - 1}{\cosh \zeta R - 1}$$

(21)

Solving for $r$, we get

$$u (\cosh \zeta R - 1) = \cosh \zeta r - 1,$$

(22)

which finally gives

$$r = F^{-1}(u) = \frac{acosh((u(\cosh \zeta R - 1) + 1))}{\zeta}. $$

(23)

### 4.5 Generating uniformly distributed points in 3-dimensional hyperbolic sphere

Here we also describe generation of uniformly distributed points in a 3-dimensional hyperbolic sphere.

**Theorem 5 (Uniform point in 3-dimensional hyperbolic sphere).** Suppose the point with polar coordinates $(r, \theta, \phi)$ is uniformly distributed in a 3-dimensional hyperbolic sphere of radius $R$ centered at the origin. Then the probability density function of $r$ is given by

$$f(r) = \frac{2\zeta \cosh(2\zeta r) - 2\zeta r}{\sinh(2\zeta R) - 2\zeta R}$$

(24)

The probability density function of $\theta$ is given by

$$f(\theta) = \frac{1}{2\pi} , 0 \leq \theta \leq 2\pi.$$ 

(25)

And the probability density function of $\phi$ is given by
\[ f(\phi) = \frac{1}{\pi}, 0 \leq \phi \leq \pi. \]  

(26)

**Proof.** We observe that \( \theta \sim \text{Uniform}[0, 2\pi) \) and \( \phi \sim \text{Uniform}[0, \pi] \) by symmetry. The distribution of the radius is found by examining the volume of hyperbolic space available at distance \( r \) from the origin. The area of a 3-dimensional hyperbolic sphere of radius \( r \) in a space of curvature \( K = -\zeta^2 \) is given by [9]:

\[ A_{\zeta}(r) = \frac{\pi \zeta^2}{2} \left( \frac{1}{\zeta} \sinh 2\zeta r - 2r \right). \]  

(27)

Accordingly, the cumulative distribution function for the radius \( r \) is given by

\[
F(r) = \frac{A_{\zeta}(r)}{A_{\zeta}(R)} = \frac{\frac{\pi \zeta^2}{2} \left( \frac{1}{\zeta} \sinh 2\zeta r - 2r \right)}{\frac{\pi \zeta^2}{2} \left( \frac{1}{\zeta} \sinh 2\zeta R - 2R \right)} = \frac{1}{\zeta} \sinh 2\zeta r - 2r \left( \frac{1}{\zeta} \sinh 2\zeta R - 2R \right) \left( 1 \right) = \frac{\sinh (2\zeta r) - 2\zeta r}{\sinh (2\zeta R) - 2\zeta R}.
\]

(28)

We differentiate to get the probability density function:

\[ f(r) = F'(r) = \frac{2\zeta \cosh (2\zeta r) - 2\zeta r}{\sinh (2\zeta R) - 2\zeta R}. \]  

(29)

This completes the proof. \( \square \)

To simulate a uniform point in a 3-dimensional hyperbolic sphere, we use a uniform random variable \( u \sim \text{Uniform}[0, 1] \) and apply the inverse transform:
Figure 3: Gromov $\delta$-hyperbolicity in a 2D space of constant curvature $K = -\zeta^2$. The value of $\max\{\Delta\} = \max\{(L - M)/2\}$ as a function of quadruple diameter $L$. In a (flat) Euclidean plane with $\zeta = 0$, $\max\{\Delta\} \propto L$ (blue line), which implies that there is no upper bound for $\Delta$. In hyperbolic planes with $\zeta > 0$, $\max\{\Delta\}$ saturates as $L$ increases. The existence of the maximal value implies $\delta$-hyperbolicity. For instance, the hyperbolic plane with $\zeta = 1$ is $\delta$-hyperbolic with $\delta = \ln 2$ (green horizontal line) in accordance with [36]. The experiment uses 100,000 uniform random quadruples for each plane.

\[ u = \frac{\sinh(2\zeta r) - 2\zeta r}{\sinh(2\zeta R) - 2\zeta R} \]  

Solving for $r$ analytically is not possible, so we solve (30) using Matlab.
5 Estimating δ-hyperbolicity: An illustration

This section illustrates estimation of Gromov δ-hyperbolicity in Euclidean and hyperbolic spaces of constant curvature. In order to do this, we generate multiple quadruples of points that are uniformly distributed in a circle of radius $R$ in either Euclidean plane or hyperbolic plane with constant curvature $K = -\zeta^2$. The δ-hyperbolicity is then estimated as

$$\delta = \max_i \{\Delta_i = (L_i - M_i)/2\},$$

where the maximum is taken over all simulated quadruples indexed by $i$. We also look at the behavior of $\Delta = (L - M)/2$ as a function of various parameters of the experiment, to establish the key qualitative patterns that distinguish flat from negatively curved space. This exercise highlights some essential properties of δ-hyperbolicity and ensures that, in a situation with a known answer, our statistical approach leads to correct estimation of the Gromov δ parameter.

Recall that the flat Euclidean plane corresponds to $\delta = \infty$, an isotropic hyperbolic plane with $\zeta > 0$ corresponds to a finite $\delta$, and, in particular, $\zeta = 1$ corresponds to $\delta = \ln 2$ [36].

Our first experiment examines the behavior of $\Delta = (L - M)/2$ as a function of the quadruple diameter $L$. For that, we generate 1,000 quadruples in a circle of radius $R$, with 100 distinct values of $R$ varying between $R = 10^{-3}$ and $R = 10^2$ on a logarithmic scale (hence producing 100,000 quadruples). Then we calculate $\Delta$ and $L$ for each quadruple and plot the maximal value of $\Delta(L)$ vs. $L$ (using some binning for $L$). Section 4.4 discusses how to generate uniform points in a hyperbolic circle. The results are shown in Fig. 3. In a flat Euclidean plane, $\Delta$ increases linearly with slope 1 as a function of the diameter $L$, in accordance with a straightforward analytic analysis. In a
hyperbolic space, $\delta$ saturates and becomes a $\zeta$-dependent constant as $L$ increases, hence reflecting the curved geometry of the examined space. As the curvature (and parameter $\zeta$) decreases, the saturation onset shifts to larger values of $L$, while for smaller values of $L$ the curves get closer to the straight line of slope 1. This tells us that these spaces are essentially becoming “flatter” as $\zeta$ decreases. Lastly, we see that all curves are overlapping for small values of $L$, which means that when the quadruple points are very close to each other, the effect of negative curvature is practically unnoticeable. In other words, a sufficiently small neighborhood always has geometry of a flat (Euclidean) space.

Our second experiment examines the behavior of $\Delta$ as a function of radius $R$; see Fig. 4. For that, we generate multiple uniform quadruples in two-dimensional Euclidean space and two-dimensional hyperbolic space with constant curvature $\zeta = 1$, within circles of changing radius. First, we observe that in both flat and curved spaces, there is a positive monotone relation between $R$ and $L$. This reflects the intuitive fact that large quadruples only may appear within circles of large radius. Furthermore, for small values of radius ($R \leq 1$) the flat and curved spaces behave similarly, confirming our earlier observation that small neighborhoods always have flat geometry. However, for larger radii ($R \geq 5$) the values of maximal $\Delta$ saturate with $L$ in hyperbolic space, and keep linearly growing in Euclidean space. This is consistent with our earlier observation that curved hyperbolic geometry is best felt at large scales.

The results of this section suggest a useful benchmark – behavior of $\Delta$ as a function of $L$ – against which one can assess the results for spaces with unknown geometry. This is what we do in the next sections.
Figure 4: $\Delta$ in Euclidean plane (circles) and hyperbolic plane (stars) with $\zeta = 1$. Results for uniform quadruples within circles of different radii $R$. The experiment uses 50,000 uniform random quadruples for each value of $R$ in each space.
6 Earthquakes: Data, proximity, and networks

This section describes the earthquake catalogs examined in this study, introduces the space-time-magnitude earthquake proximity, which is the main tool of our metric analysis, and defines the earthquake proximity networks.

6.1 Gutenberg-Richter law

The main empirical observation regarding earthquakes is the Gutenberg-Richter law, which states that the number \( N \) of earthquakes with magnitude above \( m \) is approximated by

\[
\log_{10} N \approx a - bm,
\]

where \( m \geq m_0 \) and \( b \approx 1 \). The observed earthquakes closely follow this relation in different geographic regions and time ranges, although the first principles behind the law remain unsettled. Another empirical observation is that the magnitudes of consecutive earthquakes seem to be statistically independent.

One can interpret these observations via a statistical statement that the magnitude \( m \) of an earthquake, given \( m \geq m_0 \), follows an exponential distribution with parameter \( \lambda = b \ln 10 \). To see this, we rewrite the Gutenberg-Richter law as

\[
P(\text{magnitude} \geq m) = 10^{a - bm}.
\]

For the minimum magnitude \( m = m_0 \) this gives \( 10^{a - bm_0} = 1 \), which implies \( a = bm_0 \). Accordingly,

\[
P(\text{magnitude} \geq m) = 10^{bm_0 - bm} = 10^{-b(m - m_0)} = e^{-b(m - m_0) \ln 10},
\]
which we recognize as the survival function of an exponential distribution with parameter $\lambda = b \ln 10$. The empirical survival function for the examined earthquakes is shown in Fig. 5. A line that corresponds to the exponential tail decay with parameter $b = 1$, $\lambda = \ln 10$ is shown in the figure for visual comparison. One can see that the tail of the empirical distribution ($m > 2.5$) can be fairly closely approximated by the exponential distribution with $\lambda = \ln 10$. The observed deviations, in particular those seen for $m < 2.5$ are typical for the natural seismicity.

6.2 Baiesi-Paczuski proximity

An asymmetric proximity $\eta_{ij}$ between an earthquake $j$ and an earlier earthquake $i$ is defined using the approach of Baiesi and Paczuski [7]:
where \( t_{ij} = t_j - t_i \) is the time difference in seconds between the examined events, \( r_{ij} \) is the spatial distance in meters between the earthquake epicenters, \( d \) is the dimension of the epicenters, and \( b \approx 1 \) is the parameter of the Gutenberg-Richter law. In this work we use \( d = 2 \) and \( b = 1 \); accordingly, the proximity has units of \([\text{m}^2 \cdot \text{sec}]\).

Note that this proximity is not a metric (distance) since it can be equal to zero for non-identical earthquakes (when \( t_{ij} = 0 \) or \( r_{ij} = 0 \)) and it does not satisfy the triangle inequality (e.g., one always can connect any two points by two segments such that the points within the first have the same time coordinate, and the points within the second have the same spatial coordinate, resulting in zero length of the two segments). We notice that these deviations from the proper distance are solely due to the existence of distinct points with the same space or time coordinates. Roughly speaking, the proximity behaves as an asymmetric distance for large time and space separations between events, and hence can be considered as an approximation to a proper (unknown) metric in the time-space-magnitude domain of earthquakes.

An intuitive interpretation of the proximity is given in terms of a stationary homogeneous point process with exponential magnitudes and independent time, space, and magnitude components. Specifically, the proximity \( \eta_{ij} \) between points \( i \) and \( j \) equals the expected number of events in this process within the space-time cylinder between points \( i \) and \( j \) (i.e., the cylinder with the time projection \([t_i, t_j]\) and space projection being the circle centered at event \( j \) and with radius \( r_{ij} \)). Indeed, this is exactly how the Euclidean distance can be defined via a homogeneous point process with unit intensity. This set up satisfies the expectation that the value of \( \eta_{ij} \) should be small when events \( i \)
and \( j \) might be related, and it should be larger when there is no relation between earthquakes \( i \) and \( j \). Consider the case when \( N(m) \) earthquakes with magnitude above \( m \) occur independently in \( d_f \)-dimensional space and time and obey the Gutenberg-Richter relation \( \log_{10} N(m) = a - bm \). Then the expected number of events with magnitude \( m \) in time interval \( t \) and distance \( r \) from any earthquake is proportional to \( t^{d_f} 10^{-bm} \). This means that the distance \( \eta_{ij} \) is essentially the expected number (up to a constant) of earthquakes with magnitude \( m \) in the time interval \( t \) and distance \( r \) from earthquake \( j \) in a process with no clustering. When the distance \( \eta_{ij} \) is significantly smaller than the majority of pairwise distances in the catalog, then this provides motivation to consider that earthquake \( i \) is a parent to earthquake \( j \). For a comprehensive discussion of the proximity, we refer to [44, 47].

The proximity \( \eta \) has shown instrumental in various analyses of seismicity, including scale-free properties of earthquake networks [7], earthquake cluster identification and classification with respect to the physical properties of the lithosphere [45, 46, 48], discriminating between natural and human-induced seismicity [39, 48], analysis of earthquake aftershocks and foreshocks [46, 25, 17, 23], and understanding triggering processes in rock fracture [19].

In this work, we assume that the minimal time separation between events is 1sec and the minimal space separation is 1m. This means that if two events occurred at a smaller separation, we artificially make it 1s and/or 1m; such cases are, however, very rare. We then use a logarithmic version of the proximity

\[
\log_{10} \eta_{ij} = \log_{10} t_{ij} + d \log_{10} r_{ij} - b(m_i - m_{\text{max}}) \geq 0
\]

to quantify space-time-magnitude separation between pairs of earthquakes. All metric
characteristics of the earthquake quadruples reported below refer to this logarithmic quantity.

6.3 Earthquake networks

Baiesi and Paczuski [7] have applied the earthquake proximity $\eta$ to analysis of earthquake graphs (a.k.a. networks). They considered a spanning graph whose vertices correspond, one-to-one, to the set of examined earthquakes. Each earthquake (vertex) $j$ is connected to a single earlier (in time) earthquake $i$ that minimizes the proximity $\eta_{ij}$. It is easy to check that this construction results in a time-oriented tree. The tree root corresponds to the first event in the catalog. Each event (except the first one) has a single parent (earlier event to which it is connected by an edge) and may have multiple offspring (later events to which it is connected by edges). It has been shown that the out-degree distribution in such graphs follows a power law decay with index $\gamma \approx -2$.

In this work, we consider a slightly different graph construction. Specifically, we construct graph $G_{\eta_0}$ whose vertices correspond, one-to-one, to the observed earthquakes. The edges are formed between the pairs of vertices with proximity $\eta$ below a threshold $\eta_0$. In general, a graph $G_{\eta_0}$ is multi-component. Each connected component of $G_{\eta_0}$ is a subgraph such that any pair of its vertices can be connected by a path that consists of edges each of which is shorter than $\eta_0$. Any two vertices from different connected components cannot be connected by such a path. In this work, we consider such earthquake graphs and measure the distance between two vertices either as the number of edges in the shortest path connecting them (unweighted, combinatorial graph), or as the minimal total edge lengths in such a path (weighted, metric graph).
6.4 $\delta$-hyperbolicity with respect to the earthquake proximity $\eta$

In this study, we estimate $\delta$-hyperbolicity with respect to the earthquake proximity $\eta$. We denote the respective hyperbolicity parameter as $\delta_{\eta}$. We start by identifying transformations of time, space, and magnitude that have no effect on the resulting hyperbolicity.

**Theorem 6** (Invariance of $\delta_{\eta}$). *Let us utilize the earthquake proximity formula $\eta(x, y) = \log_{10}(t) + d \log_{10} r - bm_y, t_y \leq t_x, where t = t_x - t_y, r is the Euclidean distance between the two points, and d is a constant. Then, a multiplicative transformation of space or time and an additive transformation of magnitude have no effect of the value of $\delta_{\eta}$.*

**Proof.** First we consider a multiplicative transformation of the distance between points, $r$. Suppose $r' = C_r r$ for all $r$. Then

$$
\eta'(x, y) = \log_{10} t + d \log_{10} C_r r - bm_y \\
= \log_{10} t + d \log_{10} r - bm_y + d \log_{10} C_r \\
= \eta(x, y) + d \log_{10} C_r.
$$

Accordingly, we have for a quadruple $ABCD$:

$$
\frac{L' - M'}{2} = \frac{\eta'(A, B) + \eta'(C, D) - (\eta'(A, C) + \eta'(B, D))}{2} = \frac{\eta(A, B) + \eta(C, D) + 2d \log_{10} C_r - \eta(A, C) - \eta(B, D) - 2d \log_{10} C_r}{2} = \frac{L - M}{2}.
$$

Hence, a multiplicative transformation of the spatial variable $r$ does not have an effect of the value of $\delta_{\eta}$.

Next, we consider a multiplicative transformation of the time variable, $t' = C_t t$. 
Here we have

\[ \eta'(x, y) = \log_{10} C_t t + d \log_{10} r - b m_y \]
\[ = \log_{10} t + d \log_{10} r - b m_y + \log_{10} C_t \]
\[ = \eta(x, y) + \log_{10} C_t, \]  
(35)

and

\[ \frac{L' - M'}{2} = \frac{\eta'(A, B) + \eta'(C, D) - (\eta'(A, C) + \eta'(B, D))}{2} \]
\[ = \frac{\eta(A, B) + \eta(C, D) + 2 \log_{10} C_t - \eta(A, C) - \eta(B, D) - 2 \log_{10} C_t}{2} \]
\[ = \frac{L - M}{2}. \]  
(36)

Thus, a multiplicative transformation of the time variable, \( t \), does not have an effect of the value of \( \delta_\eta \).

Finally, we consider an additive transformation of the magnitude variable, \( m' = m + C_m \). Here

\[ \eta'(x, y) = \log_{10} t + d \log_{10} r - b(m_y + C_m) \]
\[ = \log_{10} t + d \log_{10} r - b m_y - b C_m \]
\[ = \eta(x, y) - b C_m, \]  
(37)
and
\[
\frac{L' - M'}{2} = \frac{\eta'(A, B) + \eta'(C, D) - (\eta'(A, C) + \eta'(B, D))}{2}
\]
\[= \frac{\eta(A, B) + \eta(C, D) - 2bCm - \eta(A, C) - \eta(B, D) + 2bCm}{2}
\]
\[= \frac{L - M}{2}.
\]

Thus, an additive transformation of the magnitude variable \( m \), does not have an
effect of the value of \( \delta_\eta \). This completes the proof.

Theorem 6 ensures that the analysis of \( \delta \)-hyperbolicity is independent of the units
selected for time, distance, and energy measurements.

7 Hyperbolic property of earthquakes

We now focus on \( \delta \)-hyperbolicity for seismicity. We use two complementary approaches
– studying the space-time-magnitude domain of earthquakes using the earthquake prox-
imity \( \eta \), and earthquake proximity graphs using weighted and unweighted graph dis-
tances.

7.1 Synthetic catalogs

We start with analysis of synthetic catalogs, which allows us to generate quadruples,
with a wide range of spatial, time, and magnitude components. Specifically, we work
with a homogeneous Poisson model of seismicity. Every catalog is comprised of earth-
quakes that have uniform spatial coordinates in a circle of radius \( R \), a uniform temporal
coordinate on time interval \([0, T]\), and an exponential magnitude coordinate. The space,
time, and magnitude components are independent. We use such synthetic catalogs to
Figure 6: $\Delta$ vs $R$ in a synthetic earthquake catalog. The experiment uses 10,000 uniform random quadruples for each value of $R$.

examine the dependence of $\Delta = (L - M)/2$ with respect to the range of each catalog component – space, time, and magnitude.

First, we inspect the relationship between $\Delta$ and the catalog radius $R$. We consider a range of radii to generate the spatial coordinates, and for each radius we generate the time coordinates uniformly on an interval $[0, T]$ with $T = 50$ yrs, and the magnitude coordinates from an exponential distribution that is truncated by a maximal magnitude $m_{\text{max}} = 6$. We create 10,000 quadruples for each radius and compute $\Delta$ for each quadruple, see Fig. 6. We see that $\Delta$ is independent of the spatial extent of the catalog.

Similarly, we look at the relationship between $\Delta$ and the time span of the catalog. We consider a range of durations $T$ to generate the time coordinate on the interval
[0, T], and for each T we generate the spatial coordinates in a circle with constant radius $R = 100\text{km}$ and the magnitude coordinates from an exponential distribution that is truncated by a maximal magnitude $m_{\text{max}} = 6$. We create 10,000 quadruples for each upper time limit and compute $\Delta$ for each quadruple, see Fig. 7. We see that $\Delta$ is independent of the time span of the catalog.

Finally, we explore the relation between $\Delta$ and the magnitude range. We consider a range of upper magnitude truncation boundaries $m_{\text{max}}$. For each $m_{\text{max}}$, we generate the spatial coordinates in a circle with radius $R = 100\text{km}$ and generate the time coordinates uniformly on an interval $[0, T]$ with $T = 50\text{ yrs}$. We generate 10,000 quadruples for each maximal magnitude value and compute $\Delta$ for each quadruple, see Fig. 8. We
observe that $\Delta$ is only weakly dependent on $m_{\text{max}}$, showing a slight increase for $m_{\text{max}}$ within the interval $[1, 2]$. For larger values of $m_{\text{max}} > 2$, the value of $\Delta$ again stabilizes.

In summary, our experiments (Figs. 6, 7, 8) suggest that $\Delta = (L - M)/2$ can take a range of values within the interval $[0, 2.5]$. The distribution of $\Delta$ is independent of spatial and time range of the examined earthquakes. The distribution depends weakly on the range of examined magnitudes, and remains bounded from above for the physically realistic magnitudes (recall that the largest recorded earthquake, that occurred in 1960 in Chile, had magnitude 9.5).

Next, we analyze the behavior of $\Delta$ in a synthetic catalog with respect to the diameter of the quadruple $L$. Recall (Fig. 3) that this behavior is closely related to the $\delta$-hyperbolicity. We create 100,000 quadruples where each point has spatial coordinates

---

### Table

<table>
<thead>
<tr>
<th>Magnitude, $m_{\text{max}}$</th>
<th>Maximum</th>
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<th>97.5th Percentile</th>
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<td>$10^{-2}$</td>
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</tr>
<tr>
<td>$10^{-1}$</td>
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<td></td>
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</tr>
<tr>
<td>$10^0$</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$10^1$</td>
<td></td>
<td></td>
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</table>

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Figure 8: $\Delta$ vs $m_{\text{max}}$ in a synthetic earthquake catalog. The experiment uses 10,000 uniform random quadruples for each value of $m_{\text{max}}$. 
generated uniformly in a circle with radius $R = 100\text{km}$, temporal coordinate generated uniformly on an interval $[0, T]$ with $T = 37\text{yrs}$ (to match that in the catalog of Southern California), and magnitude coordinate generated exponentially and truncated by a range of maximal magnitudes $m_{\text{max}}$. The range of maximal magnitudes is needed to produce a wide enough range of quadruple diameters. We then compute $\Delta = (L - M)/2$ for each quadruple. For each diameter $L$ (up to some binning), we compute the maximum of $\Delta$. The results are shown in Fig. 9. We see that the overall trend in both figures is similar to that of Fig. 3 in that $\Delta$ increases for smaller values of $L < 28$, and then stabilizes as $L$ increases above $L = 28$. This hints at hyperbolic geometry of the earthquake space-time-magnitude domain with the earthquake proximity $\eta$. Notice that in the earthquake domain, we have to vary the maximal magnitude over almost two orders ($0.1 < m_{\text{max}} < 7$) to obtain a wide range of quadruple diameters $L$. A fixed maximal magnitude always corresponds to a narrow range of $L$, insufficient to see the increase and stabilization of $\Delta = \Delta(L)$.

7.2 San Jacinto Fault Zone

In this study we examine geometry of earthquakes in Southern California. We work with the relocated catalog of Hauksson et al. [26] extended for the period 1981–2017. The catalog is available via the SCEC Data Center\(^1\). We focus primarily on the San Jacinto Fault Zone, the most active fault zone of the examined region. The same qualitative results are obtained for other subregions of Southern California and other seismically active regions. The examined catalog contains 18,972 earthquakes with magnitudes ranging from 1.50 to 5.43. For each registered earthquake $i$, the catalog reports its occurrence time $t_i$; location comprised of latitude $\lambda_i$, longitude $\phi_i$, and depth $z_i$; and

\(^1\)http://www.data.scec.org/research-tools/downloads.html
Figure 9: Maximum $\Delta$ vs $L$ in a synthetic earthquake catalog. The experiment uses 100,000 uniform random quadruples for each value of $m_{\text{max}}$; this results in 800,000 quadruples total.

Magnitude (a logarithmic measure of energy) $m_i$. The map of the examined catalog is shown in Fig. 10. The time-latitude projection of the examined earthquakes is shown in Fig. 11.

7.2.1 Earthquake proximity

We now examine the earthquake catalog of Southern California. We extract 10,000,000 quadruples from the catalog and for each quadruple compute the value of $\Delta$ and the corresponding diameter $L$. For each $L$, we compute the maximum and 99th, 97.5th, and 95th percentiles, see Fig. 12. As we know from the synthetic catalog experiments, a catalog with a fixed space, time, and magnitude range can only produce a limited range of quadruple diameters $L$. Accordingly, we have to interpret the dependence $\Delta(L)$ with
Figure 10: The study area in Southern California. Gray dots show the epicenters of earthquakes with magnitude $m \geq 1.5$ from the catalog of Hauksson et al. [26] extended to 2017. The red dots show the examined earthquakes within the San Jacinto Fault Zone. Black lines show the major faults.

Figure 11: The time-latitude plot of the examined earthquakes.

respect to this limited range. The range of quadruple diameters in this experiment is $38 < L < 45$. First, we observe that $\Delta$ slightly increases with $L$. This increase,
Figure 12: $\Delta$ vs $L$ for real earthquake catalog of Southern California. The experiment uses 10,000,000 uniform random quadruples.

however, is very different from what we observed in Euclidean space. The slope of a linear approximation to the empirical curve $\Delta(L)$ is significantly lower than 1, expected in a flat space. The observed value of $\Delta$ in all quadruples is below 3. Moreover, for the max $\Delta(L)$, we see stabilization for larger $L$ ($L > 42$). This behavior is reminiscent of the behavior of $\Delta$ in a hyperbolic space.

7.2.2 Combinatorial networks

We now shift our focus to $\delta$-hyperbolicity in earthquake graphs. The graph is constructed by connecting the earthquakes (vertices) with proximity $\log_{10} \eta_{ij} < 15$. Then, we individually examine each connected subgraph with more than 500 vertices. From each selected subgraph, we select 10,000,000 random independent quadruples and compute the respective $\Delta$ and $L$. 

Figure 13: $\Delta$ vs $L$ for unweighted (combinatorial) earthquake graph. The experiment uses 10,000,000 uniform random quadruples.

We start with the combinatorial case, where the distance between vertices is measured as the number of edges in the minimal path between them. The results are shown in Fig. 13 and graph statistics are shown in Table 3. Here, the range of observed quadruple diameters is $10 < L < 45$. The value of $\Delta$ is below 5.5 for all examined quadruples. Furthermore, $\Delta(L)$ increases with $L$ within the interval $10 < L < 30$, and then stabilizes (and even slightly decreases for the largest quadruple diameters). The slope of the curve $\Delta(L)$ in the steepest part is not exceeding 0.1, which is smaller than the unit slope expected in a flat space.

7.2.3 Metric networks

The results for metric graph, where the edge length equals the respective proximity, are shown in Fig. 14 and graph statistics are shown in Table 3. Here, the range of quadruple
diameters is $100 < L < 700$. The observed values of $\Delta$ are bounded from above by 80. The values of $\Delta$ increase with $L$ within the interval $100 < L < 400$, and then stabilize (end even decrease for the largest diameters). The slope of the curve $\Delta(L)$ in the steepest part does not exceed 0.2.

Overall, the qualitative behavior of $\Delta(L)$ in combinatorial and metric graphs is very similar and is reminiscent of that expected in a hyperbolic space. Accordingly, we conclude that the large-scale geometry of earthquake proximity graph is hyperbolic.

![Graph](image)

Figure 14: $\Delta$ vs $L$ for weighted (metric) earthquake graph. The experiment uses 10,000,000 uniform random quadruples.

Figure 15 compares the actual earthquake proximity $\log_{10} \eta$ with that observed in combinatorial and metric graph for multiple pairs of earthquakes. This experiment uses the proximity threshold $\log_{10} \eta < 12$ and considers all connected subgraphs with at least 50 vertices. The clusters of (red) points in the plot correspond to the pairs of earthquakes separated by distinct numbers of edges in a metric graph (between 1 and 7). For events
separated by a single edge, the actual proximity equal the graph separation; for other events, the actual proximity is smaller than the graph separation. The plot provides basic intuition behind the change of scale between Fig. 12 that refers to the actual earthquake proximity, and Figs. 13 and 14 that refer to unweighted and weighted graph analyses, respectively.
<table>
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<th>Number of events</th>
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<th>Min mag</th>
<th>Max mag</th>
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<td>37</td>
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<td>2,311</td>
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<td>5.00</td>
<td>9.10</td>
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</table>

Table 2: Summary statistics of the earthquake catalogs examined in this work

Figure 16: The area of study for each of the earthquake catalogs.
Figure 16: The area of study for each of the earthquake catalogs cont.
Figure 16: The area of study for each of the earthquake catalogs cont.
Figure 16: The area of study for each of the earthquake catalogs cont.
Figure 17: The time-latitude plot of the examined earthquakes for each catalog.
Figure 17: The time-latitude plot of the examined earthquakes for each catalog cont.
Figure 18: The time-magnitude plot of the examined earthquakes for each catalog.
Figure 18: The time-magnitude plot of the examined earthquakes for each catalog cont.
Figure 19: Empirical survival function $P(\text{magnitude} > m)$ for the examined earthquakes in each catalog.
Figure 19: Empirical survival function $P(\text{magnitude} > m)$ for the examined earthquakes in each catalog cont.
Figure 19: Empirical survival function $P(\text{magnitude} > m)$ for the examined earthquakes in each catalog cont.
Figure 19: Empirical survival function $P(\text{magnitude} > m)$ for the examined earthquakes in each catalog cont.
Figure 20: $\Delta$ vs $L$ for each of the studied catalogs. Each experiment uses 10,000,000 uniform random quadruples.
Figure 20: $\Delta$ vs $L$ for each of the studied catalogs cont. Each experiment uses 10,000,000 uniform random quadruples.
Figure 20: $\Delta$ vs $L$ for each of the studied catalogs cont. Each experiment uses 10,000,000 uniform random quadruples.
Figure 20: $\Delta$ vs $L$ for each of the studied catalogs. Each experiment uses 10,000,000 uniform random quadruples.
Figure 21: $\Delta$ vs $L$ for unweighted (combinatorial) earthquake graphs for each catalog. Each experiment uses 10,000,000 uniform random quadruples for each connected subgraph.
Figure 21: $\Delta$ vs $L$ for unweighted (combinatorial) earthquake graphs for each catalog cont. Each experiment uses 10,000,000 uniform random quadruples for each connected subgraph.
Figure 21: $\Delta$ vs $L$ for unweighted (combinatorial) earthquake graphs for each catalog cont. Each experiment uses 10,000,000 uniform random quadruples for each connected subgraph.
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Figure 22: $\Delta$ vs $L$ for weighted (metric) earthquake graphs for each catalog. The experiment uses 10,000,000 uniform random quadruples for each connected subgraph.
Figure 22: $\Delta$ vs $L$ for weighted (metric) earthquake graphs for each catalog cont. The experiment uses 10,000,000 uniform random quadruples for each connected subgraph.
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<td>San Jacinto Fault Zone</td>
<td>Unweighted</td>
<td>9784</td>
<td>76389</td>
<td>11.4</td>
<td>0.70</td>
<td>5.50</td>
<td>0.50</td>
</tr>
<tr>
<td>San Jacinto Fault Zone</td>
<td>Weighted</td>
<td>9784</td>
<td>76389</td>
<td>161.7</td>
<td>0.70</td>
<td>76.6</td>
<td>7.50</td>
</tr>
<tr>
<td>ETAS</td>
<td>Unweighted</td>
<td>9392</td>
<td>53287</td>
<td>7.40</td>
<td>0.66</td>
<td>4.50</td>
<td>0.28</td>
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<td>ETAS</td>
<td>Weighted</td>
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<td>53287</td>
<td>138.1</td>
<td>0.66</td>
<td>83.38</td>
<td>4.93</td>
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<tr>
<td>Landers Sequence</td>
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<td>3738</td>
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<td>0.69</td>
<td>3.00</td>
<td>0.02</td>
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<td>Weighted</td>
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<td>3738</td>
<td>54.1</td>
<td>0.69</td>
<td>36.92</td>
<td>0.22</td>
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<tr>
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<td>8599</td>
<td>3.7</td>
<td>0.68</td>
<td>2.50</td>
<td>0.04</td>
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<tr>
<td>Southern California Seismicity</td>
<td>Weighted</td>
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<td>8599</td>
<td>46.0</td>
<td>0.68</td>
<td>29.70</td>
<td>0.44</td>
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<tr>
<td>North–West Pacific Ocean</td>
<td>Unweighted</td>
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<td>168694</td>
<td>3.2</td>
<td>0.70</td>
<td>2.00</td>
<td>0.14</td>
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<tr>
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<td>Weighted</td>
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<td>168694</td>
<td>58.6</td>
<td>0.70</td>
<td>39.03</td>
<td>2.20</td>
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</tbody>
</table>

Table 3: Statistics for unweighted and weighted graphs for each earthquake catalog.
7.3 Other Catalogs

Here we perform the hyperbolic analyses of Sections 7.2.1, 7.2.2, and 7.2.3 for several alternative earthquake catalogs. The corresponding graph statistics are summarized in Table 3.

7.3.1 ETAS

The ETAS catalog [25] was created from the ETAS model which belongs to the class of Marked Point Processes (MPP). In this process, the focus of the analysis is the conditional intensity \( m(t, f, m|H_t) \) of a process \( Z_t = t_i, f_i, m_i \) given its history \( H_t = (t_i, f_i, m_i : t_i < t) \) up to time \( t \), where \( t_i \) is earthquake occurrence times, \( f_i \) the coordinates of the epicenter, and \( m_i \) the magnitudes. The conditional intensity completely specifies the process \( Z_t \). For a comprehensive model definition please refer to [45] and [25]. This catalog covers a time period of about 22 years and contains 27,855 earthquakes with magnitudes ranging from 2.50 to 7.33. The map of the examined catalog is shown in Fig. 16a. The time-latitude projection of the examined earthquakes is shown in Fig. 17a. The time-magnitude projection of the earthquakes is shown in Fig. 18a and the empirical survival function of the earthquake magnitudes in shown in Fig. 19a.

As in Section 7.2.1, we select 10,000,000 quadruples from the catalog to compute \( \delta \), see Fig. 20a. The earthquake graph constructed by connecting the earthquakes (vertices) with proximity \( \log_{10} \eta_{ij} < 20 \). This results in only one subgraph of over 500 nodes, from which we select 1,000,000 quadruples. The corresponding plots for the unweighted graph distance and weighted graph distance are shown in Figs. 21a and 22a, respectively.
7.3.2 Landers sequence

The Landers sequence catalog covers seismicity within the fault zone of the Landers earthquake. This event occurred on June 28th, 1992 near the town of Landers, California. The 7.3 magnitude earthquake had a significant impact in Southern California, namely a string of additional events that occurred both before and after the Landers earthquake. The catalog of these events and others covers the time period from 1981–2014. It contains 66,682 earthquakes with magnitudes ranging from 0.00 to 7.30. The map of the examined catalog is shown in Fig. 16b. The time-latitude projection of the examined earthquakes is shown in Fig. 17b. The time-magnitude projection of the earthquakes is shown in Fig. 18b and the empirical survival function of the earthquake magnitudes in shown in Fig. 19b. As in Section 7.2.1, we select 10,000,000 quadruples from the catalog to compute $\delta$, see Fig. 20b. The earthquake graph constructed by connecting the earthquakes (vertices) with proximity $\log_{10} \eta_{ij} < 20$. This results two subgraphs of over 500 nodes, from which we select 1,000,000 quadruples, totaling 2,000,000 quadruples. The corresponding plots for the unweighted graph distance and weighted graph distance are shown in Figs. 21b and 22b, respectively.

7.3.3 Southern California Seismicity

The Southern California Seismicity catalog contains events from the time period from 1981–2019. The catalog contains 123,275 earthquakes with magnitudes ranging from 2.00 to 7.30. The map of the examined catalog is shown in Fig. 16c. The time-latitude projection of the examined earthquakes is shown in Fig. 17c. The time-magnitude projection of the earthquakes is shown in Fig. 18c and the empirical survival function of the earthquake magnitudes in shown in Fig. 19c. As in Section 7.2.1, we select 10,000,000
quadruples from the catalog to compute $\delta$, see Fig. 20c. The earthquake graph constructed by connecting the earthquakes (vertices) with proximity $\log_{10} \eta_{ij} < 13$. This results five subgraphs of over 500 nodes, from which we select 1,000,000 quadruples, totaling 5,000,000 quadruples. The corresponding plots for the unweighted graph distance and weighted graph distance are shown in Figs. 21c and 22c, respectively.

7.3.4 North–West Pacific Ocean

The North–West Pacific Ocean catalog [33] covers time period from 2000–2015. It contains 2,311 earthquakes with magnitudes ranging from 5.00 to 9.10. For each registered earthquake $i$, the catalog reports its occurrence time $t_i$; location comprised of latitude $\lambda_i$, longitude $\phi_i$, and depth $z_i$; and magnitude (a logarithmic measure of energy) $m_i$. The map of the examined catalog is shown in Fig. 16d. The time-latitude projection of the examined earthquakes is shown in Fig. 17d. The time-magnitude projection of the earthquakes is shown in Fig. 18d and the empirical survival function of the earthquake magnitudes in shown in Fig. 19d. As in Section 7.2.1, we select 10,000,000 quadruples from the catalog to compute $\delta$, see Fig. 20d. The earthquake graph constructed by connecting the earthquakes (vertices) with proximity $\log_{10} \eta_{ij} < 20$. This results in only one subgraph of over 500 nodes, from which we select 1,000,000 quadruples. The corresponding plots for the unweighted graph distance and weighted graph distance are shown in Figs. 21d and 22d, respectively.

8 Discussion

This work examines the large scale geometric property of the space of observed earthquakes (Fig. 10,11) equipped with the Baiesi-Paczuski proximity (32). We estimate
Gromov $\delta$-hyperbolicity (Sect. 4.1) and find that in all conducted experiments the $\delta$ parameter is well bounded from above, and does not tend to increase with the linear dimension of the examined region. These properties characterize a hyperbolic metric space (Fig. 3) and suggest that a negatively curved hyperbolic geometry underlies the space-time-magnitude distribution of seismicity.

Our estimations are done in three complementary ways – in the time-space-magnitude domain $D$ of earthquakes equipped with Baiesi-Paczuski proximity $\eta$ (Figs. 12, 20), in combinatorial proximity graphs (Figs. 13, 21), and in metric proximity graphs (Figs. 14, 22). That all estimations consistently suggest $\delta$-hyperbolicity contributes to the robustness of our empirical results.

The stationary and homogeneous Poisson synthetic catalog also exhibit hyperbolic property (Sect. 7.1, Fig. 9). This suggests that the earthquake hyperbolicity is attributed to the general properties of the Baiesi-Paczuski earthquake proximity $\eta$ rather than complex clustering and interactions of the observed earthquakes (see Fig. 11). At the same time, the values of quadruple diameter and $\Delta$ statistic reported in synthetic catalogs (Fig. 9) are slightly different from those for the observed earthquakes (Fig. 12). The effects of earthquake space-time clustering and other deviations from a homogeneous Poisson field on the hyperbolic property is an interesting problem that will be explored elsewhere.

The suggested hyperbolic geometry of earthquake field has the following immediate consequences and interpretations:

(a) The complex heterogeneous space-time-energy characteristics of earthquakes are represented via a uniform distribution of points in a hyperbolic space. This interpretation needs to be further explored. It might be particularly useful for examining individual aftershock sequences and/or swarms.
(b) The power law degree distributions in earthquake networks [7] is explained via the results of Krioukov et al. [31].

(c) A useful insight is provided into the geometry of earthquake interactions [40] that can be interpreted and studied via hyperbolic geometry and curved space-time geodesics.

(d) The hyperbolic embedding suggests a natural neighborhood of an earthquake in space-time-energy domain. This domain can be used as a guide in analysis of aftershock/foreshock domains, and can facilitate aftershock/foreshock/swarm identification and earthquake declustering problem.

It remains an open problem to test the suggested hyperbolicity of the earthquake space using other models and data, and probably expand this framework to other phenomena and conceptual models related to seismicity. The latter include solar flare statistics, rock fracture (acoustic emission experiments), and a range of models explored within the self-organized criticality (SOC) framework.
References


