The Homotopy Theory of Commutative dg Algebras and Representability Theorems for Lie Algebra Cohomology

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Abstract

Building on the seminal works of Quillen [12] and Sullivan [16], Bousfield and Gugenheim [3] developed a "homotopy theory" for commutative differential graded algebras (cDGA) in order to study the rational homotopy theory of topological spaces. This "homotopy theory" is a certain categorical framework, invented by Quillen, that provides a useful model for the non-abelian analogs of the derived categories used in classical homological algebra.

In this masters thesis, we use K. Brown’s generalization [5] of Quillen’s formalism to present a homotopy theory for the category of semi-free, finite-type cDGA over a field $k$ of characteristic 0. In this homotopy theory, the "weak homotopy equivalences" are a refinement of those used by Bousfield and Gugenheim. As an application, we show that the category of finite-dimensional Lie algebras over $k$ faithfully embeds into our homotopy category of cDGA via the Chevalley-Eilenberg construction. Moreover, we prove that Lie algebra cohomology with coefficients in a trivial module is representable in this homotopy category, in analogy with the classical representability theorem for singular cohomology of CW complexes. Finally, we show that central extensions of Lie algebras are recovered within our homotopical framework as certain principal cofiber sequences.
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1 Introduction

The quintessential characteristic of homotopy theory is the replacement a rigid notion of equivalence with a more flexible one. This can be achieved by localizing the relevant category with respect to the class of desired equivalences. The process of localizing categories shares many similarities with the process of localizing of rings. In both processes, we look for the smallest category, resp. ring, containing the original category, resp. ring, such that the desired morphisms, resp. elements, are invertible. Thus, if it exists, the localization, written as \( C[W^{-1}] \), of a category \( C \) with respect to some class of morphisms \( W \), is defined by the following universal property: Any functor \( F: C \to D \) that carries the class \( W \) to isomorphisms factors uniquely through \( C[W^{-1}] \).

For arbitrary \( C \) and \( W \) we can sometimes construct \( C[W^{-1}] \) by formally inverting the morphisms in \( W \). That is, we define \( C[W^{-1}] \) to be the category with the same objects as \( C \) but with morphisms defined as reduced zig zags of morphisms in \( C \)

\[
X \xrightarrow{f_1} \cdots \xleftarrow{w_1} \cdots \xrightarrow{f_i} \cdots \xleftarrow{w_j} Y
\]

where \( w_k \in W \). The arrows pointing to the left provide the formal inverses to the morphisms in \( W \). This is a clunky category to work with, but that is to be expected since \( C \) and \( W \) are generic and the only information used in its construction is the universal property. But if we know more about the behavior of the class \( W \) in relation to \( C \), we can obtain a cleaner description of \( C[W^{-1}] \).

1.1 Homotopy theory as non-abelian homological algebra

Let \( \text{Ch}_{\geq 0}(R) \) be the category of chain complexes of \( R \)-modules for some ring \( R \). We can present the localization of \( \text{Ch}_{\geq 0}(R) \) with respect to the class of quasi isomorphisms in the form of the derived category of \( R \)-modules, written as \( D(R) \). The objects of the derived category are the same as \( \text{Ch}_{\geq 0}(R) \), but the morphisms in \( D(R) \) between two complexes, \( M \) and \( N \), are equivalence classes of spans. A span from \( M \) to \( N \) is a pair of chain maps

\[
M \leftarrow \tilde{M} \rightarrow N
\]

in which the left arrow is a quasi isomorphism. If \( M \) is a complex of projective modules, then
the set of equivalence classes of spans from $M$ to any complex $N$, can be presented as the set of chain homotopy classes of chain maps from $M$ to $N$. This category satisfies the above universal property, so $D(R)$ can be considered the localization. However, the derived category is equipped with additional structure, which allows one to deduce that

$$\text{hom}_{D(A)}(X, Y[n]) \cong \text{Ext}_A^n(X, Y)$$

Yet, the classical construction of the derived category does not make sense within the context of non-abelian categories. One such solution to this is the theory of model categories, introduced by Quillen in [13]. A model category is a category $M$ (with small limits and colimits), equipped with three classes of maps called cofibrations, fibrations and weak equivalences subject to a handful of axioms. It is customary to call the localization of $M$ with respect to the class of weak equivalences the homotopy category, which we denote by $\text{Ho}(M)$. Within this theory the bifibrant objects in $M$ play an important role, i.e. the objects in $M$ such that the map from the initial object is a cofibration and the map to the terminal object is a fibration. The morphisms in the homotopy category of the full subcategory of bifibrant objects are in one-to-one correspondence with certain equivalence classes of morphisms in $M$. Moreover, every object in $M$ is weakly equivalent to a bifibrant object. Thus, we can show this category is equivalent to the homotopy category of $M$. Analogous to the derived category, the homotopy category is more than just the localization. For example, if $M$ is pointed then we can define a notion of a loop and a suspension endofunctor on the homotopy category. In this case, Quillen [13] showed that certain hom-sets in the homotopy category of such model categories correspond to familiar notions of “cohomology groups”

### 1.2 Brown categories

Since the abstract framework of model categories requires that the underlying category has small limits and colimits, one must use a different approach to study the homotopy theory of a category if it is not closed with respect to limits and colimits. One such structure is the notion of a category of fibrant objects (CFO), which was introduced by Brown in [5]. A CFO is a category $C$ (with finite products) equipped with a class of fibrations and weak equivalences subject to axioms similar to that of a model category. One major distinction between these two sets of axioms is that every object in
a CFO is required to be fibrant, i.e. the unique map from every object in \( C \) into the terminal object must be a fibration. One can define a similar structure on a category with finite coproducts equipped with cofibrations and weak equivalences subject to the dualized axioms of a CFO. We will refer to categories with this structure as \textit{Brown categories}. A prototypical Brown category is the category of finite-type CW complexes where the cofibrations are retracts of inclusions and the weak equivalences are homotopy equivalences.

\subsection{Sullivan algebras and rational homotopy theory}

Rational homotopy theory is the study of simply connected spaces up to rational weak equivalence. In other words, it is the study of the category of simply connected spaces localized with respect to the maps, \( f : X \to Y \), such that \( f_* \otimes \text{id}_\mathbb{Q} : \pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q} \) is an isomorphism for all \( n \). There is an equivalent algebraic formulation of this category given by the homotopy category associated to the Bousfield-Gugenheim [3] (BG) model structure on commutative differential graded algebras (cdgas) over \( \mathbb{Q} \). In this model structure, the fibrations are degree wise surjections, and the weak equivalences are the quasi isomorphisms. The definition of a cofibration involves the notion of a \textit{Sullivan algebra}. A Sullivan algebra \( S(V) \) is a cdga whose underlying commutative graded algebra is isomorphic to the symmetric algebra on a graded vector space \( V \) equipped with an increasing filtration that is compatible with the differential on \( S(V) \) in a certain way. A cofibration in the (BG) model structure is an inclusion \( A \hookrightarrow A \otimes S(V) \) of cdgas where \( S(V) \) is a Sullivan algebra. Thus the cofibrant objects are exactly the Sullivan algebras.

\subsection{Chevalley-Eilenberg algebras of finite dimensional Lie algebras}

For any finite dimensional Lie algebra \( \mathfrak{g} \) over a field of char 0 we can define a differential on the graded symmetric algebra of \( \mathfrak{g}^* \) concentrated in degree 1 by dualizing its bracket. The fact that the bracket satisfies the Jacobi identity implies that the differential squares to zero. This cdga contains information about the Lie algebra cohomology of \( \mathfrak{g} \) in the sense that the cohomology of the cdga is the cohomology of \( \mathfrak{g} \) with coefficients in the ground field. This cdga is called the \textit{Chevalley-Eilenberg algebra} (CE algebra) of \( \mathfrak{g} \), and the assignment of a Lie algebra to its CE algebra defines a functor. In fact, this functor faithfully embeds the category of Lie algebras into the the category of cdgas. One
might be tempted to believe that the CE algebra of a Lie algebra is a Sullivan algebra, but this is not the case. A CE algebra is a Sullivan algebra iff $\mathfrak{g}$ is nilpotent. Hence, if we wish to study all Lie algebras via their CE algebras, the BG model structure on cdga may not be the best fit.
2 Summary of main results

2.1 The Brown category of Chevalley-Eilenberg algebras

Upon reviewing relevant algebraic definitions and facts in Sec. 3, we introduce (see Def. 4.4) the category, CEA\textsubscript{alg}, of Chevalley-Eilenberg algebras (CE algebras). A CE algebra is a semi free, finite type commutative differential graded algebra (cdga) \((S(V), \delta)\), generated by a positively graded vector space \(V\). A morphism of CE algebras is simply a cdga morphism. To every CE algebra \((S(V), \delta)\) we can associate a cochain complex \((V, \delta^1_V)\) where the differential is defined by pre and post composing \(\delta\) with the inclusion \(V \hookrightarrow S(V)\) and the projection \(S(V) \rightarrow V\), respectively. Similarly, to every morphism of CE algebras \(F: (S(V), \delta_V) \rightarrow (S(W), \delta_W)\) we associate a morphism of cochain complexes \(F^1_1: (V, \delta^1_V) \rightarrow (W, \delta^1_W)\) where \(F^1_1\) is obtained by pre and post composing \(F\) with the inclusion \(V \hookrightarrow S(V)\) and the projection \(S(W) \rightarrow W\). We use this associated cochain map in Def. 4.9 to characterize two classes of morphisms in CEA\textsubscript{alg}. We say that \(F\) is a weak equivalence iff \(F^1_1\) is a quasi isomorphism of cochain complexes, and we say that \(F\) is a cofibration iff \(F^1_1\) is injective in all degrees greater than or equal to 2. In Thm. A.6 of the Appendix, it is shown that weak equivalence of CE algebras induces a quasi isomorphism of cdgas, but the converse is false in general (see Sec. 4.3).

In Prop. 4.11, we show that every strict morphism of CE algebras can be factored into a cofibration followed by a weak equivalence. In particular, there exists a such a factorization of the fold map \(\nabla: S(V) \otimes S(V) \rightarrow S(V)\). Our main theorem below and Brown’s Factorization Lemma (see Lemma 4.3) imply that any morphism of CE algebras can be factored into a cofibration followed by weak equivalence.

We provide an explicit description of the pushout of a strict cofibration along an arbitrary morphism of CE algebras in Prop. 4.14. Using this proposition and Lemma 4.16 we show that the pushout of an arbitrary cofibration along any morphism exists in CEA\textsubscript{alg}. Moreover, we prove the pushout of an (acyclic) cofibration is itself an (acyclic) cofibration.

Although the category CEA\textsubscript{alg} is a full subcategory of cdga, which is both complete and cocomplete, it is not closed under products. For this reason CEA\textsubscript{alg} cannot form a model category. It is however, closed under finite coproducts (see Prop. ??). However, the main novel result of this thesis
establishes that this homotopical structure can be modeled as a Brown category:

**Theorem.** The category $\text{CEAlg}$, of $\text{CE}$ algebras over a field of characteristic 0 and cdga morphisms between them, admits the structure of a Brown category of cofibrant objects in which a morphism 

$$F: (S(V), \delta_V) \to (S(W), \delta_W)$$

is:

- a weak equivalence iff $F_1^1: (V, \delta_V^{-1}) \to (W, \delta_W^{-1})$ is a quasi isomorphism of cochain complexes

- a cofibration iff $F_1^1: (V, \delta_V^{-1}) \to (W, \delta_W^{-1})$ is injective in all degrees greater than 1

where $F_1^1 = \text{pr}_W F|_V$, $\delta_V^{-1} = \text{pr}_V \delta_V|_V$, and $\delta_W^{-1} = \text{pr}_W \delta_W|_W$.

Let us mention a few initial remarks about this result. First since there are quasi isomorphisms of cdgas that aren’t weak equivalences, the homotopy category of $\text{CE}$ algebras, $\text{Ho}(\text{CEAlg})$, is distinct from the homotopy category induced by the Bousfield Gugenheim model structure on cdgas, $\text{Ho}(\text{cdga})$.

Second, one of the potential applications we have in mind for this alternative homotopical structure is to study (non-abelian) Lie algebra cohomology via representable functors out of $\text{Ho}(\text{CEAlg})$, in analogy with the classical E. Brown representability theorem for the singular cohomology of CW complexes [4]. As a first step, in this thesis we focus on recovering the classical Chevalley-Eilenberg cohomology within this homotopy theoretic context. Indeed, we prove that the category of finite dimensional Lie algebras embeds faithfully into $\text{Ho}(\text{CEAlg})$. In contrast, we show in Sec. 4.3 that there is no such embedding into $\text{Ho}(\text{cdga})$. Moreover since a $\mathfrak{g}$-module, for some Lie algebra $\mathfrak{g}$, is a vector space $M$ equipped with a Lie algebra homomorphism $\mathfrak{g} \to \text{End}_k(M)$ where $\text{End}_k(M)$ is given the commutator bracket, the category of $\mathfrak{g}$-modules can be realized as a category under $\text{Ho}(\text{CEAlg})$. These facts suggest that $\text{Ho}(\text{CEAlg})$ is a suitable environment in which to study Lie algebras.

Finally we note that there is a similar homotopical structure on a class of infinite dimensional topological semi free cdgas, due to Lazarev [11], that shares many of the same properties of $\text{Ho}(\text{CEAlg})$. 
The advantage of the finite type algebras is that they have well-defined spatial realizations as certain simplicial manifolds [9], in analogy with the fact that finite-dimensional Lie algebras can be integrated into Lie groups.

2.2 Representability of Lie algebra cohomology

In the second part of this thesis, we begin in section 5 by introducing the notion of suspension in a pointed Brown category \( C \). Analogous to the topological case, the suspension \( \Sigma X \) of an object \( X \) in a Brown category is a cogroup object in the homotopy category. Thus, for all \( X \in C \), the functor \( [\Sigma X, -] \) factors through the category of groups, where \([-, -]\) denotes the set of morphisms in the homotopy category.

For \( n \geq 1 \) and \( M \) a finite dimensional vector space, we denote by \( K(M, n) \) the symmetric algebra on \( M^* \) concentrated in degree \( n \) with trivial differential. We show, in Prop. 5.12, that \( \Sigma K(M, n + 1) \) is weakly equivalent to \( K(M, n) \). Thus, \([K(M, n), (S(V), \delta)]\) is a group for all CE algebras \((S(V), \delta)\). In particular we prove the following:

**Theorem.** Let \( C(g) \) be the Chevalley-Eilenberg algebra of a finite dimensional Lie algebra \( g \). If we consider \( M \) as a trivial \( g \)-module, then for all \( n \geq 1 \) there is a natural isomorphism of abelian groups

\[
H^n(g, M) \cong [K(M, n), C(g)]
\]

This result is analogous to E. Brown’s representability theorem for singular cohomology [4], which states that \( H^n(X, A) \) is naturally isomorphic as a group to \([X, K(A, n)]\), for any pointed CW complex \( X \) and Eilenberg-Maclane space \( K(A, n) \). Just as our group structure arises from the suspension functor, the group structure on \([X, K(A, n)]\) is given by the fact that \( \Omega K(A, n + 1) \simeq K(A, n) \).

2.3 Classification of principal cofibrations

Finally, motivated by the definition of a principal fiber sequence given in [1], we define, in Sec. 5.1, the notion of a principal cofiber sequence in a pointed Brown category. In a manner similar to classifying principal \( G \)-bundles over a CW complex, we show that the set of equivalence classes
of principal cofiber sequences with fixed classifying space \( K(M, n + 1) \), base \( C(\mathfrak{g}) \), and cofiber \( K(M, n) \) are in one to one correspondence with \([K(M, n + 1), C(\mathfrak{g})]\) for all \( n \geq 1 \). In addition, in the \( n = 1 \) case, we show that equivalence classes of these principal cofiber sequences are in one to one correspondence with equivalence classes of central extensions of the Lie algebra \( \mathfrak{g} \) by \( M \). Thus the classic correspondence between the second Lie algebra cohomology group with coefficients in a trivial module and central extensions fits into the following commutative diagram of bijections between sets:

\[
\begin{array}{ccc}
\{ \text{Central extensions of } \mathfrak{g} \text{ by } M \} / \sim & \overset{\text{Prop. 5.19}}{\approx} & H^2(\mathfrak{g}, M) \\
\text{Prop. 5.20} & \overset{\approx}{\sim} & \text{Thm. 5.14} \\
\{ \text{Principal cofiber sequences with classifying space } K(M, 2), \text{ base } C(\mathfrak{g}), \text{ and cofiber } K(M, 1) \} / \sim & \overset{\text{Prop. 5.16}}{\approx} & [K(M, 2), C(\mathfrak{g})]
\end{array}
\]

Hence the well-established bijection between central extensions and \( H^2 \) of a Lie algebra is recovered by the homotopy theory of CEAlg. Moreover, the above diagram for the \( n = 1 \) case suggests that, for the \( n \geq 2 \) case, we interpret the sequence of bijections

\[
H^n(\mathfrak{g}, M) \approx [K(M, n + 1), C(\mathfrak{g})] \approx \left\{ \text{Principal cofiber sequences with classifying space } K(M, n + 1) \text{ base } C(\mathfrak{g}), \text{ and cofiber } K(M, n) \right\} / \sim
\]

as a realization of higher degree Lie algebra cohomology classes as “higher degree” extensions (i.e. cofiber sequences) of \( \mathfrak{g} \) by more general CE algebras.
3 Preliminaries

3.1 Conventions

Throughout this thesis, we will adopt the following conventions and notations:

- \( \mathbb{k} \) denotes a field of characteristic zero
- We define \( |x| := n \) if \( x \in V^n \) for some graded vector space \( V \)
- \( V[k] \) denotes the graded vector space with \( V[k]^n := V^{n-k} \) for a graded vector space \( V \)
- Non-graded vector spaces will be considered as graded vector spaces considered in degree 0
- The category of commutative dg algebras and dg algebra morphisms is denoted by cdga.
- The category of semi-free, finite-type commutative dg algebras and dg algebra morphisms is denoted by CEA Alg.

3.2 Projective model structure on \( \text{Ch}^{\geq 0} \)

For a more thorough introduction to homological algebra, we recommend Weibel’s book [19], and for more on the model category structure on complexes, we recommend Goerss and Schemmerhorn’s expository article [8].

Recall that a **cochain complex** \( (D, d) \) over \( \mathbb{k} \) is a \( \mathbb{N} \)-graded \( \mathbb{k} \)-vector space \( D = \bigoplus_{n \geq 0} D^n \) equipped with a degree 1 map of graded vector spaces \( d: D \to D \), called the **differential**, with the property that \( d^2 = 0 \). The condition that the differential squares to 0 implies that \( \text{im} \ d \subseteq \text{ker} \ d \).

Hence we define the **cohomology** of a cochain complex \( (D, d) \) to be the \( \mathbb{N} \)-graded \( \mathbb{k} \)-vector space

\[
H^*(D) = \bigoplus_{n \geq 0} H^n(D)
\]

where

\[
H^n(D) := \frac{(\text{ker} \ d: D^n \to D^{n+1})}{(\text{im} \ d: D^{n-1} \to D^n)}
\]

A morphism of cochain complexes \( f: (D, d_D) \to (E, d_E) \) is a degree 0 map of graded vector spaces \( f: D \to E \) with the property that \( d_E f = f d_D \). A morphism of complexes \( f \) is said to be injective (resp. surjective) in degree \( n \) iff \( f|_{D^n}: D^n \to E^n \) is injective (resp. surjective). Every
morphism of complexes $f$ induces a degree 0 map of graded vector spaces $H^*(f): H^*(D) \to H^*(E)$. If $H^*(f)$ is an isomorphism we say that $f$ is a **quasi isomorphism**. We denote the category of cochain complexes and cochain morphisms by $\text{Ch}^{\geq 0}$.

**Example 3.1.**

1. If $(D, d_D)$ and $(E, d_E)$ are complexes then their **direct sum** $(D \oplus E, d_\oplus)$ is defined as $D \oplus E := \bigoplus_{n \geq 0} D^n \oplus E^n$ with differential $d_\oplus(x, y) := (d_D(x), d_E(y))$. With the obvious projections and inclusions, $(D \oplus E, d_\oplus)$ is the categorical product and coproduct in $\text{Ch}^{\geq 0}$, respectively.

2. If $(D, d_D)$ and $(E, d_E)$ are complexes then their **tensor product** $(D \otimes E, d_\otimes)$ is defined as the graded vector space whose degree $n$ component is

$$(D \otimes E)^n := \bigoplus_{i+j=n} D^i \otimes E^j$$

The differential is defined as $d_\otimes(x \otimes y) := d_D(x) \otimes y + (-1)^{|x|} x \otimes d_E(y)$.

3. If $(D, d_D)$ and $(E, d_E)$ are complexes then the **mapping complex** $(\text{Map}(D, E), \partial)$ is graded vector space $\text{Map}(D, E) := \bigoplus_{n \geq 0} \text{Map}^n(D, E)$, where $\text{Map}^n(D, E)$ is the $\mathbb{k}$-vector space of degree $n$ maps from $D$ to $E$, equipped with the differential given by

$$\partial(f) = d_E f - (-1)^{|f|} f d_D$$

There are three classes of cochain maps that are of particular interest to us. The class of quasi isomorphisms, the class of injections in degrees $n \geq 1$, and the class of surjections in all degrees. With these distinguished classes, the category $\text{Ch}^{\geq 0}$ admits the structure of a model category. However, for our purposes it will be more convenient to work with the following reformulation of this model category.

**Definition 3.2.** Let $\text{Ch}^{\geq 1}$ denote the category of **positively graded cochain complexes** $(D, d_D)$ over $\mathbb{k}$, i.e. $D^0 = 0$. Then a morphism $f : (D, d_D) \to (E, d_E)$ is called

- **A weak equivalence** iff $f$ is a quasi isomorphism
- **A cofibration** iff $f$ is injective in all degrees $n \geq 2$
• A fibration iff $f$ is surjective in all degrees

We denote the category $\text{Ch}^{\geq 1}$ equipped with these three classes by $\text{Ch}^{\geq 1}_{\text{proj}}$.

**Theorem 3.3.** The category $\text{Ch}^{\geq 1}_{\text{proj}}$ admits the structure of a model category.

We refer to this structure the **projective model structure** on the category of positively graded cochain complexes. The interested reader can find a variation of the proof that $\text{Ch}^{\geq 1}_{\text{proj}}$ is a model category in [8, Thm. 1.5].

### 3.3 Commutative differential graded algebras

For those readers who desire a more in depth exposition of graded algebra, we recommend Hess’ article [10] and Felix, Halperin, and Thomas’ book [6, Sec. 1.3].

A **commutative graded algebra** $(A, \mu, \eta)$ over $k$ is a $\mathbb{N}$-graded $k$-vector space $A = \bigoplus_{n \geq 0} A^n$ equipped with two degree 0 maps of graded vector spaces

$$\eta: k \to A$$

called the **unit**, and

$$\mu: A \otimes A \to A, \quad x \otimes y \mapsto xy$$

called **multiplication**, such that the following are satisfied:

1. $\mu$ is associative, i.e. $(xy)z = x(yz)$

2. $\mu$ is graded commutative, meaning $xy = (-1)^{|x||y|}yx$

3. $\mu$ is unital, that is TFDC

$$\begin{align*}
\kappa \otimes A & \xrightarrow{\eta \otimes \text{id}_A} A \otimes A & A \otimes A & \xleftarrow{\text{id}_A \otimes \eta} A \otimes \kappa \\
 & & \downarrow{\mu} & \downarrow{\mu} \\
 & & A & A
\end{align*}$$

**Notation 3.4.** Let $1_A = \eta(1)$. 
Condition 3 implies that there exists a $1_A \in A^0$ such that $1_A x = x = x 1_A$. A morphism of commutative graded algebras $F: (A, \mu_A, \eta_A) \to (B, \mu_B, \eta_B)$ is a degree 0 map of graded vector spaces such that $F \eta_A = \eta_B$ and $F \mu_A = \mu_B (F \otimes F)$.

**Definition 3.5.** A left $A$-module over a commutative graded algebra $(A, \mu, \eta)$ is a graded vector space $M$ equipped with a degree 0 linear map

$$A \otimes M \to M, \quad a \otimes m \mapsto am$$

such that $(aa')m = a(a'm)$ for all $a, a' \in A$ and $m \in M$. A right $A$-module is a graded vector space $M$ equipped with a degree 0 linear map $M \otimes A \to M$ that satisfies the analogous associative property. We refer to a graded vector space that is both a left and right module as a module.

**Example 3.6.** Let $F: (A, \mu_A, \eta_A) \to (B, \mu_B, \eta_B)$ be a morphism of commutative graded algebras. Then $B$ is an $A$-module given by the assignments $a \otimes b \mapsto F(a)b$ and $b \otimes a \mapsto bF(a)$ for all $a \in A$ and $b \in B$.

**Definition 3.7.** A commutative differential graded algebra (cdga) $(A, \delta, \mu, \eta)$ over $\mathbb{k}$ consists of a cochain complex $(A, \delta)$ over $\mathbb{k}$ equipped with two cochain maps $\eta: (\mathbb{k}, 0) \to (A, \delta)$ and $\mu: (A, \delta) \otimes (A, \delta) \to (A, \delta)$, such that conditions 1-3 above are satisfied. A morphism of cdgas $F: (A, \delta_A, \mu_A, \eta_A) \to (B, \delta_B, \mu_B, \eta_B)$ is a cochain map $F: (A, \delta_A) \to (B, \delta_B)$ such that $F \eta_A = \eta_B$ and $F \mu_A = \mu_B (F \otimes F)$. We denote the category of cdgas and cdga morphisms by $\text{cdga}$.

**Notation 3.8.** We will write a cdga as $(A, \delta)$ leaving the multiplication and unit implicit, and similarly, we will write a commutative graded algebra as $A$ leaving the multiplication and unit implicit.

**Example 3.9.** Let $(A, \delta_A)$ and $(B, \delta_B)$ be cdgas. The tensor product of these cdgas $(A \otimes B, \delta_\otimes)$ is defined as the tensor product of cochain complexes with multiplication given by

$$(a \otimes b)(a' \otimes b') := (-1)^{|b||a'|} aa' \otimes bb'$$

and unit given by $\eta_\otimes(1) := 1_A \otimes 1_B$. This construction gives the categorical coproduct in $\text{cdga}$. Indeed, given two morphisms of cdgas (we suppress the differentials) $\psi_A: A \to Q$ and $\psi_B: B \to Q$ we can define a morphism of cdgas $\tilde{\psi}: A \otimes B \to Q$ by $\tilde{\psi}(a \otimes b) = \psi_A(a) \psi_B(b)$, and inclusions of
cdgas $\iota_A : A \to A \otimes B$ and $\iota_B : B \to A \otimes B$ by $\iota_A(a) = a \otimes 1_B$ and $\iota_B(b) = 1_A \otimes b$. It is easy to check that the necessary maps commute and that $\tilde{\psi}$ is unique.

**Remark 3.10.** Note that a cdga is a commutative graded algebra equipped with a differential $\delta$ that satisfies the following for all $x, y \in A$

$$\delta(xy) = \delta(x)y + (-1)^{|x|}x\delta(y)$$

The above relation is called the (graded) Leibniz rule. Hence the differential on a cdga is an example of a derivation.

**Definition 3.11.** A derivation of degree $n$ is a degree $n$ linear map $F : A \to M$, where $A$ is a commutative graded algebra and $M$ is an $A$-module, such that

$$F(aa') = F(a)a' + (-1)^{|a|}aF(a')$$

for all $a, a' \in A$.

### 3.4 Semi-free cdgas

**Definition 3.12.** The tensor algebra, $T(V)$, of a $\mathbb{N}$-graded $\mathbb{k}$-vector space $V$ is defined to be $T(V) := \bigoplus_{n\geq0} T^n(V)$ where $T^0(V) := \mathbb{k}$ and for $n \geq 1$, $T^n(V) := V \otimes \cdots \otimes V$ is the $n$-fold tensor product. Multiplication in $T(V)$ is given by the assignment

$$(x_1 \otimes \cdots \otimes x_i)(y_1 \otimes \cdots \otimes y_j) \mapsto x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j$$

and the unit is given by

$$\mathbb{k} \xrightarrow{\text{id}} T^0(V) \hookrightarrow T(V)$$

The multiplication in $T(V)$ is associative and unital, but is not graded commutative.

**Definition 3.13.** The symmetric algebra $S(V)$ of a $\mathbb{N}$-graded $\mathbb{k}$-vector space $V$ is defined as $S(V) := T(V)/I$ where $I$ is the two sided ideal generated by $\{x \otimes y - (-1)^{|x||y|}y \otimes x \mid x, y \in V\}$. We write $x_1 \vee \cdots \vee x_n$ to denote the equivalence class represented by $x_1 \otimes \cdots \otimes x_n$ in $S(V)$.

Equivalently, if $I^n = I \cap T(V)$, then $S(V) = \bigoplus_{n\geq0} S^n(V)$ where $S^n(V) := T^n(V)/I^n$. Since $I^0 = 0 = I^1$, $S^0(V) = \mathbb{k}$ and $S^1(V) = V$. Note that $S^n(V)$ corresponds to words of length $n$
and not degree \( n \) elements. In general, \(|x_1 \lor \cdots \lor x_n| = |x_1| + \cdots + |x_n|\). \( S(V) \) naturally has the structure of a commutative graded algebra. The multiplication and unit are inherited from the tensor algebra \( T(V) \). That is, multiplication is given by the assignment

\[
(x_1 \lor \cdots \lor x_i)(y_1 \lor \cdots \lor y_j) \mapsto x_1 \lor \cdots \lor x_i \lor y_1 \lor \cdots \lor y_j
\]

and the unit is given by

\[
\mathbb{k} \xrightarrow{\text{id}} S^0(V) \hookrightarrow S(V)
\]

**Proposition 3.14.** Let \( V \) be a graded vector space. Then \( S(V) \) is the free commutative graded algebra on \( V \). That is, for any degree 0 linear map \( f : V \to A \), where \( A \) is a commutative graded algebra, there exists a unique morphism of commutative graded algebras \( F : S(V) \to A \) such that \( F|_V = f \).

**Proof.** Let \( n \geq 1 \) and let \( x_1 \lor \cdots \lor x_n \in S^n(V) \). Then we define

\[
F(x_1 \lor \cdots \lor x_n) = F(x_1) \cdots F(x_n)
\]

and \( F(1_{\mathbb{k}}) = 1_A \).

\( \square \)

There exists a similar method of extending degree \( n \) graded linear maps to derivations.

**Proposition 3.15.** Let \( V \) be a graded vector space and let \( M \) be a \( S(V) \)-module. Then for any degree \( n \) linear map \( f : V \to M \), there exists a degree \( n \) derivation \( F : S(V) \to M \) such that \( F|_V = f \).

**Proof.** Let \( n \geq 1 \) and let \( x_1 \lor \cdots \lor x_n \in S^n(V) \). Then we define

\[
F(x_1 \lor \cdots \lor x_n) = \sum_{i=1}^{n} (-1)^{n||x_1 \lor \cdots \lor x_{i-1}||} x_1 \cdots x_{i-1} \delta(x_i)x_{i+1} \cdots x_n
\]

and \( F(1_{\mathbb{k}}) = 0 \).

\( \square \)

Let \( V \) be a \( \mathbb{k} \)-vector space and \( n \geq 2 \). Recall that the \( n^{th} \textbf{-exterior power} \), \( \Lambda^n(V) \), of \( V \) is defined to be the quotient of the \( n \)-fold tensor product, \( V \otimes \cdots \otimes V \), by the subspace generated by

\[
\{x_1 \otimes \cdots \otimes x_n - \text{sgn}(\sigma)x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \mid \sigma \in S_n\}
\]
We denote the equivalence class represented by \( x_1 \otimes \cdots \otimes x_n \) in \( \Lambda^n(V) \) as \( x_1 \wedge \cdots \wedge x_n \). If \( V \) is finite dimensional and with basis \( \{e_1, \ldots, e_m\} \), then \( \Lambda^n V \) has dimension \( \binom{m}{n} \) with basis given by
\[
\{ e_{i_1} \wedge \cdots \wedge e_{i_n} \mid i_1 < \cdots < i_n \}
\] (1)

The exterior power can also be characterized by the following universal property. Recall that a function \( f: V \times \cdots \times V \to W \) is said to be **alternating** if \( f(x_1, \ldots, x_n) = 0 \) when \( x_i = x_j \) for \( i \neq j \). Then every multilinear alternating function \( f: V \times \cdots \times V \to W \) induces a unique \( \mathbb{k} \)-linear map \( f: \Lambda^n V \to W \). Suppose that \( f: \Lambda^j V \to \mathbb{k} \) and \( g: \Lambda^k V \to \mathbb{k} \). We define \( f \wedge g: \Lambda^{j+k} V \to \mathbb{k} \) to be the \( \mathbb{k} \)-linear map given by
\[
(f \wedge g)(x_1 \wedge \cdots \wedge x_{j+k}) = \sum_{\sigma \in \text{Shuf}(j,k)} \text{sgn}(\sigma) f(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(j)}) g(x_{\sigma(j+1)} \wedge \cdots \wedge x_{\sigma(j+k)})
\] (2)
where,
\[
\text{Shuf}(j, k) = \{ \sigma \in S_{j+k} \mid \sigma(1) < \cdots < \sigma(j) \text{ and } \sigma(j+1) < \cdots < \sigma(j+k) \}
\]

As shown in Warner’s book [18], if \( V \) has basis \( \{e_1, \ldots, e_n\} \) and \( \{\theta^1, \ldots, \theta^n\} \) is the corresponding dual basis for \( V^* \), then
\[
(\theta^1 \wedge \cdots \wedge \theta^n)(e_1 \wedge \cdots \wedge e_n) = 1
\]

The **exterior algebra** of \( V \) is the commutative graded algebra \( \Lambda(V) := \bigoplus_{n \geq 0} \Lambda^n(V) \) where \( \Lambda^0(V) := \mathbb{k} \) and \( \Lambda^1(V) := V \). The multiplication is given by the assignment
\[
(x_1 \wedge \cdots \wedge x_i)(y_1 \wedge \cdots \wedge y_j) \mapsto x_1 \wedge \cdots \wedge x_i \wedge y_1 \wedge \cdots \wedge y_j
\]
and the unit is given by
\[
\mathbb{k} \xrightarrow{\text{id}} \Lambda^0(V) \hookrightarrow \Lambda(V)
\]

**Proposition 3.16.** Let \( V \) be a \( \mathbb{k} \)-vector space, and let \( V[1] \) denote the graded vector space consisting of \( V \) concentrated in degree 1. Then there is an isomorphism of graded algebras
\[
S(V[1]) = \Lambda(V)
\]

**Proof.** Note that since \( V[1] \) is concentrated in degree 1, word length of an element in \( S(V[1]) \) cor-
responds to its degree, and the two sided ideal $I \triangleleft T(V[1])$ in Def. 3.13 is generated by \( \{x \otimes y + y \otimes x \mid x, y \in V\} \). Since $S_n$ is generated by \( \{(12), (23), \ldots, (n-1 \, n)\} \), we have that

\[
T^n(V[1]) \cap I = \{x_1 \otimes \cdots \otimes x_n - sgn(\sigma)x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \mid \sigma \in S_n\}
\]

Moreover the multiplication and the unit in both commutative graded algebras are given by concatenation of words and inclusion, respectively.

\[\Box\]

**Notation 3.17.** For the remainder of this thesis we will assume that all graded vector spaces $V$ are positively graded, i.e. $V^n = 0$ for all $n \leq 0$, unless stated otherwise.

**Definition 3.18.** A semi-free cdga over $k$ is a cdga $(S(V), \delta)$ whose underlying commutative graded algebra is isomorphic to $S(V)$ with the canonical multiplication and unit for some positively graded $k$-vector space $V = \bigoplus_{n \geq 1} V^n$.

Let $F: S(V) \to S(W)$ be a graded linear map. We can decompose $F$ by precomposing and post composing with various combinations of graded vector space inclusions and projections as follows

\[
S^j(V) \hookrightarrow \bigoplus_{n \geq 0} S^n(V) \xrightarrow{F} \bigoplus_{n \geq 0} S^n(W) \twoheadrightarrow S^k(W)
\]

We denote this composition by

\[
F^k_j: S^j(V) \to S^k(W)
\]

By the universal property of the product and coproduct the collection $\{F^k_j\}_{j \geq 1, k \geq 1}$ determines $F$.

Suppose that $S(V)$ and $S(W)$ are equipped with their canonical multiplication and unit. Then if $F$ is a morphism of commutative graded algebras, it is the unique extension of $F|_V: V \to S(W)$.

Since $F|_V = \sum_{n \geq 1} F^n_1$, it is completely determined by the collection $\{F^n_1\}_{n \geq 1}$. Note that this sum is not infinite since $F^n_1(x) = 0$ for all $n \geq |x| + 1$. We call the collection $\{F^n_1\}_{n \geq 1}$ the **structure maps** of $F$ and refer to $n$ as the **arity** of $F^n_1$. Moreover since $F$ is respects multiplication, we have

\[
F^k_j(x_1 \vee \cdots \vee x_j) = \sum_{k_1 + \cdots + k_j = k} F^{k_1}_1(x_1) \vee \cdots \vee F^{k_j}_1(x_j)
\]

$F$ is called **strict morphism** iff $F^n_1 = 0$ for all $n \geq 2$. 
**Composition**

Let $F : S(V) \to S(W)$ and $G : S(W) \to S(U)$ be morphisms of commutative graded algebras. Then $GF : S(V) \to S(U)$ is the unique extension of $GF|_V = \sum_{n \geq 0} (GF)^n$ where

$$(GF)^n_1 = \sum_{k=1}^{n} G^n_k F^k_1$$  \hspace{1cm} (4)

**Differentials**

Suppose that $(S(V), \delta)$ is a cdga. Then since $\delta$ is a derivation, it is the unique extension of $\delta|_V = \sum_{n \geq 1} \delta^n_1$. Hence it is determined by the collection of degree 1 graded vector space maps $\{\delta^n_1\}_{n \geq 1}$. Again we call the collection $\{\delta^n_1\}_{n \geq 1}$ the structure maps of $\delta$ and refer to $n$ as the arity of $\delta^n_1$. Observe that $\delta \circ \delta$ is a derivation since

$$\delta \circ \delta(x \lor y) = \delta\left(\delta(x) \lor y + (-1)^{|x|} x \lor \delta(y)\right)$$
$$= \delta \circ \delta(x) \lor y + (-1)^{|x|+1} \delta(x) \lor \delta(y) + (-1)^{|x|} \delta(x) \lor \delta(y) + (-1)^{2|x|} x \lor \delta \circ \delta(y)$$
$$= \delta \circ \delta(x) \lor y + x \lor \delta \circ \delta(y)$$
$$= \delta \circ \delta(x) \lor y + (-1)^{|x||\delta \circ \delta|} x \lor \delta \circ \delta(y)$$

Thus $\delta \circ \delta$ is determined by its restriction to $V$. Therefore, $\delta \circ \delta = 0$ iff $(\delta \circ \delta)_1^n = \sum_{k=1}^{n} \delta^n_k \delta^k_1 = 0$ for all $n \geq 1$.

Additionally suppose that $F : (S(V), \delta_V) \to (S(W), \delta_W)$ is a morphism of commutative graded algebras. Then $F \delta_V$ and $\delta_W F$ are both derivations, where $S(W)$ is given the structure of an $S(V)$-module as in Ex. 3.6, and are thus determined by their restriction to $V$. Therefore $F$ is a cdga morphism iff

$$(F \delta_V)_1^n = \sum_{k=1}^{n} F^n_k \delta^k_1 = \sum_{k=1}^{n} \delta_W^n k F^k_1 = (\delta_W F)_1^n$$

for all $n \geq 1$.

**3.5 Lie algebras**

For more on Lie algebras and Lie algebra cohomology, we recommend Weibel’s book [19].

**Definition 3.19.** A Lie algebra over $\mathbb{k}$ is a $\mathbb{k}$-vector space $\mathfrak{g}$ equipped with a bilinear alternating
map \([\cdot, \cdot]: \wedge^2 g \to g\), called the **bracket**, such that the bracket satisfies the **Jacobi identity**, i.e.

\[
[x, [y, z]] = [[x, y], z] + [y, [x, z]]
\]

**A Lie algebra homomorphism is a \(\mathbb{k}\)-linear map** \(f: g \to h\) **such that** \(f([x, y]) = [f(x), f(y)]\).

Observe that since the bracket is alternating, \([x, y] = -[y, x]\) for all \(x, y \in g\).

**Example 3.20.**

1. Given any associative \(\mathbb{k}\)-algebra \(A\) we can define a bracket on \(A\) as follows,

\[
[x, y] := xy - yx \quad \text{for all } x, y \in A
\]

This bracket is called the **commutator bracket**.

2. Let \(V\) be a \(\mathbb{k}\)-vector space. We can make \(V\) into a Lie algebra by defining the bracket for any \(x, y \in V\) to be \([x, y] = 0\). Such a Lie algebra is called **abelian**.

**\(g\)-modules**

**Definition 3.21.** **A \(g\)-module** is a \(\mathbb{k}\)-vector space \(M\) and a Lie algebra homomorphism \(\phi: g \to \text{End}(M)\) where \(\text{End}(M)\) is equipped with the commutator bracket. **For** \(g \in g\) and \(x \in M\) we write \(\phi(g)(x)\) as \(gx\).

The fact that \(\phi\) is a Lie algebra homomorphism implies that

\[
[g, g']x = g(g'x) - g'(gx)
\]

for all \(g, g' \in g\), \(x \in M\).

**Example 3.22.**

1. The Jacobi identity makes the assignment \(g \mapsto [g, -]\) into a Lie algebra homomorphism \(g \to \text{End}(g)\). Hence \(g\) is a \(g\)-module.
2. Let \( V \) be a vector space. Then the trivial map map \( 0 : g \rightarrow \text{End}(V) \) defines a Lie algebra homomorphism and hence gives \( V \) the structure of a \( g \)-module. We call such a module as a **trivial** \( g \)-module.

**The Chevalley-Eilenberg functor**

A Lie algebra \( g \) is said to be **finite dimensional** if its underlying vector space is finite dimensional. Given a finite dimensional Lie algebra \( g \) it is easily verified using the basis constructed in (1) that 

\[
(\bigwedge^2 g)^* \cong \bigwedge^2 g^*,
\]

where \((-)^*\) denotes the \( k \)-linear dual. Thus the bracket on \( g \) defines a linear map 

\[
[-, -]^* : g^* \rightarrow \bigwedge^2 g^*.
\]

**Definition 3.23.** The **Chevalley-Eilenberg algebra** of a finite dimensional Lie algebra \( g \) is the semi-free cdga given by \((S(g^*[1]), \delta)\), where \( g^*[1] \) is the graded vector space consisting of \( g^* \) concentrated in degree 1, whose differential defined as

\[
\delta_{g_1}^n = \begin{cases} 
[-, -]^* & n = 2 \\
0 & \text{otherwise}
\end{cases}
\]

We denote this cdga by \((C(g), \delta_g)\).

A straightforward calculation shows that the Jacobi identity implies that \( \delta_{g_2} \delta_{g_1} = 0 \), and hence \( \delta_g \) is a differential.

Let \( f : g \rightarrow h \) be a Lie algebra homomorphism. From this we define a strict cdga morphism \( C(f) : (C(h), \delta_h) \rightarrow (C(g), \delta_g) \) where \( C(f)^1 = f^* : h^* \rightarrow g^* \). Since \( f \) is a Lie algebra homomorphism TFDC

\[
\begin{array}{ccc}
\bigwedge^2 g & \xrightarrow{\bigwedge^2 f} & \bigwedge^2 h \\
[-, -] & \downarrow f & [-, -] \\
g & \xrightarrow{f} & h
\end{array}
\]

Thus \( \delta_{g_1}^2 C(f)^1 = C(f)^2 \delta_{h_1}^2 \).

**Definition 3.24.** The **Chevalley-Eilenberg functor** \( C : \text{LieAlg} \rightarrow \text{cdga} \) is the contravariant functor
defined by the assignment \( g \mapsto (C(g), \delta_g) \) on objects and the assignment

\[
(f: g \to h) \mapsto \left( C(f): (C(h), \delta_h) \to (C(g), \delta_g) \right)
\]

where \( C(f) \) is the strict cdga morphism given by the \( \mathbb{K} \)-linear dual of \( f \).

**Proposition 3.25.** The Chevalley-Eilenberg functor \( C: \text{LieAlg} \to \text{cdga} \) is full and faithful.

**Proof.** If \( C(f) = C(g) \) for some Lie algebra homomorphisms \( f, g: g \to h \), then \( C(f)^n_1 = C(g)^n_1 \) for all \( n \geq 1 \). Thus, \( f^* = C(f)^1_1 = C(g)^1_1 = g^* \) which implies that \( f = g \).

Now let \( F: (C(h), \delta_h) \to (C(g), \delta_g) \) be a morphism in cdga. For degree reasons \( F \) must be strict. Moreover, since \( F \) commutes with the differentials, we have

\[
\begin{array}{ccc}
\Lambda^2 h^* & \xrightarrow{F^2_2} & \Lambda^2 g^* \\
[-,-]^* & \downarrow & [-,-]^* \\
h^* & \xrightarrow{F_1^1} & g^*
\end{array}
\]

Therefore, \( F_1^{1*}: g \to h \) defines a Lie algebra homomorphism such that \( C(F_1^{1*}) = F \).

\[ \square \]

**Remark 3.26.** Let \( V \) be a finite dimensional vector space and suppose that \( (S(V[1]), \delta) \) is a cdga. For degree reasons the structure maps of the differential must be zero except in arity 2. Since \( (\Lambda^2 V)^* \cong \Lambda^2 V^* \) one can define a bilinear alternating map \( \delta_2^{1*}: \Lambda^2 V^* \to V^* \). The fact that \( \delta \) is a differential implies that \( \delta_2^{1*} \) satisfies the Jacobi identity. Thus, the image of the Chevalley-Eilenberg functor is all of the semi-free cdgas that are the symmetric algebras on a finite dimensional vector spaces concentrated in degree 1.

**Lie algebra cohomology**

**Definition 3.27.** The **Lie algebra cohomology groups** \( H^*(g, M) \) of a Lie algebra \( g \) with coefficients in the \( g \)-module \( M \) are the cohomology groups of the complex \( \left( \text{hom}_k(\Lambda^* g, M), d \right) \), where the differential

\[
d_n : \text{hom}_k(\Lambda^n g, M) \to \text{hom}_k(\Lambda^{n+1} g, M)
\]
is defined as
\[ d_0 f(x) := xf(1_k) \]
and for all \( n \geq 1 \)
\[ d_n f(x_1 \wedge \cdots \wedge x_{n+1}) = - \sum_{i=1}^{n+1} (-1)^{i+1} x_i f(x_1 \wedge \cdots \wedge \tilde{x_i} \wedge \cdots \wedge x_{n+1}) \]
\[ -\sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge \cdots \wedge \tilde{x_i} \wedge \cdots \wedge \tilde{x_j} \wedge \cdots \wedge x_{n+1}) \]

Remark 3.28. We will show that differential \( \delta_\mathfrak{g} \) on \( C(\mathfrak{g}) \) agrees with the differential \( d \) on \( \text{hom}(\Lambda^* \mathfrak{g}, \mathbb{k}) \)
where \( \mathbb{k} \) is a trivial \( \mathfrak{g} \)-module. Let \( n \geq 2 \). Since the structure maps of \( \delta_\mathfrak{g} \) are zero except in arity 2, the differential \( \delta_\mathfrak{g} : \Lambda^n \mathfrak{g}^* \to \Lambda^{n+1} \mathfrak{g}^* \) is
\[ \delta_\mathfrak{g}(f_1 \wedge \cdots \wedge f_n) = \sum_{i=1}^{n} (-1)^{i-1} f_1 \wedge \cdots \wedge \delta_\mathfrak{g}^2 f_i \wedge \cdots \wedge f_n \]
Because \( |\delta_\mathfrak{g}^2 f_i| = 2 \) this can be written as
\[ = \sum_{i=1}^{n} (-1)^{i-1} \delta_\mathfrak{g}^2 f_i \wedge f_1 \wedge \cdots \wedge \tilde{f_i} \wedge \cdots \wedge f_n \]
Recall the definition of \( f \wedge g \in (\Lambda^{j+k} V)^* \) given in (2), where \( f \in (\Lambda^j V)^* \) and \( g \in (\Lambda^k V)^* \). Hence, if we consider \( (f_1 \wedge \cdots \tilde{f_i} \wedge \cdots \wedge f_n) \in (\Lambda^{n-1} \mathfrak{g})^* \),
\[ \delta_\mathfrak{g}(f_1 \wedge \cdots \wedge f_n)(x_1 \wedge \cdots \wedge x_{n+1}) = \]
\[ = \sum_{i=1}^{n} \sum_{\sigma \in \text{Shuf}(2, n-1)} (-1)^{i-1} \text{sgn}(\sigma) f_i([x_{\sigma(1)}, x_{\sigma(2)}])(f_1 \wedge \cdots \tilde{f_i} \wedge \cdots \wedge f_n)(x_{\sigma(3)} \wedge \cdots \wedge x_{\sigma(n+1)}) \]
The shuffles, \( \sigma \in \text{Shuf}(2, n-1) \), are given by a pair of integers \( 1 \leq j < k \leq n+1 \) such that
\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & n+1 \\
j & k & 1 & \cdots & \hat{j} & \cdots & \hat{k} & \cdots & n+1
\end{pmatrix}
\]
and the sign of such a permutation is \( (-1)^{j+k+1} \). Hence the sum above can be written as
\[ \sum_{i=1}^{n} \sum_{j<k} (-1)^{i-1} (-1)^{j+k+1} f_i([x_j, x_k])(f_1 \wedge \cdots \tilde{f_i} \wedge \cdots \wedge f_n)(x_1 \wedge \cdots \tilde{x_j} \cdots \tilde{x_k} \wedge \cdots \wedge x_{n+1}) \]
We will now consider \((f_1 \wedge \cdots \wedge f_n) \in \text{hom}(\Lambda^n g, k)\) with \(k\) a trivial \(g\)-module, and apply the differential defined in Def. 3.27 to it.

\[
d_n(f_1 \wedge \cdots \wedge f_n)(x_1 \wedge \cdots \wedge x_{n+1}) =
\]
\[
= - \sum_{j<k} (-1)^{j+k} (f_1 \wedge \cdots \wedge f_n)([x_j, x_k] \wedge x_1 \wedge \cdots \hat{x}_j \cdots \wedge x_{n+1})
\]
\[
(5)
\]

For convenience we define

\[
(y_1^{jk} \wedge \cdots \wedge y_n^{jk}) := ([x_j, x_k] \wedge x_1 \cdots \hat{x}_j \cdots \wedge x_{n+1})
\]

Thus (5) can be written as

\[
= - \sum_{j<k} \sum_{\sigma \in \text{Shuf}(1, \ldots, 1)} (-1)^{j+k} sgn(\sigma) f_1(y_{\sigma(1)}^{jk}) \cdots f_n(y_{\sigma(n)}^{jk})
\]

Note that \(\text{Shuf}(1, \cdots, 1) = S_n\). We will partition \(S_n\) into \(n\) subsets characterized by what they send to 1.

\[
= - \sum_{j<k} \sum_{i=1}^{n} \sum_{\sigma \in S_n, \sigma(i)=1} (-1)^{j+k} sgn(\sigma) f_1(y_1^{jk}) f_1(y_{\sigma(1)}^{jk}) \cdots f_n(y_{\sigma(n)}^{jk})
\]

(6)

Let \(\sigma \in S_n\) such that \(\sigma(i) = 1\). Then, if we identify \(S_{n-1}\) with the set of bijections between \(\{1, \cdots, i-1, \cdots n\}\) and \(\{2, \cdots, n\}\),

\[
\sigma = (12)(23) \cdots (i-1)i \sigma'
\]

for some \(\sigma' \in S_{n-1}\). Hence, \(sgn(\sigma) = (-1)^{i-1} sgn(\sigma')\). Thus (6) can be written as

\[
= - \sum_{j<k} \sum_{i=1}^{n} \sum_{\sigma' \in S_{n-1}} (-1)^{j+k} (-1)^{i-1} sgn(\sigma') f_1(y_1^{jk}) f_1(y_{\sigma'(1)}^{jk}) \cdots f_i(y_i^{jk}) \cdots f_n(y_{\sigma'(n)}^{jk})
\]

\[
= - \sum_{j<k} \sum_{i=1}^{n} (-1)^{j+k} (-1)^{i-1} f_i(y_i^{jk}) \sum_{\sigma' \in S_{n-1}} sgn(\sigma') f_1(y_{\sigma'(1)}^{jk}) \cdots f_i(y_i^{jk}) \cdots f_n(y_{\sigma'(n)}^{jk})
\]

\[
= - \sum_{j<k} \sum_{i=1}^{n} (-1)^{j+k} (-1)^{i-1} f_i([x_j, x_k])(f_1 \wedge \cdots \hat{f}_i \cdots \wedge f_n)(y_2^{jk} \wedge \cdots y_n^{jk})
\]

\[
= - \sum_{j<k} \sum_{i=1}^{n} (-1)^{j+k} (-1)^{i-1} f_i([x_j, x_k])(f_1 \wedge \cdots \hat{f}_i \cdots \wedge f_n)(x_1 \wedge \cdots \hat{x}_j \cdots \hat{x}_k \cdots \wedge x_{n+1})
\]

Therefore, if we consider \(k\) as a trivial \(g\)-module, \(\delta_g = d_n : \text{hom}(\Lambda^n g, k) \rightarrow \text{hom}(\Lambda^{n+1} g, k)\).
Hence

\[ H^*(\mathfrak{g}, \mathfrak{k}) = H^*(C(\mathfrak{g}), \delta) \]
4 The Brown category of Chevalley- Eilenberg algebras

In this section we introduce the category, CEAlg, of Chevalley-Eilenberg algebras along with two classes of morphisms called cofibrations and weak equivalences. We show that this category along with these two classes satisfy the axioms of a Brown category of cofibrant objects. In [5], Brown introduces the framework of a category of fibrant objects for a category with finite products. However, CEAlg is not closed under finite products. It is closed under finite coproducts, and for this reason, we dualize the framework established by Brown to a category of cofibrant objects, which we will refer to as a Brown category.

4.1 Axioms for a Brown Category

**Definition 4.1.** Let \( C \) be a category with finite coproducts endowed with two classes of morphisms called weak equivalences and cofibrations. A morphism that is both a weak equivalence and a cofibration is called an acyclic cofibration. Then \( C \) is a Brown Category of cofibrant objects for a homotopy theory iff:

1. All isomorphisms in \( C \) are acyclic cofibrations

2. The class of cofibrations is closed under composition and the class of weak equivalences satisfies the two out of three axiom. i.e. given any two composable morphisms \( f, g \) such that any two of \( f, g, \) or \( gf \) are weak equivalences then so is the third

3. The pushout of an (acyclic) cofibration along any morphism in \( C \) exists, and (acyclic) cofibrations are preserved under pushout. That is, if \( f \) is an (acyclic) cofibration and the following pushout square exists,

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Z \\
\downarrow{f} & & \downarrow{j} \\
Y & \xrightarrow{\_} & Y \sqcup_X Z
\end{array}
\]

and \( j \) is also an (acyclic) cofibration.

4. For every object \( X \in C \) there exists a cylinder object. i.e. there exists a (not necessarily
functorial) factorization

\[
X \coprod X \xrightarrow{(i_0, i_1)} \text{Cyl}(X) \xrightarrow{s} X
\]

of the fold map \(\nabla : X \coprod X \to X\) into a cofibration followed by a weak equivalence. We denote the composition of inclusion into the first coordinate with \((i_0, i_1)\) by \(i_0\), and similarly denote the inclusion into the second coordinate followed by \((i_0, i_1)\) as \(i_1\).

5. Every object is cofibrant. That is, the unique morphism from the initial object into any object is a cofibration.

**Example 4.2.** If \(M\) is a model category, then its full subcategory of cofibrant objects is a Brown category. This implies that \(\text{Ch}^{\geq 1}_{\text{proj}}\) is a brown category since every object in \(\text{Ch}^{\geq 1}_{\text{proj}}\) is cofibrant. Recall that the cofibrant objects in the Bousfield Gugenheim model structure on cdga are exactly the Sullivan algebras. Hence the category of Sullivan algebras is a Brown category.

The following lemma is a useful fact about morphisms in a Brown Category.

**Lemma 4.3.** Let \(C\) be a Brown category and \(f\) be any morphism in \(C\). Then \(f\) can be factored into \(f = rj\), where \(j\) is a cofibration and \(r\) is a weak equivalence.

**Proof.** Let \(f : X \to Y\) be a morphism in \(C\) and let \(X \coprod Y \xrightarrow{(i_0, i_1)} \text{Cyl}(X) \xrightarrow{s} X\) be a cylinder object for \(X\). We denote the pushout of \((i_0, i_1)\) along \(f \coprod \text{id} : X \coprod Y \to Y \coprod X\) by \(Z\), which exists because \((i_0, i_1)\) is a cofibration. Thus there exists a commutative of the form

\[
\begin{array}{ccc}
X \coprod X & \xrightarrow{f \coprod \text{id}} & Y \coprod X \\
\downarrow \scriptstyle{(i_0, i_1)} & & \downarrow \\
\text{Cyl}(X) & \xrightarrow{r} & Z
\end{array}
\]

\[
\exists! r
\]

Let \(j\) be the composition \(X \xhookrightarrow{} Y \coprod X \to Z\). Then \(j\) is a cofibration, \(r\) is a weak equivalence, and \(f = rj\). See [5] for details.

\[\square\]
4.2 Chevalley-Eilenberg algebras

**Definition 4.4.** A semi-free cdga \((S(V), \delta)\) is said to be **finite-type** if \(V^n\) is finite dimensional for all \(n\). We call a semi-free, finite-type cdga a Chevalley-Eilenberg algebra (CE algebra). We denote the category of CE algebras and cdga morphisms between them by CEAlg.

**Example 4.5.**

1. Let \((D, d) \in \text{Ch}^{\geq 0}\) be a finite-type cochain complex (not necessarily positively graded). We can define a differential \(\delta\) on \(S(D[1])\) by \(\delta^n_1 = d_{n+1}\) and \(\delta^n_1 = 0\) for \(n \geq 2\). Then \((S(D[1]), \delta)\) is a CE algebra.

2. Let \(\mathfrak{g}\) be a finite dimensional Lie algebra. Then its Chevalley-Eilenberg algebra \((C(\mathfrak{g}), \delta_\mathfrak{g})\) (see Def. 3.23) is a CE algebra.

**Proposition 4.6.** The category CEAlg is closed under finite coproducts.

**Proof.** Let \((S(V), \delta)\) and \((S(V'), \delta')\) be two CE algebras. In Ex. 3.9 we show that their coproduct in cdga is given by \((S(V) \otimes S(V'), \delta_\otimes)\) with inclusion maps

\[ \iota : (S(V), \delta) \longrightarrow (S(V) \otimes S(V'), \delta_\otimes) \leftarrow (S(V'), \delta') : \iota' \]

defined as \(\iota(x) := x \otimes 1\) and \(\iota'(y) := 1 \otimes y\). One can check that the linear map

\[ V \oplus V' \rightarrow S(V) \otimes S(V'), \quad (x, y) \mapsto x \otimes 1 + 1 \otimes y \]

lifts to an isomorphism \(S(V \oplus V') \cong S(V) \otimes S(V')\) in the category of graded commutative algebras. Moreover, since \(V\) and \(V'\) are finite-type, \(V \oplus V'\) is finite-type. Therefore, \((S(V) \otimes S(V'), \delta_\otimes)\) is a CE algebra.

\[ \square \]

**Remark 4.7.** It is not difficult to show that the isomorphism constructed above \(S(V \oplus V') \cong S(V) \otimes S(V')\) is natural. Throughout this paper we will frequently use this isomorphism to blur the distinction between \((x, y)\) and \(x \otimes 1 + 1 \otimes y\).
Let \((S(V), \delta_V)\) be a CE algebra. Then since \(\delta_V\) squares to zero, \(\delta_V^1 \circ \delta_V^1 = 0\). Thus \((V, \delta_V^1)\) is a cochain complex. Let \(F: (S(V), \delta_V) \to (S(W), \delta_W)\) be a morphism in CEAlg. Then the fact that \(F\) commutes with the differentials implies that \(F^1_1 \delta_V^1 = \delta_W^1 F^1_1\). Hence the assignments \((S(V), \delta_V) \mapsto (V, \delta_V^1)\) and \(F \mapsto F^1_1\) induce a well defined functor. We call this functor the \textbf{functor of indecomposables} and we denote it by

\[
T^* : \text{CEAlg} \to \text{Ch}^{\geq 1}
\]

**Proposition 4.8.** A cdga morphism \(F: (S(V), \delta_V) \to (S(W), \delta_W)\) is an isomorphism iff \(F^1_1 : (V, \delta_V^1) \to (W, \delta_W^1)\) is an isomorphism of cochain complexes.

**Proof.** If \(F: (S(V), \delta_V) \to (S(W), \delta_W)\) is an isomorphism then there exists a \(G: (S(W), \delta_W) \to (S(V), \delta_V)\) such that \(GF = \text{id}_{S(V)}\) and \(FG = \text{id}_{S(W)}\). Thus \(F^1_1 G^1_1 = \text{id}_V\) and \(G^1_1 F^1_1 = \text{id}_W\), and hence \(F^1_1\) is invertible.

Now suppose that there exists an inverse \((F^1_1)^{-1} : (W, \delta_W^1) \to (V, \delta_V^1)\) of \(F^1_1\). We will construct a left inverse \(G: (S(W), \delta_W) \to (S(V), \delta_V)\) for \(F\) as follows. Let \(G^1_1 = (F^1_1)^{-1}\), then \(\delta_V^1 G^1_1 = G^1_1 \delta_W^1\). Let \(n \geq 2\) and suppose that for all \(k < n\) that \(G^k_1: W \to S^k(V)\) is defined and that

\[
\sum_{i=1}^{k} \delta_V^i G^k_1 = \sum_{i=1}^{k} G^k_1 \delta_W^i
\]

Then we define

\[
G^n_1 := - \sum_{i=2}^{n} G^n_i F^i_1 G^1_1
\]

Thus \((GF)^n_1 = \sum_{i=2}^{n} G^n_i F^i_1 + G^1_1 F^1_1 = 0\). Using (7) and the fact that \(\delta_V^1 G^1_1 = G^1_1 \delta_W^1\) implies \(\delta_V^n G^n_1 = G^n_1 \delta_W^n\), it is not hard to show that \((\delta_W G)^n_1 = (G \delta_V)^n_1\). Therefore if \(G\) is the unique lift of \(\{G^n_1\}_{n \geq 1}\) to a cdga morphism, then \(GF = \text{id}_{S(V)}\).

Similarly we can build a right inverse \(H: (S(W), \delta_W) \to (S(V), \delta_V)\) for \(F\) defined as the unique lift of \(\{H^n_1\}_{n \geq 1}\) where

\[
H^n_1 = \begin{cases} 
(F^1_1)^{-1} & n = 1 \\
- \sum_{i=1}^{n-1} H^n_i F^i_1 H^1_1 & n > 1
\end{cases}
\]

Then \(G = GFH = H\) is a two sided inverse for \(F\).
The prior proposition shows that the functor $T^*$ completely characterizes isomorphisms in $\text{CEAlg}$, and this property motivates the following definition.

**Definition 4.9.** We say that a morphism $F$ in $\text{CEAlg}$ is a weak equivalence (resp. cofibration) iff $T^*(F)$ is a weak equivalence (resp. cofibration) in $\text{Ch}^1_{\text{proj}}$. That is, $F$ is a weak equivalence iff $F^1_1$ is a quasi isomorphism of cochain complexes, and $F$ is a cofibration iff $F^1_1$ is injective in all degrees $n \geq 2$.

### 4.2.1 Factorization of CE algebra morphisms

We first dualize the construction of Goerss and Schemmerhorn in [8] to factor a morphism in $\text{Ch}^1_{\text{proj}}$ into a cofibration followed by a weak equivalence. We then lift this construction to factor strict morphisms of CE algebras using techniques of Rogers [14]. Since the fold map is strict, this will establish the existence of cylinder objects.

**Factoring morphisms in $\text{Ch}^1_{\text{proj}}$**

Let $f : (D, d_D) \to (E, d_E)$ be a map of positively graded cochain complexes. Define a complex $(I(D), d_{I(D)}) \in \text{Ch}^1_{\text{proj}}$ as

$$I(D)^n := \begin{cases} 
\{0\} \oplus D^2 & n = 1 \\
D^i \oplus D^{i+1} & n \geq 2
\end{cases}$$

with differential $d_{I(D)}(x, y) := (y, 0)$. Observe that the degree $-1$ map of graded vector spaces $h : I(D) \to I(D)$ defined as $h(x, y) = (0, x)$ serves as a contraction homotopy on $(I(D), d_{I(D)})$ in the sense that $id_{I(D)} = d_{I(D)}h + hd_{I(D)}$. Thus $(I(D), d_{I(D)})$ is acyclic. Let $\gamma : (D, d_D) \to (I(D), d_{I(D)}) \in \text{Ch}^1_{\text{proj}}$ be defined as

$$\gamma(x) = \begin{cases} 
(0, d_D(x)) & |x| = 1 \\
(x, d_D(x)) & |x| \geq 2
\end{cases}$$

Then $f$ can be factored as

$$D \xrightarrow{f, \gamma} E \oplus I(D) \xrightarrow{\text{pr}_E} E$$
\((f, \gamma)\) is a cofibration in \(\text{Ch}_{\text{proj}}^{\geq 1}\) since it is injective in all degrees greater than 1, and \(\text{pr}_E\) is a weak equivalence in \(\text{Ch}_{\text{proj}}^{\geq 1}\) since \(I(D)\) is acyclic.

**Factoring strict morphisms in CEAlg**

We lift the above factorization to the category CEAlg. First, we record an elementary lemma from homological algebra.

**Lemma 4.10.** Let \((D, d_D)\) and \((E, d_E)\) be cochain complexes of vector spaces. Let \((\text{Map}(D, E), \partial)\) be the mapping complex from Example 3.1. Then if \(E\) is acyclic, \(\text{Map}(D, E)\) is acyclic as well.

**Proof.** If \(E\) is acyclic, then since our base ring is a field, there exists a contracting chain homotopy \(h: E \rightarrow E\) satisfying \(id_E = d_E h + h d_E\). One then observes that the degree \(-1\) map

\[ h_*: \text{Map}(D, E) \rightarrow \text{Map}(D, E), \quad h_*(f) := h \circ f \]

is a contracting chain homotopy on the mapping complex. \(\square\)

**Proposition 4.11.** Let \(F: (S(V), \delta) \rightarrow (S(V'), \delta')\) be a strict morphism between CE algebras. Then \(F\) can be factored in CEAlg as

\[ (S(V), \delta) \xrightarrow{J} (S(\tilde{V}), \tilde{\delta}) \xrightarrow{P} (S(V'), \delta') \]

Where \(J\) is a cofibration and \(P\) is a weak equivalence.

**Proof.** Let the differential on \(S(I(V))\) be the strict differential induced by \(d_{I(V)}\) and define \(\tilde{V} := V' \oplus I(V)\) so that \(S(\tilde{V}) \cong S(V') \otimes S(I(V))\). Let \(\tilde{\delta}\) be the differential on \(S(\tilde{V})\) induced by the tensor product. Define \(P: S(\tilde{V}) \rightarrow S(V')\) be the strict map with \(P_1^1 = \text{pr}_{V'}\). A simple check shows that this defines a morphism of CE algebras, which is in fact a weak equivalence because \(\text{pr}_{V'}\) is a quasi isomorphism of cochain complexes. We define \(J_1: V \rightarrow V' \oplus I(V)\) as \(J_1 := (F_1^1, \gamma)\) where \(\gamma\) is defined in equation (8). Then observe that \(P_1^1 J_1^1(x) = F_1^1(x) = F(x)\) for all \(x \in V\). It remains to construct the higher arity structure maps of \(J\) satisfying \(P_n^1 J_1^n = 0\) for all \(n \geq 2\). We proceed by induction on \(n\).
Consider $V$ and $S^2(\tilde{V})$ as cochain complexes with differentials $\delta_1^2$ and $\tilde{\delta}_2^2$ respectively. Define degree 1 map of graded vector spaces $q_2 : V \rightarrow S^2(\tilde{V})$ by

$$q_2 = J_2^2 \delta_1^2 - \tilde{\delta}_1^2 J_1^1$$

Now observe that $P_1^1 J_1^1 = F_1^1$ implies that $P_n^m J_n^m = F_n^m$ for all $n \geq 1$ by equations (3) and (4). Thus, since $F$ and $P$ are strict morphisms of CE algebras, we have

$$P_2^2 q_2 = P_2^2 J_2^2 \delta_1^2 - P_2^2 \tilde{\delta}_1^2 J_1^1$$
$$= P_2^2 J_2^2 \delta_1^2 - \delta_2^2 P_1^1 J_1^1$$
$$= F_2^2 \delta_1^2 - \delta_1^2 F_1^1$$
$$= 0$$

Hence $q_2 : V \rightarrow \ker P_2^2$. By Prop. A.4 in the Appendix, the fact that $P_1^1 : (\tilde{V}, \delta_1^1) \rightarrow (V', \delta_1^1)$ is a quasi isomorphism implies that $P_n^m : (S^n(\tilde{V}), \delta_n^m) \rightarrow (S^n(V'), \delta_n^m)$ is a quasi isomorphism for all $n \geq 1$. A long exact sequence of cohomology shows that $(\ker P_n^m, \delta_n^m)$ is acyclic for all $n \geq 1$, and in particular when $n = 2$.

Let $(\Map(V, \ker P_2^2), \partial)$ be the mapping complex as defined in Ex. 3.1. Since $(\ker P_2^2, \tilde{\delta}_2^2)$ is acyclic, Lemma 4.10 implies that the mapping complex is acyclic. It is easily verified that $q_2$ is a 1-cocycle in this complex. Hence there exists a degree 0 map of graded vector spaces $J_1^2 : V \rightarrow \ker P_2^2$ such that

$$\partial J_1^2 = \tilde{\delta}_2^2 J_2^1 - J_2^1 \delta_1^1 = J_2^2 \delta_1^2 - \tilde{\delta}_1^2 J_1^1 = q_2$$

Thus $\tilde{\delta} J_1^2 = (J \delta)_1^2$ and we have completed our base case.

Let $n > 2$ and suppose that for all $1 < k < n$ that $J_k^k$ is defined, $(\tilde{\delta} J)_k^k = (J \delta)_k^k$, and $P_k^k J_k^k = 0$. We follow the same strategy as the base case. That is, we define a degree 1 map of graded vector spaces and show that it is a 1-cocycle in $(\Map(V, \ker P_n^m), \partial)$. Let

$$q_n := \sum_{k=2}^{n} J_k^m \delta_1^k - \sum_{k=1}^{n-1} \tilde{\delta}_k^m J_1^k$$

We first show that $q_n$ lands in $\ker P_n^m$. First observe that for $k < n$, $P_n^m J_k^m = 0$. Indeed by
equation (3) we have

\[ P_n^{m} J_{n,k}^{i} (x_1 \vee \cdots \vee x_k) = P_n^{m} \left( \sum_{n_1 + \cdots + n_k = n} J_{n_1}^{n_1} (x_1) \vee \cdots \vee J_{n_k}^{n_k} (x_k) \right) \]

\[ = \sum_{n_1 + \cdots + n_k = n} P_{n_1}^{m_1} J_{n_1}^{n_1} (x_1) \vee \cdots \vee P_{n_k}^{m_k} J_{n_k}^{n_k} (x_k) \]

Since \( k < n \), each summand will have a term with \( n_i \geq 2 \) for some \( i = 1, \ldots, k \). The inductive hypothesis implies that \( P_{n_i}^{m_i} J_{n_i}^{1} = 0 \), and hence \( P_{n_i}^{m_i} J_{n}^{n_i} = 0 \) for all \( k < n \). Additionally, note that the fact that \( P \) is strict and commutes with the differentials implies that \( P_{n}^{m} \delta_{k}^{n} = (P\delta)_{k}^{n} = (\delta P)_{k}^{n} = \delta_{k}^{m} P_{k}^{k} \). Therefore,

\[ P_{n}^{m} q_{2} = \sum_{k=2}^{n} P_{n}^{m} J_{k}^{n} \delta_{1}^{k} - \sum_{k=1}^{n-1} P_{n}^{m} \delta_{k}^{n} J_{1}^{k} \]

\[ = \sum_{k=2}^{n} P_{n}^{m} J_{k}^{n} \delta_{1}^{k} - \sum_{k=1}^{n-1} \delta_{k}^{n} P_{k}^{k} J_{1}^{k} \]

\[ = P_{n}^{m} J_{n}^{n} \delta_{1}^{1} - \delta_{n}^{m} P_{1}^{1} J_{1}^{1} \]

\[ = F_{n}^{m} \delta_{1}^{1} - \delta_{n}^{m} F_{1}^{1} \]

\[ = 0 \]

Hence \( q_{n} : V \to \ker P_{n}^{m} \).

By Prop. A.9 in the Appendix, \( q_{n} \) is a cocycle in the mapping complex. Thus, since \( (\text{Map}(V, \ker P_{n}^{m}), \partial) \) is acyclic, there exists a degree 0 map \( J_{n}^{n} : V \to \ker P_{n}^{m} \) such that \( \partial J_{n}^{n} = q_{n} \). Therefore \( (J\delta)_{1}^{1} = (J\delta)_{1}^{1} \) and \( P_{n}^{m} J_{1}^{1} = 0 \), and we have completed our inductive step.

Hence, if \( J \) is the unique lift of \( \{J_{n}^{n}\}_{n \geq 1} \) to a morphism of graded algebras, then the above implies that \( J \) is a morphism of cdgas, and so we have \( PJ = F \) with \( J \) a cofibration and \( P \) a weak equivalence.

\[ \square \]

**Corollary 4.12.** For every CE algebra \((S(V), \delta)\) there exists a factorization of the fold map \( \nabla : (S(V) \otimes S(V), \delta_{\otimes}) \to (S(V), \delta) \) into a cofibration followed by a weak equivalence. Thus for every object in CEAAlg there exists a cylinder object.

**Remark 4.13.** For the convenience of the reader we will explicitly describe the cylinder object
resulting from the application of Prop. 4.11 to the fold map $\nabla: (S(V) \otimes S(V), \delta_{\otimes}) \to (S(V), \delta)$. Let $\widehat{\text{Cyl}}(V) \colonequals S(V) \otimes S(I(V \oplus V))$. That is, $\widehat{\text{Cyl}}(V)$ is the symmetric algebra on the graded vector space

$$V \oplus I(V \oplus V)^n = \begin{cases} V^1 \oplus 0 \oplus 0 \oplus V^2 & n = 1 \\ V^n \oplus V^n \oplus V^{n+1} \oplus V^{n+1} & n \geq 2 \end{cases}$$

We denote the given factorization by

$$(S(V) \otimes S(V), \delta_{\otimes}) \xrightarrow{(i_0, i_1)} (\widehat{\text{Cyl}}(V), \widehat{\delta}) \xrightarrow{s} (S(V), \delta)$$

The arity 1 differential on $\widehat{\text{Cyl}}(V)$ is given by

$$\widehat{\delta}_1^1(v, x, y, a, b) = (\delta_1^1(v), a, b, 0, 0)$$

The arity 1 structure maps of $i_0$ and $i_1$ are

$$i_{01}^1(v) = \begin{cases} (v, 0, 0, 0, \delta_1^1(v), 0) & |v| = 1 \\ (v, v, 0, \delta_1^1(v), 0) & |v| \geq 2 \end{cases}$$

$$i_{11}^1(v) = \begin{cases} (v, 0, 0, 0, \delta_1^1(v)) & |v| = 1 \\ (v, v, v, 0, \delta_1^1(v)) & |v| \geq 2 \end{cases}$$

Finally $s$ is the strict morphism defined by the projection of the first coordinate.

Our main theorem (Thm. 4.18) in conjunction with Lemma 4.3 imply that an arbitrary morphism in CEA$\text{lg}$ can be factored into a cofibration followed by a weak equivalence.

### 4.2.2 Pushouts in CEA$\text{lg}$

We begin by recalling the construction of pushouts in cdga. Then we provide an alternative description of the pushout in $\text{Ch}_{\text{proj}}^{\geq 1}$ of a cofibration along an arbitrary morphism which we will lift to CEA$\text{lg}$. We then use a version of Vallette’s [17] "strictification" lemma, due to Rogers [14, Lemma 3.9], to prove that the pushout in CEA$\text{lg}$ of an arbitrary (acyclic) cofibration along any morphism is an (acyclic) cofibration.
Let $A' \xleftarrow{F} A \xrightarrow{G} A''$ be a diagram in cdga. The pushout of this diagram is

$$
\begin{array}{ccc}
A & \xrightarrow{G} & A'' \\
\downarrow{F} & & \downarrow{\pi'} \\
A & \xrightarrow{\iota} & A' \otimes_A A''
\end{array}
$$

where $A' \otimes_A A''$ is the quotient of $A' \otimes A''$ by the ideal generated by $\{F(a) \otimes 1 - 1 \otimes G(a) | a \in A\}$. The maps $\iota$ and $\iota'$ are the canonical inclusions followed by the quotient map, i.e. $\iota'(a') = \overline{a} \otimes 1$ and $\iota''(a'') = 1 \otimes \overline{a''}$.

**Pushout in $\text{Ch}^{\geq 1}_{\text{proj}}$**

Let $(D', d') \xleftarrow{f} (D, d) \xrightarrow{g} (D'', d'')$ be a diagram in $\text{Ch}^{\geq 1}_{\text{proj}}$ where $f$ is a cofibration. Recall that $D' \oplus_D D'' := D' \oplus D''/I$ where $I = \{(f(z), -g(z)) \in D' \oplus D'' | z \in D\}$ and let $\pi: (D' \oplus D'', d_{\oplus}) \rightarrow (D' \oplus_D D'', d_{\oplus})$ denote the projection of complexes. Since $f$ is injective in all degrees $n \geq 2$ there exists a partial section $\sigma: D' \rightarrow D$ of graded vector spaces such that

$$f \sigma(x) = \begin{cases} 
0 & |x| = 1 \\
\sigma & |x| \geq 2
\end{cases}$$

We now define another positively graded vector space $\tilde{D}$ by

$$
\tilde{D}^n := \begin{cases} 
(D^{1} \oplus D''^1)/I^1 & n = 1 \\
\ker \sigma^n \oplus D^{mn} & n \geq 2
\end{cases}
$$

where $I^1 = \{(f(z), -g(z)) | z \in D^1\}$. We also define a pair of degree 0 linear maps $k: D' \oplus D'' \rightarrow \tilde{D}$ and $l: \tilde{D} \rightarrow D' \oplus_D D''$ by

$$k(x, y) = \begin{cases} 
\pi^1(x, y) & |(x, y)| = 1 \\
(x - f \sigma(x), g \sigma(x) + y) & |(x, y)| \geq 2
\end{cases}$$

and

$$l(\tilde{x}) = \begin{cases} 
\sigma(\tilde{x}) & |\tilde{x}| = 1 \\
\tilde{x} & |\tilde{x}| \geq 2
\end{cases}$$
\[ l(x,y) = \begin{cases} 
\tau(x,y) & |(x,y)| = 1 \\
(x,y) & |(x,y)| \geq 2 
\end{cases} \]

where \( \pi^1 \) is the restriction of the projection above to \( D'^1 \oplus D''^1 \), and \( \tau \) is a section of \( \pi^1 \) in the sense that \( \tau \) is a \( k \)-linear map such that \( \pi^1 \tau = \text{id}_{D'^1 \oplus D''^1} \). We will then define a differential \( \overline{d} \) on \( \overline{D} \) as \( \overline{d} := kd \cdot l \).

\( \overline{d} \) is in fact a differential. First observe since \( \tau \) is a section of \( \pi \), we there exists some \( z \in D^1 \) such that \( l(x,y) = (x,y) + (f(z),-g(z)) \) for all \( (x,y) \in \overline{D}^1 \). In fact, for all \( (x,y) \in D^1 \oplus D''^1 \), there exists a \( z \in D \) such that

\[ lk(x,y) = (x,y) + (f(z),-g(z)) \]

where \( z \) is given explicitly as \( z = -\sigma(x) \) for elements of degree \( n \geq 2 \). Since \( f \) and \( g \) are cochain maps it is easy to show that

\[ kd \cdot (f(z),-g(z)) = 0 \]

Therefore, on degree 1 elements

\[ \overline{d}^2(x,y) = kd \cdot lk \cdot l(x,y) \]

\[ = kd \cdot lk \cdot ((x,y) + (f(z),-g(z))) \]

\[ = kd \cdot lk \cdot (x,y) \]

\[ = kd \cdot lk(d'x,d''y) \]

\[ = kd \cdot ((x,y) + (f(z'),-g(z')))) \]

\[ = kd \cdot (d'x,d''y) \]

\[ = 0 \]

and on elements of degree \( n \geq 2 \) a similar calculation shows that \( \overline{d}^2 = 0 \). Thus \((\overline{D}, \overline{d})\) is a cochain complex.

We now show that \( k: (D' \oplus D'', d_{\oplus}) \rightarrow (\overline{D}, \overline{d}) \) and \( \pi l: (\overline{D}, \overline{d}) \rightarrow (D' \oplus D'' \oplus D_0, \overline{d}_{\oplus}) \) are cochain
maps. By (4.2.2) and (4.2.2) we have
\[
\tilde{d}k(x, y) = k d \circ l k(x, y) = k d \circ ((x, y) + (f(z), -g(z))) = k d \circ (x, y)
\]
Hence \(k\) is a cochain map. Observe that (4.2.2) implies that \(\pi l k = \pi\). Thus since \(\pi\) is a cochain map
\[
\pi l \tilde{d} = \pi l k d \circ l = \pi d \circ l = d \circ \pi l, \text{ so } \pi l \text{ is also a cochain map.}
\]
It is not hard to check that the following diagram in \(Ch_{\text{proj}}^{\geq 1}\) commutes.

\[
\begin{array}{c}
(D, d) \xrightarrow{g} (D'', d'') \\
\downarrow f \downarrow \quad \downarrow k \\
(D', d') \xrightarrow{k l'} \quad \tilde{D} \xrightarrow{\pi l} (D' \oplus_D D'', \tilde{d}) \\
\downarrow \pi \downarrow \quad \downarrow \pi l \\
(D', d') \oplus (D'' \oplus_D \tilde{d})
\end{array}
\]

Thus it remains to show that \(\pi l\) is unique. Suppose that \(\epsilon: \tilde{D} \rightarrow (D' \oplus_D D'', \tilde{d})\) is another map that fits in this diagram. Note that in degree 1, \(\tilde{D}^1 = (D' \oplus_D D'')^1\) and
\[
\epsilon k l'(x) = \epsilon(x, 0) = (x, 0)
\]
\[
\epsilon k l''(y) = \epsilon(0, y) = (0, y)
\]
Hence \(\epsilon\) is the identity in degree 1 and so is \(\pi l\). In degrees \(n \geq 2\) we have
\[
\epsilon k l'(x) = \epsilon(x - f \sigma(x), g \sigma(x)) = (x, 0)
\]
\[
\epsilon k l''(y) = \epsilon(0, y) = (0, y)
\]
Thus since \(x \in \ker \sigma\) we have \(\epsilon(x, y) = \epsilon(x - f \sigma(x), g \sigma(x) + y) = (x, y) = \pi l(x, y)\). Therefore \(\pi l\) is unique, and (9) is indeed a pushout. Moreover, since \(f\) is injective in degree \(n \geq 2\), \(k l''\) is too.

**Pushouts of strict cofibrations in CEA\(g\)**

We lift the above construction of the pushout in \(Ch_{\text{proj}}^{\geq 1}\) to CEA\(g\).

**Proposition 4.14.** Let \(F: (S(V), \delta) \rightarrow (S(V'), \delta')\) be a strict cofibration of CE algebras and \(G: (S(V), \delta) \rightarrow (S(V''), \delta'')\) be an arbitrary morphism in CEA\(g\). Then the pushout of \(F\) along
$G$ exists in CEAlg.

Proof. We begin by applying the above construction to the diagram $(V', \delta'^1) \xleftarrow{F_1} (V, \delta^1) \xrightarrow{G_1} (V'', \delta'^{1})$. That is, we construct a partial section $\sigma : V' \to V$ of $F_1$ and define a graded vector space $\tilde{V}$ as

$$
\tilde{V}^n = \begin{cases} 
(V'^1 \oplus V''^1)/I^1 & n = 1 \\
\ker \sigma^n \oplus V'^{mn} & n \geq 2 
\end{cases}
$$

where $I^1 = \{(F_1^1(z), -G_1^1(z)) \mid z \in V^1\}$. Observe that for degree reasons $G(z) = G_1^1(z)$ for all $z \in V^1$ and since $F$ is strict $I^1 = \{F(z) \otimes 1 - 1 \otimes G(z) \mid z \in V^1\}$. We will now lift $k$ and $l$ to algebra morphisms. We define $K : S(V') \otimes S(V'') \to S(\tilde{V})$ to be the unique lift of $\{K_1^n\}_{n \geq 1}$ where $K_1^1 = k$ and

$$K_1^n(x, y) = \begin{cases} 
0 & |(x, y)| = 1 \\
1 \otimes G_1^n \sigma(x) & |(x, y)| \geq 2 
\end{cases}
$$

We define $L : S(\tilde{V}) \to S(V') \otimes S(V'')$ as the strict morphism with $L_1^1 = l$.

We now will show that the following diagram commutes

$$
\begin{array}{ccc}
S(V) & \xrightarrow{G} & S(V'') \\
\downarrow F & & \downarrow K'^{'''} \\
S(V') & \xrightarrow{K'^{''}} & S(\tilde{V}) \\
\downarrow \pi L & & \downarrow \pi \gamma \\
S(V') \otimes_{S(V')} S(V'') & \xrightarrow{\pi} & S(\tilde{V}) \\
\end{array}
$$

Since (9) commutes, it suffices to check the commutativity of the higher arity structure maps of the diagram above. Moreover, for degree reasons, it suffices to check the commutativity of the higher arity structure maps only on elements of degree $n \geq 2$. Let $z \in V$ with $|z| \geq 2$. Then for $n \geq 2$ we have

$$(K'l'F)^1_1^n(z) = K_1^n l'^1_1 F_1^1(z) = 1 \otimes G_1^n(\sigma F_1^1(z)) = 1 \otimes G_1^n(z) = K_1^n l'^m_n G_1^n(z)$$

However, since $K'l''$ is strict, $K_1^n l'^{mn}_n G_1^n(z) = (K'l'^n G)^1_1^n(z)$. Hence the inner square commutes. $\pi L K'l'' = \pi \gamma$ because (9) commutes and $K'l''$, $\pi L$, and $\pi \gamma$ are all strict. Finally since $K'l'(x) = \cdots$
\[(x - F\sigma(x)) \otimes 1 + 1 \otimes G\sigma(x)\] for all \(x \in V'\) with \(|x| \geq 2\), we conclude that \(\pi L K' = \tilde{\iota}'\) and that the diagram above commutes.

Similar to the construction in \(\mathrm{Ch}^{\geq 1}_{\text{proj}}\), we define the differential on \(S(\tilde{V})\) by \(\tilde{\delta} := K\delta \otimes L\) where \(\delta\) is the differential on \(S(V') \otimes S(V'')\) induced by the tensor product. An analogous computation will show that \(\tilde{\delta} \circ \tilde{\delta} = 0\).

Next we show that \(K\) and \(\pi L\) are cdga morphisms. Note that for \((x, y) \in V' \oplus V''\) with degree \(n \geq 2\), we have

\[
\tilde{\delta}K(x, y) = K\delta \otimes LK(x, y)
= K\delta \left( (x, y) - F\sigma(x) \otimes 1 + 1 \otimes G\sigma(x) \right)
= K\delta(x, y) + K\left( -F\delta\sigma(x) \otimes 1 + 1 \otimes G\delta\sigma(x) \right)
= K\delta(x, y)
\]

For \(|(x, y)| = 1\) we have \(LK(x, y) = (x, y) + (F^1_1(z), -G^1_1(z))\) for some \(z \in V^0\). We leave it to the reader to show that \(K\delta \otimes (F^1_1(z), -G^1_1(z)) = 0\) by using the following facts

\[
\delta^{n2}_1G^1_1(z) = G^2_1\delta^1_1(z) + G^2_2\delta^2_1(z) - \delta^{n2}_1G^2(z) = G^2_1\delta^1_1(z) + G^2_2\delta^2_1(z)
\]

\[
K^2_2 \left( F^2_2\delta^2_1(z) \otimes 1 \right) = K^2_2 \left( 1 \otimes G^2_2\delta^2_1(z) \right)
\]

\[
K^2_1 \left( F^1_1\delta^1_1(z) \otimes 1 \right) = K^2_1 \left( 1 \otimes G^1_1\delta^1_1(z) \right) + K^2_2 \left( 1 \otimes G^2_1\delta^1_1(z) \right) = K^2_2 \left( 1 \otimes G^2_1\delta^1_1(z) \right)
\]

Hence for all \((x, y) \in V'^1 \oplus V''^1\),

\[
\tilde{\delta}K(x, y) = K\delta \otimes LK(x, y)
= K\delta \left( (x, y) + (F^1_1(z), -G^1_1(z)) \right)
= K\delta(x, y)
\]

To see that \(\pi L\) is a cdga morphism observe that \(\pi LK = \pi\). Thus \(\pi L\tilde{\delta} = \pi LK\delta \otimes L = \pi\delta \otimes L = \tilde{\delta} \otimes \pi L\). Moreover \(\pi L\) is the unique lift of the unique linear map \(\pi l\) in (9) and is therefore unique. \(\square\)

**Corollary 4.15.** Let \(F: (S(V), \delta) \to (S(V'), \delta')\) be a strict (acyclic) cofibration and \(G: (S(V), \delta) \to (S(V''), \delta'')\) be an arbitrary morphism in \(\text{CAlg}\). Then the map induced by the pushout of \(F\) along
$G$ is an (acyclic) cofibration.

**Proof.** The functor of indecomposables, $T^*$, sends the pushout square constructed in Proposition 4.14 to a pushout square in $\text{Ch}^\geq_1$. Since $F^1_1$ is an (acyclic) cofibration in $\text{Ch}^\geq_1$ and pushouts in $\text{Ch}^\geq_1$ preserve (acyclic) cofibrations, the arity 1 structure map of the induced CE algebra morphism, $(Kt'')_1$, is an (acyclic) cofibration in $\text{Ch}^\geq_1$. Hence, $Kt'' : (S(V''), \delta'') \to (S(\tilde{V}), \tilde{\delta})$ is an (acyclic) cofibration.

\[\square\]

**Pushouts of arbitrary (acyclic) cofibrations in CEAlg**

We begin with the "strictification" lemma which we will use to reduce the pushout of an arbitrary (acyclic) cofibration to the pushout of a strict (acyclic) cofibration.

**Lemma 4.16.** Let $F : (S(V), \delta) \to (S(V'), \delta')$ be an arbitrary cofibration in CEAlg. Then there exists a CE algebra $(S(V'), \hat{\delta})$ and an isomorphism $\Phi : (S(V'), \delta') \to (S(V'), \hat{\delta})$ such that $\Phi F$ is strict with $(\Phi F)_1^1 = F_1^1$.

**Proof.** By hypothesis $F_1^1$ is injective in all degrees $n \geq 2$. Again, we choose a partial section of graded vector spaces $\sigma : V' \to V$ such that

\[\sigma F_1^1(x) = \begin{cases} 0 & |x| = 1 \\ x & |x| \geq 2 \end{cases}\]

We define a degree 0 linear map $\Phi_1^1 : V' \to V'$ to be the identity, so that $\Phi_1^1 F_1^1 = F_1^1$. It remains to construct the higher arity structure maps of $\Phi$ so that $(\Phi F)_n^1 = 0$ for all $n \geq 2$, and we do so by induction on $n$.

Let $n = 2$. Define

\[\Phi_1^2 := -\Phi_2^2 F_1^2 \sigma\]

Now observe that since $F_1^2$ is zero on degree 1 elements, we have

\[(\Phi F)_1^2 = \Phi_2^2 F_1^2 + \Phi_1^2 F_1^1 = \Phi_2^2 F_1^2 - \Phi_2^2 F_1^2 \sigma F_1^1 = 0\]
Let \( n > 2 \), and suppose for all \( 1 < k < n \) that \( \Phi_1^k \) is defined and \( (\Phi F)_1^k = 0 \). Then define

\[
\Phi_1^n := - \sum_{k=2}^{n} \Phi_1^n F_1^k \sigma
\]

Thus \( \Phi_1^n \) is well defined and \( (\Phi F)_1^n = 0 \).

Let \( \Phi : S(V') \to S(V') \) be the unique lift of \( \{ \Phi_1^n \}_{n \geq 1} \) to a morphism of commutative graded algebras. Since \( \Phi_1^1 \) is an isomorphism, we can use the construction in Prop. 4.8 to build an inverse for \( \Phi \). We use this inverse to define the differential \( \delta := \Phi \delta' \Phi^{-1} \). This clearly defines a differential and promotes \( \Phi : (S(V'), \delta') \to (S(V'), \delta) \) to an isomorphism of CE algebras. Moreover, by construction \( \Phi F \) is strict with \( (\Phi F)_1^1 = F_1^1 \).

\[
\square
\]

**Proposition 4.17.** Let

\[
(S(V'), \delta') \xleftarrow{F} (S(V), \delta) \xrightarrow{G} (S(V''), \delta'')
\]

be a diagram in \( \text{CEAlg} \) where \( F \) is an (acyclic) cofibration and \( G \) is arbitrary. Then,

1. the pushout of this diagram exists in \( \text{CEAlg} \)

2. the morphism induced by the pushout of \( F \) along \( G \) is also an (acyclic) cofibration.

**Proof.** Suppose \( F \) is an (acyclic) cofibration. Then the strictification lemma implies that there exists a differential \( \tilde{\delta} \) on \( S(V') \) and a CE algebra isomorphism \( \Phi : (S(V'), \delta') \to (S(V'), \tilde{\delta}) \) such that the composition of \( F \) with \( \Phi \) is strict with \( (\Phi F)_1^1 = F_1^1 \). Hence \( \Phi F \) is a strict (acyclic) cofibration. Prop. 4.14 implies that there exists of pushout square in \( \text{CEAlg} \) of the form

\[
\begin{array}{ccc}
(S(V), \delta) & \xrightarrow{G} & (S(V''), \delta'') \\
\Phi F \downarrow & & \downarrow j'' \\
(S(V'), \tilde{\delta}) & \xrightarrow{j'} & (S(V), \tilde{\delta})
\end{array}
\]

Note that \( j'' \) is an (acyclic) cofibration by Cor 4.17. We now show that the following diagram is a
pushout square which will complete our proof.

\[
\begin{array}{c}
(S(V), \delta) \xrightarrow{G} (S(V''), \delta'') \\
\downarrow F & \Downarrow j'' \\
(S(V'), \delta') \xrightarrow{j' \Phi} (S(\tilde{V}), \tilde{\delta})
\end{array}
\]

(10)

Indeed, suppose that there exists \((S(W), \delta_W) \in \text{CEAlg}\) along with a pair of maps \(\Gamma': (S(V'), \delta') \to (S(W), \delta_W)\) and \(\Gamma'': (S(V''), \delta'') \to (S(W), \delta_W)\) such that \(\Gamma'' F = \Gamma' G\). Then since \(\Phi\) is invertible, there exists a unique morphism \(\tilde{\Gamma}: (S(\tilde{V}), \tilde{\delta}) \to (S(W), \delta_W)\) such that TFDC

Thus \(\tilde{\Gamma}\) is the map that witnesses the universal property of \((S(\tilde{V}), \tilde{\delta})\) in diagram (10).

\[
\begin{array}{c}
(S(V), \delta) \xrightarrow{G} (S(V''), \delta'') \\
\downarrow \Phi F & \Downarrow \Gamma'' \\
(S(V'), \delta) \xrightarrow{\Gamma'} (S(\tilde{V}), \tilde{\delta}) \\
\Downarrow \Gamma' \Phi^{-1}
\end{array}
\]

Thus \(\tilde{\Gamma}\) is the map that witnesses the universal property of \((S(\tilde{V}), \tilde{\delta})\) in diagram (10).

\[
\begin{array}{c}
(S(V), \delta) \xrightarrow{G} (S(V''), \delta'') \\
\downarrow \Phi F & \Downarrow \Gamma'' \\
(S(V'), \delta) \xrightarrow{\Gamma'} (S(\tilde{V}), \tilde{\delta}) \\
\Downarrow \Gamma' \Phi^{-1}
\end{array}
\]

\[
\begin{array}{c}
(S(V), \delta) \xrightarrow{G} (S(V''), \delta'') \\
\downarrow \Phi F & \Downarrow \Gamma'' \\
(S(V'), \delta) \xrightarrow{\Gamma'} (S(\tilde{V}), \tilde{\delta}) \\
\Downarrow \Gamma' \Phi^{-1}
\end{array}
\]

\[
\begin{array}{c}
(S(V), \delta) \xrightarrow{G} (S(V''), \delta'') \\
\downarrow \Phi F & \Downarrow \Gamma'' \\
(S(V'), \delta) \xrightarrow{\Gamma'} (S(\tilde{V}), \tilde{\delta}) \\
\Downarrow \Gamma' \Phi^{-1}
\end{array}
\]

\[
\begin{array}{c}
(S(V), \delta) \xrightarrow{G} (S(V''), \delta'') \\
\downarrow \Phi F & \Downarrow \Gamma'' \\
(S(V'), \delta) \xrightarrow{\Gamma'} (S(\tilde{V}), \tilde{\delta}) \\
\Downarrow \Gamma' \Phi^{-1}
\end{array}
\]

Theorem 4.18. The category CEAlg admits the structure of a Brown category of cofibrant objects in which a morphism \(F: (S(V), \delta_V) \to (S(W), \delta_W)\) is:

- A weak equivalence iff \(F^1: (V, \delta_V^1) \to (W, \delta_W^1)\) is a quasi isomorphism of cochain complexes
- A cofibration iff \(F^1: (V, \delta_V^1) \to (W, \delta_W^1)\) is injective in all degrees greater than 1

Proof. Because the weak equivalences and cofibrations are characterized by the by their image in \(\text{Ch}_{\text{proj}}^{\geq 1}\), via \(T^*\), axioms 1, 2, and 5 are satisfied. Prop. 4.17 satisfies axiom 3 and Cor. 4.12 satisfies axiom 4.
4.3 Homotopy category of Chevalley-Eilenberg algebras

In this section we show that the localization of a Brown category with respect to its weak equivalences, which we call the homotopy category, exists by explicitly constructing its sets of morphisms. Since CEAlg is a category of cofibrant objects, we have provided the dual of the corresponding statements pertaining to categories of fibrant objects. However, the proofs of such statements will be omitted as they follow immediately from [5]. We then will describe certain sets morphisms in the homotopy category associated to CEAlg. Using this description we then will then show that the category of finite dimensional Lie algebras embeds faithfully into the homotopy category of CEAlg, and show that no such embedding exists into the homotopy category associated to the Bousfield Guggenheim model structure on cdga.

**Homotopy category of a Brown category**

**Definition 4.19.** Let $C$ be a Brown category. Then the **homotopy category** of $C$, written as $\text{Ho}(C)$, is a category equipped with a functor $\gamma: C \to \text{Ho}(C)$ such that for every functor $F: C \to D$ that inverts weak equivalences, there exists a unique functor $\tilde{F}: \text{Ho}(C) \to D$ such that $\tilde{F}\gamma = F$.

Within the abstract framework of Brown category there is a notion of homotopy that is reminiscent of the usual notion of homotopy from topology.

**Definition 4.20.** Two morphisms $f, g: X \to Y$ in a Brown category $C$ are said to be **homotopic** iff there exists a map

$$H: \text{Cyl}(X) \to Y$$

for some cylinder object of $X$ such that $Hi_0 = f$ and $Hi_1 = g$. If $f$ and $g$ are homotopic we write $f \simeq g$.

By combining cylinder objects as is done in [13] we can show that the relation $\simeq$ defines an equivalence relation on $\text{hom}_C(X, Y)$. The following proposition describes some useful facts about homotopy.

**Proposition 4.21.** Suppose that $f \simeq g: X \to Y$. Then

1. for any morphism $u: Y \to Z$ we have $uf \simeq ug: X \to Z$. 
2. for any morphism \( u : W \to X \) and for any cylinder object \( \text{Cyl}(W) \), there exists a an acyclic cofibration \( k : Y \to Z \) such that \( kf u \simeq kgu : W \to Z \) via a homotopy \( H : \text{Cyl}(W) \to Z \).

Proof. Follows from Proposition 1 in [5].

In order to get our hands on the homotopy category we first make an approximation. We define an equivalence relation on \( \text{hom}_C(X, Y) \) by \( f \sim g \) iff there exists a weak equivalence \( k : Y \to Z \) such that \( kf \simeq kg \). Note that this relation identifies homotopic maps. By using the above proposition one can easily verify that this equivalence relation respects composition. Thus we define \( \pi C \) be the category with the same objects as \( C \) and whose morphisms are equivalence classes of morphisms in \( C \) under the relation \( \sim \). We define the class of weak equivalences in \( \pi C \) to be the image of the the weak equivalences in \( C \) under the canonical functor \( C \to \pi C \).

Proposition 4.22. The category \( \pi C \) admits a calculus of left fractions, in the sense of [7], with respect to the class of weak equivalences. That is,

1. Given any weak equivalence \( k : W \to X \) and any morphism \( f : W \to Y \), there exists a weak equivalence \( k' : Y \to Z \) and a morphism \( f' : X \to Z \) such that the following diagram is homotopy commutative.

\[
\begin{array}{ccc}
W & \xrightarrow{f} & Y \\
\downarrow{k} & & \downarrow{k'} \\
X & \xrightarrow{f'} & Z
\end{array}
\]

2. Given a weak equivalence \( k : W \to X \) and a pair of parallel morphisms \( f, g : X \to Y \) such that \( fk \simeq gk \), there exists a weak equivalence \( k' : Y \to Z \) such that \( k'f \simeq k'g \).

Proof. Follows from Proposition 2 in [5].

Fix an object \( Y \in \pi C \). Consider the category \( \mathcal{F}_Y \) whose objects are weak equivalences \( [k] : Y \to Y' \) and whose morphisms are from \([k']\) to \([k]\) are commutative diagrams in \( \pi C \) of the form

\[
\begin{array}{ccc}
Y' & \xrightarrow{[f]} & Y'' \\
\downarrow{[k]} & & \downarrow{[k']} \\
Y & & \end{array}
\]
If $X$ and $Y$ are objects of $C$, let

$$[X, Y] := \lim_{\to} \hom_{\pi C}(X, Y')$$

where the directed limit is indexed by $Y' \in \mathcal{F}_Y$.

**Proposition 4.23.** If $C$ is a Brown category and $X, Y \in C$, then

$$\hom_{\Ho(C)}(X, Y) \cong [X, Y]$$

**Proof.** Follows from Theorem 1 in [5].

---

**Morphisms of $\Ho(\CEAlg)$ into $C(g)$**

For the remainder of this thesis we will denote a CE algebra by $S(V)$ leaving the differential, multiplication and unit implicit. We will write $C(g)$ for the CE algebra of a finite dimensional Lie algebra equipped with its usual differential (see Def. 3.23). Let $\pi \CEAlg$ be the quotient category of $\CEAlg$ by the relation $F \sim G$ iff there exists a weak equivalence $K$ such that $KF \cong KG$. We write $[F]$ for the equivalence class represented by $F$ in $\pi \CEAlg$.

In the following sequence of propositions we will obtain a tangible description of $[S(V), C(g)]$.

We begin with a few technical results.

**Proposition 4.24.** Let $K : S(V) \to S(W)$ be an acyclic cofibration. Then there exists a morphism $\Psi : S(W) \to S(V)$ of CE algebras such that $\Psi K = \id_{S(V)}$.

**Proof.** Follows immediately from Prop. A.1 in the Appendix.

**Corollary 4.25.** Let $C(g)$ be the Chevalley-Eilenberg algebra of a finite dimensional Lie algebra. Then for every weak equivalence $K : C(g) \to S(W)$ there exists morphism $\Psi : S(W) \to C(g)$ such that $\Psi K = \id_{C(g)}$.

**Proof.** Since $C(g)$ is the symmetric algebra on a vector space concentrated in degree 1, all morphisms in $\CEAlg$ with domain $C(g)$ are cofibrations.
Lemma 4.26. Let $C(\mathfrak{g})$ be the Chevalley-Eilenberg algebra of a finite dimensional Lie algebra. Suppose that the following diagram commutes in $\pi\text{CEAlg}$

$$
\begin{array}{ccc}
S(W) & \xrightarrow{[F]} & S(W') \\
\downarrow{[K]} & & \downarrow{[K']}
\end{array}
$$

where $K$ and $K'$ are weak equivalences. Then there exists morphisms $\Psi : S(W) \to C(\mathfrak{g})$ and $\Psi' : S(W') \to C(\mathfrak{g})$ such that $\Psi K = \text{id}_{C(\mathfrak{g})} = \Psi' K'$ and $[\Psi] = [\Psi' F]$ in $\pi\text{CEAlg}$.

Proof. By Cor. 4.25 there exists left inverses $\Psi : S(W) \to C(\mathfrak{g})$ and $\Psi' : S(W') \to C(\mathfrak{g})$ of $K$ and $K'$ respectively. Since the diagram commutes in $\pi\text{CEAlg}$, there exists a weak equivalence $G : S(W') \to S(U)$ such that $G K' \simeq GFK$. $G K' : C(\mathfrak{g}) \to S(U)$ is a weak equivalence, so there exists a left inverse $\Gamma : S(U) \to C(\mathfrak{g})$. Thus,

$$
\Psi K = \text{id}_{C(\mathfrak{g})} = \Gamma G K' \simeq \Gamma GFK
$$

Hence, by part 2 of Prop. 4.22, there exists a weak equivalence $G' : C(\mathfrak{g}) \to S(U')$ such that $G' G F \simeq G' \Psi$. Therefore $[\Psi] = [GFK]$ in $\pi\text{CEAlg}$.

Now observe that $G K' = \text{id}_{C(\mathfrak{g})} = \Psi' K'$. Again by part 2 of Prop. 4.22, there exists a weak equivalence $G'' : C(\mathfrak{g}) \to S(U'')$ such that $G'' G \simeq G'' \Psi'$. Then by Prop. 4.21 there exists a weak equivalence $G''' : S(U') \to S(U'')$ such that $G''' G'' G F \simeq G''' G'' \Psi' F$. Therefore since $G'' G''$ is a weak equivalence, $[GFK] = [\Psi' F]$. So $[\Psi] = [\Psi' F]$ in $\pi\text{CEAlg}$.

\[ \square \]

Proposition 4.27. Let $C(\mathfrak{g})$ be the Chevalley-Eilenberg algebra of a finite dimensional Lie algebra, and $S(V)$ be a CE algebra. Then,

$$
[S(V), C(\mathfrak{g})] \cong \text{hom}_{\pi\text{CEAlg}}(S(V), C(\mathfrak{g}))
$$

Proof. Let $\mathcal{J}_{C(\mathfrak{g})}$ be the category whose objects are the weak equivalences $[K] : C(\mathfrak{g}) \to S(W)$ in $\pi\text{CEAlg}$ and whose morphisms are defined as above in (11). For each $[K] \in \mathcal{J}_{C(\mathfrak{g})}$ choose a weak equivalence, $K : C(\mathfrak{g}) \to S(V) \in \text{CEAlg}$, representing it, and for every weak equivalence $K$ choose
a section, i.e. a CE algebra morphism \( \Psi : S(W) \to C(\mathfrak{g}) \) such that \( \Psi \kappa = \text{id}_{C(\mathfrak{g})} \). Consider the diagram \( \text{hom}_{\pi\text{CEAlg}}(S(V), -) : \mathcal{J}_{C(\mathfrak{g})} \to \text{Set} \). Then the collection

\[
\{ \Psi_* : \text{hom}_{\pi\text{CEAlg}}(S(V), S(W)) \to \text{hom}_{\pi\text{CEAlg}}(S(V), C(\mathfrak{g})) \}_{\mathcal{J}_{C(\mathfrak{g})}}
\]

defines a cocone with vertex \( \text{hom}_{\pi\text{CEAlg}}(S(V), C(\mathfrak{g})) \). Indeed the previous lemma implies that TFDC in \( \pi\text{CEAlg} \)

\[
\begin{array}{ccc}
S(W) & \xrightarrow{[F]} & S(W') \\
| & \downarrow{[\Psi]} & \downarrow{[\Psi']} \\
C(\mathfrak{g}) & & \\
\end{array}
\]

where \( F \) is a morphism from \( [K'] : C(\mathfrak{g}) \to S(W) \) to \( [K] : C(\mathfrak{g}) \to S(W') \) in \( \mathcal{J}_{C(\mathfrak{g})} \). Since the diagram lives in the category \( \text{Set} \), its direct limit is given by the disjoint union

\[
\bigsqcup_{S(W) \in \mathcal{J}_{C(\mathfrak{g})}} \text{hom}_{\pi\text{CEAlg}}(S(V), S(W))
\]

modulo some equivalence relation. Because all identity maps are weak equivalences, \( [\text{id}] : C(\mathfrak{g}) \to C(\mathfrak{g}) \in \mathcal{J}_{C(\mathfrak{g})} \), and hence \( \text{hom}_{\pi\text{CEAlg}}(S(V), C(\mathfrak{g})) \) is a summand of colimit. Thus there exists an inclusion \( \text{hom}_{\pi\text{CEAlg}}(S(V), C(\mathfrak{g})) \hookrightarrow [S(V), C(\mathfrak{g})] \). Using the fact that every section \( \Psi \) is a morphism in \( \mathcal{J}_{C(\mathfrak{g})} \) from \( [\text{id}] : C(\mathfrak{g}) \to C(\mathfrak{g}) \) to \( [K] : C(\mathfrak{g}) \to S(W) \), and that the identity is the only possible section of \( \text{id} : C(\mathfrak{g}) \to C(\mathfrak{g}) \), one can show that the inclusion \( \text{hom}_{\pi\text{CEAlg}}(S(V), C(\mathfrak{g})) \hookrightarrow [S(V), C(\mathfrak{g})] \) is the function witnessing the universal property of \( \text{hom}_{\pi\text{CEAlg}}(S(V), C(\mathfrak{g})) \).

\[\square\]

We will now show that the equivalence relation \( \sim \) agrees with a simpler relation on \( \text{hom}_{\text{CEAlg}}(S(V), C(\mathfrak{g})) \).

Recall the construction in Remark 4.13 of the cylinder object

\[
(S(V) \otimes S(V), \delta_{\otimes}) \xrightarrow{(i_0, i_1)} (\overline{C_y}(V), \delta) \xrightarrow{s} (S(V), \delta)
\]

For an arbitrary CE algebra \( S(V) \) and a Chevalley-Eilenberg algebra of a Lie algebra \( C(\mathfrak{g}) \), define a relation on the set \( \text{hom}_{\text{CEAlg}}(S(V), C(\mathfrak{g})) \) by, \( F \cong_{\text{Cyl}} G \) iff there exists a morphism \( H : \overline{C_y}(V) \to C(\mathfrak{g}) \) such that \( Hi_0 = F \) and \( Hi_1 = G \).
**Proposition 4.28.** Let $\mathcal{C}(\mathfrak{g})$ be the Chevalley-Eilenberg algebra of a finite dimensional Lie algebra, and $S(V)$ be a CE algebra. If $F, G : S(V) \to \mathcal{C}(\mathfrak{g})$ are CE algebra morphisms, then $F \simeq G$ iff $F \simeq_{\text{Cyl}} G$. Hence $\simeq_{\text{Cyl}}$ is an equivalence relation.

**Proof.** Clearly $F \simeq_{\text{Cyl}} G$ implies that $F \simeq G$. Now suppose that $F \simeq G$. Then by Prop. 4.21 there exists a weak equivalence $K : \mathcal{C}(\mathfrak{g}) \to S(W)$ such that $KF \simeq KG$ via $\text{Cyl}(V)$. By Cor. 4.25 there exists a $\Psi : S(W) \to \mathcal{C}(\mathfrak{g})$ such that $\Psi K = \text{id}_{\mathcal{C}(\mathfrak{g})}$. Hence $F = \Psi TF \simeq \Psi TG = G$ via $\text{Cyl}(V)$. Therefore $F \simeq_{\text{Cyl}} G$. 

**Proposition 4.29.** Let $\mathcal{C}(\mathfrak{g})$ be the Chevalley-Eilenberg algebra of a finite dimensional Lie algebra, and $S(V)$ be a CE algebra. Then

$$[S(V), \mathcal{C}(\mathfrak{g})] \cong \text{hom}_{\text{CEAlg}}(S(V), \mathcal{C}(\mathfrak{g}))/ \simeq_{\text{Cyl}}$$

**Proof.** In the previous proposition we have shown that $\simeq_{\text{Cyl}}$ and $\simeq$ agree on $\text{hom}_{\text{CEAlg}}(S(V), \mathcal{C}(\mathfrak{g}))$. We will now show $\simeq$ and $\sim$ also agree. Then,

$$\text{hom}_{\text{CEAlg}}(S(V), \mathcal{C}(\mathfrak{g})) = \text{hom}_{\text{CEAlg}}(S(V), \mathcal{C}(\mathfrak{g}))/ \simeq_{\text{Cyl}}$$

and Prop. 4.27 will yield the desired result.

Let $F, G : S(V) \to \mathcal{C}(\mathfrak{g})$ be morphisms of CE algebras. Suppose that $F \simeq G$. Then since $\text{id}_{\mathcal{C}(\mathfrak{g})}$ is a weak equivalence, $F \sim G$. On the other hand suppose that $F \sim G$. Thus there exists a weak equivalence $K : \mathcal{C}(\mathfrak{g}) \to S(W)$ such that $KF \simeq KG$. By Cor. 4.25 there exists a morphism $\Psi$ such that $\Psi K = \text{id}_{\mathcal{C}(\mathfrak{g})}$. Hence $F = \Psi TF \simeq \Psi TG = G$.

**Embedding LieAlg in Ho(CEAlg)**

We will now use Prop. 4.29 to show that the Chevalley-Eilenberg functor faithfully embeds the category of finite dimensional Lie algebras into $\text{Ho}(\text{CEAlg})$.

**Proposition 4.30.** Let $\mathcal{C}(\mathfrak{h})$ and $\mathcal{C}(\mathfrak{g})$ be Chevalley-Eilenberg algebras of finite dimensional Lie algebras. If $F, G : \mathcal{C}(\mathfrak{h}) \to \mathcal{C}(\mathfrak{g})$ are CE algebra morphisms then $F \simeq_{\text{Cyl}} G$ iff $F = G$. 

Proof. Clearly if \( F = G \) then \( F \simeq_{\mathcal{C}Y} G \). Now suppose that \( F \simeq_{\mathcal{C}Y} G \). Then there exists a homotopy \( H: \mathcal{C}Y(h) \to C(g) \) such that, \( Hi_0 = F \) and \( Hi_1 = G \). Observe that \( \mathcal{C}Y(h) \) is the symmetric algebra on \( h^* \) concentrated in degree 1, and that \( i_0^1 = i_1^1 \) (see Remark 4.13). Since \( i_0 \) and \( i_1 \) must be strict for degree reasons, \( i_0 = i_1 \). Therefore \( F = Hi_0 = Hi_1 = G \).

\[ \square \]

**Corollary 4.31.** Let \( h \) and \( g \) be finite dimensional Lie algebras. Then

\[ \text{hom}_{\text{LieAlg}}(g, h) \cong [C(h), C(g)] \]

**Proof.** Recall Prop. 3.25 which states that \( C: \text{LieAlg} \to \text{CEAlg} \) is full and faithful. Prop. 4.29 and the prior proposition imply that

\[ \text{hom}_{\text{LieAlg}}(g, h) \cong \text{hom}_{\text{CEAlg}}(C(h), C(g)) \cong [C(h), C(g)] \]

\[ \square \]

\textbf{LieAlg does not embed in Ho(cdga)}

Let \( g \) be the two dimensional \( k \)-vector space spanned by \( \{x, y\} \). Define a bracket on \( g \) by

\[ [a_1 x + a_2 y, b_1 x + b_2 y] = (a_1 b_2 - a_2 b_1) x \]

Let \( \phi: g \to k \) be the linear transformation such that \( x \mapsto 0 \) and \( y \mapsto 1 \). If we consider \( k \) as an abelian Lie algebra, then \( \phi \) is a Lie algebra homomorphism. Moreover, \( C(\phi): C(k) \to C(g) \) is a quasi isomorphism of cdgas. It is not however a weak equivalence. Thus, in the homotopy category associated to the BG model structure on cdga we have the following bijection

\[ \text{hom}_{\text{Ho(cdga)}}(C(g), C(g)) \xrightarrow{C(\phi)^*} \text{hom}_{\text{Ho(cdga)}}(C(k), C(g)) \]

Let \( \psi_1, \psi_2: g \to g \) be two Lie algebra morphisms defined by

\[ \psi_1(ax + by) = by \]

\[ \psi_2(ax + by) = bx + by \]
Note that $C(\phi^* (C(\psi_1))) = C(\phi^* (C(\psi_2)))$. Therefore the functor,

$$ C: \text{LieAlg} \rightarrow \text{Ho}(\text{cdga}) $$

is not faithful.
5 Representability Theorems

In this section we will prove results that are analogous to E. Brown’s representability theorem for the cohomology of CW complexes [4] and the classification of principal $G$ bundles over a CW complex. Specifically, we will show that the cohomology functor $H^n(-, M) : \text{LieAlg} \to \text{AbGrp}$, where $M$ is a trivial $\mathfrak{g}$-module and $n \geq 1$, is representable as a functor $\text{Ho}(\text{CEAlg}) \to \text{AbGrp}$, and we will show that set of all principal cofiber sequences with a specific base and cofiber can be classified by a set of morphisms in $\text{Ho}(\text{CEAlg})$. First we outline several homotopical constructions for a pointed Brown category.

5.1 Suspensions, cogroups, and principal cofiber sequences in Brown categories

Motivated by the homotopy theory of pointed CW complexes, Brown defines the notion of a loop functor and fiber sequences for pointed categories of fibrant objects in [5]. We dualize these constructions to a suspension functor and cofiber sequences for a pointed category of cofibrant objects. Then following [1] we introduce the notion of a principal cofiber sequence.

Suspension in Brown Categories

**Definition 5.1.** Let $\mathcal{C}$ be a pointed Brown category with zero object $\ast$. The **suspension** of an object $X \in \mathcal{C}$ is defined to be the pushout of

$$
\begin{array}{ccc}
\text{Cyl}(X) & \leftarrow & X \coprod X \\
\downarrow^{i_0,i_1} & & \downarrow \\
\ast & & 
\end{array}
$$

where $\text{Cyl}(X)$ is any cylinder object of $X$. We denote the suspension of $X$ by $\Sigma X$.

As shown in [5], different cylinder objects yield weakly equivalent suspensions. Thus, the assignment $X \mapsto \Sigma X$ defines a functor $\Sigma : \text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{C})$. There is an alternative method to obtain the suspension of an object $X$ as a sequence of pushouts utilizing the cone of $X$.

**Definition 5.2.** Let $\mathcal{C}$ be a pointed Brown category with zero object $\ast$. The **cone** of an object $X \in \mathcal{C}$ is defined to be the pushout of

$$
\begin{array}{ccc}
\text{Cyl}(X) & \leftarrow & X \\
\downarrow^{i_0} & & \\
\ast & & 
\end{array}
$$

where $\text{Cyl}(X)$ is any cylinder object of $X$. We denote the cone of $X$ by $\text{Cone}(X)$. 

**Remark 5.3.** Let \( q : \text{Cyl}(X) \to \text{Cone}(X) \) be the map induced by the pushout. Then the composition \( qi_1 : X \to \text{Cone}(X) \) is a cofibration and the pushout of

\[
\text{Cone}(X) \xleftarrow{qi_1} X \to *
\]

is \( \Sigma X \). Indeed, this fact is easily verified by using the construction in Lemma 4.3 to factor the trivial map \( X \to * \), and the pushout pasting lemma.

**Cogroup structure on the suspension**

The pinch map \( \Sigma X \to \Sigma X \lor \Sigma X \), makes the suspension of a pointed CW complex into a cogroup object in the homotopy category. Similarly, in any pointed Brown category \( \mathcal{C} \), \( \Sigma X \) is a cogroup object in \( \text{Ho}(\mathcal{C}) \) for all \( X \in B \). The proof of this fact as well as the construction of the comultiplication can be found in [5, Sec. 4, Thm. 3]. However, we provide the definition of a cogroup object, as well as a useful proposition pertaining to cogroup objects.

**Definition 5.4.** Let \( \mathcal{C} \) be a category with finite coproducts and terminal object \( * \in \mathcal{C} \). A cogroup object in \( \mathcal{C} \) is an object \( X \) equipped with three morphisms \( \Delta : X \to X \coprod X \), called comultiplication, and \( \epsilon : X \to * \), called the counit, and \( j : X \to X \) called inversion such that the following diagrams commute

1. Associativity

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \coprod X \\
\downarrow{\Delta} & & \downarrow{id \coprod \Delta} \\
X \coprod X & \xrightarrow{\Delta \coprod id} & X \coprod X \coprod X
\end{array}
\]

2. Left/Right Unit laws

\[
\begin{array}{ccc}
* \coprod X & \xrightarrow{id \coprod \epsilon} & X \coprod X \\
\downarrow{\Delta} & & \downarrow{id \coprod \epsilon} \\
X & \xrightarrow{\Delta} & X \coprod *
\end{array}
\]
3. Invertibility

\[
\begin{array}{ccc}
X & \xrightarrow{(j, \text{id})} & X \sqcup X \\
\downarrow\text{id} & & \downarrow\Delta \\
X & & X \\
\end{array}
\]

**Proposition 5.5.** Let \( C \) be a category with finite coproducts and terminal object. If \( X \in C \) is a cogroup object then \( \text{hom}_C(X, Y) \) is a group for all \( Y \in C \).

**Proof.** Let \( \Delta : X \to X \sqcup X \) be the comultiplication on \( X \). Then for \( f, g : X \to Y \), denote the morphism induced by the universal property by \( (f, g) : X \sqcup X \to Y \). Then \( (f, g)\Delta : X \to Y \). This construction gives us a multiplication on \( \text{hom}_C(X, Y) \). We direct the reader to Arkowitz’s book [2, Sec. 2.2] for the details.

\[\Box\]

**Cofiber Sequences**

Let \( f : X \to Y \) be a cofibration in a pointed Brown category \( C \) and let the following be a pushout square.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\ast & \xrightarrow{=} & C_f \\
\end{array}
\]

In [5] is shown that there exists a coaction in \( \text{Ho}(C) \) of \( \Sigma X \) on \( C_f \).

**Definition 5.6.** Let \( f : X \to Y \) be a cofibration in a pointed Brown category \( C \). Then,

\[
X \xrightarrow{f} Y \to C_f
\]

along with the coaction \( C_f \to \Sigma X \sqcup C_f \), is said to be a **cofiber sequence** with base \( X \) and cofiber \( C_f \). Two cofiber sequences with base \( X \) and cofiber \( C_f \) are said to be equivalent iff their respective
coactions agree and there exists a commutative diagram in Ho(C)

\[ \begin{array}{ccc}
X & \overset{\phi}{\longrightarrow} & Y \\
\downarrow{id} & & \downarrow{id} \\
X & \overset{\phi}{\longrightarrow} & Y'
\end{array} \rightarrow C_f
\]

in which the morphism \( Y \rightarrow Y' \) is an isomorphism in Ho(C).

**Remark 5.7.** Let \( q_1: Z \rightarrow \text{Cone}(Z) \) be the cofibration defined in Remark 5.3. Then for an arbitrary morphism \( \tilde{g}: Z \rightarrow X \) we obtain a cofiber sequence

\[ X \overset{g}{\longrightarrow} \text{Cone}(Z) \prod_Z X \rightarrow \Sigma Z \]

where \( g: X \rightarrow \text{Cone}(Z) \prod_Z X \) is the morphism induced by the pushout of

\[ \text{Cone}(Z) \overset{q_1}{\leftarrow} Z \overset{\tilde{g}}{\rightarrow} X \]

By the pushout pasting lemma and Remark 5.3, the cofiber of \( g \) is \( \Sigma Z \).

**Definition 5.8** (Def. 2.8, [1]). A cofiber sequence \( X \overset{f}{\rightarrow} Y \rightarrow \Sigma C_f \) is called a principal cofiber sequence iff there exists an object \( Z \in C \), called a classifying space, a morphism \( \tilde{g}: Z \rightarrow X \), called a classifying map, and two commutative diagrams in Ho(C) of the form

\[ \begin{array}{ccc}
X & \overset{g}{\longrightarrow} & \text{Cone}(Z) \prod_Z X \\
\downarrow{id} & & \downarrow{id} \\
X & \overset{f}{\longrightarrow} & Y
\end{array} \rightarrow \Sigma Z \]

\[ \begin{array}{ccc}
X \prod_{\Sigma Z} X & \longrightarrow & \Sigma X \prod_{\Sigma Z} X \\
\downarrow{\phi} & & \downarrow{\text{id} \prod \phi} \\
C_f & \longrightarrow & \Sigma X \prod_{C_f} C_f
\end{array} \]

in which the top row of (12) is the cofiber sequence obtained from \( \tilde{g} \) and the vertical maps are isomorphisms in Ho(C). We denote a principal cofiber sequence by \( Z \overset{\tilde{g}}{\rightarrow} X \overset{f}{\rightarrow} Y \rightarrow \Sigma C_f \).

Two principal cofiber sequences with classifying space \( Z \), base \( X \) and cofiber \( C_f \) are said to be
equivalent if their respective coactions agree and there exists a commutative diagram in \( \text{Ho}(C) \) of the form

\[
\begin{array}{ccc}
Z & \xrightarrow{\tilde{g}} & X \\
\downarrow\text{id} & & \downarrow\text{id} \\
Z & \xrightarrow{\tilde{g}'} & X \\
\end{array}
\begin{array}{ccc}
Y & \rightarrow & C_f \\
\downarrow\text{id} & & \downarrow\text{id} \\
Y' & \rightarrow & C_f \\
\end{array}
\]

in which \( Y \rightarrow Y' \) is an isomorphism in \( \text{Ho}(C) \).

### 5.2 Cohomology

**Notation 5.9.** Let \( M \) be a \( k \)-vector space and let \( n \geq 1 \). We define \( K(M, n) \) to be the symmetric algebra on the graded vector space consisting of \( M^* \) concentrated in degree \( n \), equipped with the trivial differential. That is, \( K(M, n) = (S(M^*[n]), 0) \).

We begin this section by proving that the functor \( H^n(-, M) : \text{LieAlg} \rightarrow \text{Set} \), where \( M \) is a trivial \( \mathfrak{g} \)-module and \( n \geq 1 \), is represented by the the functor \( [K(M, n), -] : \text{Ho}(\text{CEAlg}) \rightarrow \text{Set} \). We will then show that the functor \( [K(M, n), -] \) factors through the category of abelian groups, and that the bijection \( [K(M, n), C(\mathfrak{g})] \cong H^n(\mathfrak{g}, M) \) preserves the group structure.

**Proposition 5.10.** Let \( \mathfrak{g} \) be a finite dimensional Lie algebra and \( M \) be a finite dimensional vector space considered as a trivial \( \mathfrak{g} \)-module. There exists an bijection of sets

\[
H^n(\mathfrak{g}, M) \cong [K(M, n), C(\mathfrak{g})]
\]

for all \( n \geq 1 \).

**Proof.** Using Prop. 4.29, we identify \([K(M, n), C(\mathfrak{g})]\) with \( \text{hom}_{\text{CEAlg}}(K(M, n), C(\mathfrak{g}))/\simeq_{\mathfrak{g}} \). Let \( n \geq 1 \) and let \((\text{hom}_{(\Lambda^*\mathfrak{g}, M), d})\) be the cochain complex whose cohomology is \( H^*(\mathfrak{g}, M) \) (see Def. 3.27). Suppose that \( f : \Lambda^n\mathfrak{g} \rightarrow M \) is a cocycle. Since \( M \) is a trivial module,

\[
df(x_1 \wedge \cdots \wedge x_{n+1}) = - \sum_{i<j} (-1)^{i+j} f([x_i, x_j], \cdots, \check{x}_i, \cdots, \check{x}_j, \cdots, x_{n+1}) = 0
\]

Thus we can define a morphism of CE algebras \( F : K(M, n) \rightarrow C(\mathfrak{g}) \) as \( F^n_1 = f^* \) and \( F^k_1 = 0 \) for all \( k \neq n \). Because \( f \) is a cocycle, \( \delta^{n+1}_n F^n_1 = 0 \) which implies that \( F \) commutes with the differentials.
on $K(M, n)$ and $C(\mathfrak{g})$. One can easily reverse this argument to obtain a cocycle from a CE algebra morphism. Therefore,

$$\left( \ker d: \text{hom}_k(\mathfrak{g}, M) \to \text{hom}_k(\Lambda^2 \mathfrak{g}, M) \right) \cong \text{hom}_{\text{CEAlg}}(K(M, n), C(\mathfrak{g}))$$

(13)

Observe that since $M$ is a trivial $\mathfrak{g}$-module,

$$H^1(\mathfrak{g}, M) = \ker d: \text{hom}_k(\mathfrak{g}, M) \to \text{hom}_k(\Lambda^2 \mathfrak{g}, M)$$

Moreover, $K(M, 1)$ can be seen as the Chevalley-Eilenberg algebra of $M$ considered as an abelian Lie algebra. Thus by (13) and Prop. 4.30,

$$H^1(\mathfrak{g}, M) \cong \text{hom}_{\text{CEAlg}}(K(M, 1), C(\mathfrak{g})) \cong [K(M, 1), C(\mathfrak{g})]$$

Now let $n \geq 2$. Suppose that $F, G: K(M, n) \to C(\mathfrak{g})$ are homotopic via $\widetilde{\text{Cy}}\mathfrak{l}(M, n)$. Then the homotopy $H: \widetilde{\text{Cy}}\mathfrak{l}(M, n) \to C(\mathfrak{g})$ can be expressed as the following commutative diagram,

$$\begin{array}{ccc}
M^* \oplus M^* \oplus M^* & \xrightarrow{H^n_1} & \Lambda^n \mathfrak{g} \\
(0, \text{id}_{M^* \oplus M^*}) & & \downarrow \delta^n_{n-1} \\
M^* \oplus M^* & \xrightarrow{H^{n-1}_1} & \Lambda^{n-1} \mathfrak{g}
\end{array}$$

where $H_1^n(x, x, 0) = F_1^n(x)$ and $H_1^n(x, 0, x) = G_1^n(x)$ for all $x \in M^*$. Thus, $\delta^n_{n-1}H_1^{n-1}(x, -x) = F_1^n(x) - G_1^n(x)$. Define a $n - 1$ cochain $h: \Lambda^{n-1} \mathfrak{g} \to M$ by $h = (H_1^{n-1} j)^*$ where $j: M^* \to M^* \oplus M^*$ is the $k$-linear map given by $j(x) = (x, -x)$ for all $x \in M^*$. Therefore, if $f, g: \Lambda^n \mathfrak{g} \to M$ are the cocycles corresponding to $F$ and $G$ respectively, then $f - g = dh$.

On the other hand suppose that $f, g: \Lambda^n \mathfrak{g} \to M$ are two cocycles such that $f - g = dh$ for some $h: \Lambda^{n-1} \mathfrak{g} \to M$. As above we will define $F, G: K(M, n) \to C(\mathfrak{g})$ to be the CE algebra morphisms corresponding to $f$ and $g$ respectively, and we will build a homotopy $H: \widetilde{\text{Cy}}\mathfrak{l}(M, n) \to C(\mathfrak{g})$ from $F$ to $G$. For degree reasons $H$ has only two nontrivial structure maps, $H_1^n: M^* \oplus M^* \oplus M^* \to \Lambda^n \mathfrak{g}^*$,
which we define as

\[ H^n_1(x, 0, 0) = G(x) \]

\[ H^n_1(0, x, 0) = F(x) - G(x) \]

\[ H^n_1(0, 0, x) = 0 \]

and \( H^{n-1}_1 : M^* \oplus M^* \to \Lambda^{n-1} g^* \), which we define as

\[ H^{n-1}_1(x, 0) = h^*(x) \]

\[ H^{n-1}_1(0, x) = 0 \]

for all \( x \in M^* \). We leave it to the reader to verify that \( H \) commutes with the differentials on \( \widetilde{\text{Cyl}}(M, n) \) and \( C(g) \), which will complete the proof.

\[ \square \]

This proposition implies that \( H^n(g, M) \cong [K(M, n), C(g)] \) as sets. In order to strengthen this result to an isomorphism of groups, we will show that \( K(M, n) \) a canonical cogroup structure.

**Suspension in CEAsg**

**Notation 5.11.** Observe that CEAsg is a pointed category with zero object given by \( S(0) \), which we will denote by \( * \).

**Proposition 5.12.** Let \( M \) be a finite dimensional vector space and let \( n \geq 1 \). Then \( \Sigma K(M, n + 1) \cong K(M, n) \) in \( \text{Ho}(\text{CEAlg}) \).

**Proof.** Let \( \Sigma K(M, n + 1) \) be the pushout of

\[ \widetilde{\text{Cyl}}(M, n + 1) \xleftarrow{(i_0, i_1)} K(M, n + 1) \otimes K(M, n + 1) \to * \]

For degree reasons, \((i_0, i_1)\) must be strict. Thus by Prop. 4.14, \( \Sigma K(M, n + 1) \) is the symmetric
algebra on the graded vector space $\widetilde{M}$ where

\[
\widetilde{M}^i = \begin{cases} 
M^* & i = n + 1 \\
M^* \oplus M^* & i = n \\
0 & \text{otherwise}
\end{cases}
\]

and the differential $\widetilde{\delta}$ on $\Sigma K(M, n + 1)$ is strict where

\[
\widetilde{\delta}_1(x, y) = -x - y
\]

for all $x, y \in M^*$. We define a strict CE algebra morphism $F: \Sigma K(M, n + 1) \to K(M, n)$ with $F_1^1(x, y) = y$ in degree $n$ and $0$ elsewhere. Thus, $F_1^1: (\widetilde{M}, \widetilde{\delta}_1) \to (M^*[n], 0)$ is a quasi isomorphism of cochain complexes. Therefore, $\Sigma K(M, n + 1) \cong K(M, n)$ in $\text{Ho}(\text{CEAlg})$.

\[\]

**Proposition 5.13.** Let $n \geq 1$. The comultiplication on $K(M, n)$ induced by the suspension is the strict CE algebra morphism $\Delta: K(M, n) \to K(M, n) \otimes K(M, n)$ given by

\[
\Delta_1^1(x) = x \otimes 1 + 1 \otimes x
\]

**Proof.** Dualizing the sequence of pullbacks which yield a multiplication on the loop space of an object in [5, Sec. 4, Thm. 3], we obtain the desired result by a long but straightforward calculation. Specifically we proceed by obtaining a cylinder object by pushing out $K(M, n+1) \otimes K(M, n+1) \to \overline{\text{Cyl}}(M, n + 1)$ with itself. Then we invoke the universal property of the pushout to construct a morphism from this new cylinder into $\Sigma K(M, n + 1) \otimes \Sigma K(M, n + 1)$. Then after identifying $\Sigma K(M, n + 1)$ with $K(M, n)$ we reach the desired result.

\[\]

**Theorem 5.14.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $M$ be a trivial $\mathfrak{g}$-module. Then as groups

\[
H^n(\mathfrak{g}, M) \cong [K(M, n), C(\mathfrak{g})]
\]

for all $n \geq 1$. 

\[\]
Proof. Let \( F, G : K(M, n) \to C(\mathfrak{g}) \). Then by Prop. 5.5 the sum of \( F \) and \( G \) is given by \( (F, G)\Delta : K(M, n) \to C(\mathfrak{g}) \), where \( (F, G) : K(M, n) \otimes K(M, n) \to C(\mathfrak{g}) \) is the morphism induced by the universal property of the coproduct. For degree reasons, the structure maps of \( (F, G)\Delta \) are zero except in arity \( n \). The arity \( n \) structure map \( (F, G)^\otimes_1 \Delta^1_1 : M^* \to \Lambda^n g^* \) is given by
\[
(F, G)^\otimes_1 \Delta^1_1(x) = F^\otimes_1(x) + G^\otimes_1(x)
\]
Moreover, the assignment in Prop. 5.10 \([K(M, n), C(\mathfrak{g})] \to H^n(\mathfrak{g}, M)\) is given by \( F \mapsto (F^\otimes_1)^* \), and therefore respects the group structure.

\[
\square
\]

5.3 Classification of principal cofiber sequences

In this section we prove that the principal cofiber sequences in CEAlg with classifying space \( K(M, n+1) \), base \( C(\mathfrak{g}) \) and cofiber \( K(M, n) \) are classified by \([K(M, n+1), C(\mathfrak{g})]\) for all \( n \geq 1 \). We then introduce central extensions of Lie algebras and describe their relationship to these principal cofiber sequences when \( n = 1 \).

Remark 5.15. Given a cofiber sequence with base \( C(\mathfrak{g}) \), the coaction of \( \Sigma C(\mathfrak{g}) \) on the cofiber is trivial. Indeed, by Prop. 4.14, \( \Sigma C(\mathfrak{g}) = \ast \). Thus it is not necessary to show that the coactions of two cofiber sequences with base \( C(\mathfrak{g}) \) agree in order to conclude that they are equivalent.

Proposition 5.16. Let \( \mathfrak{g} \) be a finite dimensional Lie algebra, and let \( M \) be a finite dimensional vector space. The set of equivalent principal cofiber sequences in CEAlg with classifying space \( K(M, 2) \), base \( C(\mathfrak{g}) \), and cofiber \( K(M, 1) \) is in bijective correspondence with \([K(M, 2), C(\mathfrak{g})]\).

Proof. To obtain a cofiber sequence from a CE algebra morphism, we follow Remark 5.7. Let \( \widehat{\text{Cone}}(M, 2) \) be the pushout of
\[
\begin{array}{ccc}
\text{Cyl}(M, 2) & \xrightarrow{i_0} & K(M, 2) \\
\downarrow & & \downarrow \to *
\end{array}
\]
and let \( Q : \text{Cyl}(M, 2) \to \widehat{\text{Cone}}(M, 2) \) be the morphism induced by this pushout. Since \( i_0 \) is strict, Prop. 4.14 implies that \( \widehat{\text{Cone}}(M, 2) \) is the symmetric algebra on \( M^* \oplus M^* \) in degrees 1 and 2, with the only nontrivial differential given by the identity in arity 1, and that the composition \( Qi_1 : K(M, 2) \to \)
$\text{Cone}(M, 2)$ is the strict map given by $(Q_i)_1(x) = (-x, x)$ in degree 2. Given a morphism in CAlg, $F: K(M, 2) \rightarrow C(g)$, we obtain a principal cofiber sequence by a sequence of pushouts. Since $Q_i$ is a cofibration, we take the pushout of $Q_i$ along $F$. Because cofibrations are preserved by pushouts, we take the pushout of the induced map out of $C(g)$ along the unique map into the zero object. The following diagram illustrates this construction.

\[
\begin{array}{ccc}
K(M, 2) & \xrightarrow{F} & C(g) \\
\downarrow{Q_i} & & \downarrow{J} \\
\text{Cone}(M, 2) & \xrightarrow{P} & S(M')
\end{array}
\]

By Prop. 4.14,

\[
(M')^i = \begin{cases}
g^* \oplus M^* \oplus M^* & i = 1 \\
M^* & i = 2 \\
0 & \text{otherwise}
\end{cases}
\]

and the structure maps of the differential $\delta'$ on $S(M')$ are nontrivial in arity 1 and 2. The arity 1 structure map on degree 1 elements

\[
\delta'^1_1: g^* \oplus M^* \oplus M^* \rightarrow M^*
\]

is given by $\delta'^1_1(g, x, y) = -x - y$, and the arity 2 structure map on degree 1 elements

\[
\delta'^2_1: g^* \oplus M^* \oplus M^* \rightarrow \bigoplus_{j+k=2} \Lambda^j g^* \otimes \Lambda^k (M^* \oplus M^*)
\]

is given by $\delta'^2_1(g, x, y) = \delta_0^2(g) \otimes 1 + F_1^2(y) \otimes 1$, where $\delta_0$ is the differential on $C(g)$. On degree 2 elements of $M'$, $\delta'^2_1 = 0 = \delta'^2_1$. Moreover, morphism $J: C(g) \rightarrow S(M')$ is the strict map induced by including $g^*$ into $M'$. Hence, by Prop. 4.14,

\[
(M'')^n = \begin{cases}
M^* \oplus M^* & n = 1 \\
M^* & n = 2 \\
0 & \text{otherwise}
\end{cases}
\]

and the only nontrivial structure map of the differential $\delta''$ on $S(M'')$ is given by $\delta''_1(x, y) = -x - y$. 
for all \((x, y) \in (M^\prime)^1\). Thus as a graded vector space \(M^\prime = g^*[1] \oplus M^\prime\), and \(P: S(M^\prime) \rightarrow S(M^\prime)\) is the strict CEAAlg morphism given by projection.

We will now construct a cofiber sequence with base \(C(g)\) and cofiber \(K(M, 1)\) of the form

\[
C(g) \longrightarrow (S((g^* \oplus M^*)[1]), \delta_F) \longrightarrow K(M, 1)
\]

We define the arity 2 structure map, \(\delta_F^2: g^* \oplus M^* \rightarrow \bigoplus_{j+k=2} \Lambda^j g^* \otimes \Lambda^k M^*\), of the differential \(\delta_F\) on \(S((g^* \oplus M)[1])\) by

\[
\delta_F^2(g, x) = \delta_{g^1}(g) \otimes 1 + F^2_1(x) \otimes 1
\]

The other structure maps of \(\delta_F\) must be trivial for degree reasons. The fact that \(F\) is a CEAAlg morphism implies that \(\delta_F^3 F^2 = 0\). Hence \(\delta_F\) is a differential. Thus there exists a commutative diagram of the form

\[
\begin{array}{ccc}
C(g) & \xrightarrow{J} & (S(M^\prime), \delta^\prime) & \xrightarrow{P} & S(M^\prime) \\
\downarrow\text{id} & & \downarrow\text{R}^\prime & & \downarrow\text{R}^\prime \\
C(g) & \longrightarrow & (S((g^* \oplus M^*)[1], \delta_F) & \longrightarrow & K(M, 1)
\end{array}
\]

where \(R^\prime\) and \(R^\prime\) are the strict CEAAlg morphisms given by the assignments \((g, x, y) \mapsto (g, y)\) and \((x, y) \mapsto y\), respectively. Since \(R^\prime\) and \(R^\prime\) are weak equivalences,

\[
K(M, 2) \xrightarrow{F} C(g) \longrightarrow (S((g^* \oplus M^*)[1]), \delta_F) \longrightarrow K(M, 1)
\]

is a principal cofiber sequence.

Suppose that \(F, G: K(M, 2) \rightarrow C(g)\) are homotopic via \(H: \widetilde{Cyl}(M, 2) \rightarrow C(g)\). As shown in the proof of Prop. 5.10, \(H^1: M^* \oplus M^* \rightarrow g^*\) is a \(k\)-linear map such that

\[
\delta_{g^1}^2 H^1(x, -x) = F^2_1(x) - G^2_1(x)
\]

We define a strict algebra morphism \(\Phi: (S((g^* \oplus M^*)[1]), \delta_F) \rightarrow (S((g^* \oplus M^*)[1]), \delta_G)\) by
\[ \Phi_1^i(g, x) = (g + H_1^i(x, -x), x). \] Then TFDC

\[
\begin{array}{cccccc}
\bigoplus_{j+k=2} \Lambda^j g^* \otimes \Lambda^k M^* & \xrightarrow{\delta_{F, 1}^2} & \bigoplus_{j+k=2} \Lambda^j g^* \otimes \Lambda^k M^* \\
\Phi_1^i & & \Phi_1^i \\
\delta_{G, 1}^2 & & \delta_{G, 1}^2 \\
g^* \oplus M^* & \xrightarrow{\Phi_1^i} & g^* \oplus M^*
\end{array}
\]

Hence \( \Phi \) is a C\( \text{EAlg} \) morphism. Moreover, the following diagram commutes

\[
\begin{array}{cccccc}
K(M, 2) & \xrightarrow{id} & C(g) & \xrightarrow{\Phi} & (S((g^* \oplus M^*)[1], \delta_F) & \xrightarrow{id} & K(M, 1) \\
& & \downarrow{id} & & \downarrow{\Phi} & & \downarrow{id} \\
K(M, 2) & \xrightarrow{id} & C(g) & \xrightarrow{\Phi} & (S((g^* \oplus M^*)[1], \delta_G) & \xrightarrow{id} & K(M, 1)
\end{array}
\]

Since \( \Phi_1^i \) is an isomorphism of cochain complexes, \( \Phi \) is an isomorphism in C\( \text{EAlg} \) and thus a weak equivalence. Therefore the principal cofiber sequences corresponding to \( F \) and \( G \) are equivalent, and the assignment

\[
\left( F: K(M, 2) \to C(g) \right) \mapsto \left( K(M, 2) \xrightarrow{F} C(g) \to (S((g^* \oplus M^*)[1]), \delta_F) \to K(M, 1) \right)
\]

is well defined.

On the other hand, let \( K(M, 2) \xrightarrow{F} C(g) \to S(V) \to K(M, 1) \) be a principal cofiber sequence with classifying space \( K(M, 2) \), base \( C(g) \) and cofiber \( K(M, 1) \). We obtain a morphism in \([K(M, 2), C(g)]\) from a principal cofiber sequence by the assignment

\[
\left( K(M, 2) \xrightarrow{F} C(g) \to S(V) \to K(M, 1) \right) \mapsto \left( F: K(M, 2) \to C(g) \right)
\]

By the definition of equivalence between principal cofiber sequences, this assignment is well defined. It is easily verified that this assignment is an inverse to the assignment above. This completes our proof.

This result easily generalizes to the following theorem.

**Theorem 5.17.** Let \( g \) be a finite dimensional Lie algebra, and let \( M \) be a finite dimensional vector space. The set of equivalent principal cofiber sequences in C\( \text{EAlg} \) with classifying space \( K(M, n + \]
1), base $C(\mathfrak{g})$, and cofiber $K(M, n)$ is in bijective correspondence with $[K(M, n + 1), C(\mathfrak{g})]$ for all $n \geq 1$.

Proof. The proof when $n = 1$ is given above and the proof for $n \geq 2$ follows similarly. Given a morphism $F : K(M, n + 1) \to C(\mathfrak{g})$ we obtain a principal cofiber sequence via a sequence of pushouts. This principal cofiber sequence will be equivalent to

$$K(M, n + 1) \xrightarrow{F} C(\mathfrak{g}) \xrightarrow{S(\mathfrak{g}^*[1] \oplus M^*[n]), \delta_F} K(M, n)$$

where the structure maps of $\delta_F$ are trivial except in arity 2 and $n + 1$. In arity 2, $\delta_{F1}^2$ is $\delta_{\mathfrak{g}1}^2$ on $\mathfrak{g}^*[1]$ and zero on $M^*[n]$, and in arity $n + 1$, $\delta_{F1}^{n+1}$ is $F_1^{n+1}$ on $M^*[n]$ and zero on $\mathfrak{g}^*[1]$. The assignment of a principal cofiber sequence to a morphism in $[K(M, n + 1), C(\mathfrak{g})]$ is identical to the $n = 1$ case above.

Central extensions of Lie algebras

Definition 5.18. Let $\mathfrak{g}$ be a Lie algebra and let $M$ be a vector space. A central extension of $\mathfrak{g}$ by $M$ is a short exact sequence of Lie algebras

$$0 \to M \to \mathfrak{h} \to \mathfrak{g} \to 0$$

where $M$ is given the structure of an abelian Lie algebra, and $[m, x] = 0$ for all $m \in M$ and $x \in \mathfrak{h}$.

We say that two central extensions of $\mathfrak{g}$ by $M$ are equivalent iff there exists a Lie algebra isomorphism such that TFDC

Proposition 5.19. Let $\mathfrak{g}$ be a Lie algebra and $M$ be a vector space considered as a trivial $\mathfrak{g}$-module. Then there exists a bijection

$$H^2(\mathfrak{g}, M) \cong \{\text{central extensions of } \mathfrak{g} \text{ by } M\} / \sim$$
where the relation \( \sim \) is given by equivalence of central extensions.

**Proof.** We will provide the assignments that induce this bijection and direct the reader to [19, Thm. 7.6.3] for the details. Let \( \hom_{\kappa}(\Lambda^* g, M) \) the complex defined in Def. 3.27 whose cohomology is \( H^*(g, M) \). Suppose that

\[
0 \to M \to h \to g \to 0
\]

is a central extension. Since \( h \to g \) is surjective, we can choose a \( \kappa \)-linear section \( \sigma : g \to h \). Then the alternating bilinear function \( f : g \times g \to M \) given by

\[
f(g, g') = [\sigma(g), \sigma(g')] - \sigma([g, g'])
\]

induces a 2-cocycle \( f : \Lambda^2 g \to M \). On the other hand, if we are given a 2-cocycle, \( f : \Lambda^2 g \to M \), we can define a bracket on \( g \oplus M \) by

\[
[(g, m), (g', m')] = ([g, g'], f(g, g'))
\]

which yields a central extension

\[
0 \to M \to g \oplus M \to g \to 0
\]

\( \square \)

**Proposition 5.20.** Let \( g \) be a finite dimensional Lie algebra and let \( M \) be a finite dimensional vector space. Then the set of equivalent central extensions of \( g \) by \( M \) is in bijective correspondence with the set of equivalent principal cofiber sequences in \( \text{CEAlg} \) with classifying space \( K(M, 2) \), base \( C(g) \) and cofiber \( K(M, 1) \).

**Proof.** Let

\[
0 \to M \to h \to g \to 0
\]

be a central extension. If we apply the Chevalley-Eilenberg functor to this extension we obtain a cofiber sequence \( C(g) \to C(h) \to K(M, 1) \). Since \( h \cong g \oplus M \) as a vector space we can obtain a
degree a linear map by the composition
\[
M^* \hookrightarrow \mathfrak{g}^* \oplus M^* \overset{\delta^2_{\mathfrak{h}}}{\rightarrow} \bigoplus_{j+k=2} \Lambda^j \mathfrak{g}^* \otimes \Lambda^k M^* \overset{\text{pr}}{\rightarrow} \Lambda^2 \mathfrak{g}^*
\]
which we denote as $\tilde{F}$. Since $M$ is in the center of $\mathfrak{h}$, the arity 2 structure map of the differential $\delta_{\mathfrak{h}}$ is
\[
\delta^2_{\mathfrak{h}1}(g, x) = \delta^2_{\mathfrak{g}1}(g) \otimes 1 + \tilde{F}(x) \otimes 1
\]
Thus, since $\delta_{\mathfrak{h}2} \delta^2_{\mathfrak{h}1} = 0$, we can define a CE algebra morphism $F : K(M, 2) \rightarrow C(\mathfrak{g})$ to be the unique lift of
\[
F^n = \begin{cases} 
\tilde{F} & n = 2 \\
0 & n \neq 2
\end{cases}
\]
By the proof of Prop. 5.16 we know that
\[
K(M, 2) \overset{F}{\rightarrow} C(\mathfrak{g}) \rightarrow C(\mathfrak{h}) \rightarrow K(M, 1)
\]
is a principal cofiber sequence.

On the other hand, if we are given a principal cofiber sequence with classifying space $K(M, 2)$, base $C(\mathfrak{g})$, and cofiber $K(M, 1)$
\[
K(M, 2) \overset{F}{\rightarrow} C(\mathfrak{g}) \rightarrow S(V) \rightarrow K(M, 1)
\]
it can be easily verified that it is equivalent to
\[
K(M, 2) \overset{F}{\rightarrow} C(\mathfrak{g}) \rightarrow (S(\mathfrak{g}^* \oplus M^*)[1]), \delta_F \rightarrow K(M, 1)
\]
where $\delta_F$ is the differential defined in the proof of Prop. 5.16. Then applying Remark 3.26 to the cofiber sequence $C(\mathfrak{g}) \rightarrow (S(\mathfrak{g}^* \oplus M^*)[1]), \delta_F \rightarrow K(M, 1)$ we obtain a central extension
\[
0 \rightarrow M \rightarrow \mathfrak{g} \oplus M \rightarrow \mathfrak{g} \rightarrow 0
\]
Where the bracket on $\mathfrak{g} \oplus M$ is given by
\[
[(g', m'), (g', m')] = (\langle g, g' \rangle, F^2_{h1}(g, g'))
\]
Moreover it is obvious the the notion of equivalence is the same between central extension and principal cofiber sequences.

Thus the notion of a central extension is recovered by the notion of a principal cofiber sequence in $\text{Ho}(\text{CEAlg})$, and the correspondence between central extensions and the second cohomology group is related to the correspondence of Prop. 5.16 in the following theorem.

**Theorem 5.21.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $M$ be a finite dimensional vector space considered as a trivial $\mathfrak{g}$-module. Then the following collection of bijections commutes

\[
\begin{array}{ccc}
\{\text{Central extensions of } \mathfrak{g} \text{ by } M\} / \sim \\ \cong \quad \text{Prop. 5.19} \\
& H^2(\mathfrak{g}, M) \\
\cong \quad \text{Thm. 5.14} \\
\{ \text{Principal cofiber sequences with classifying space } \\
K(M, 2), \text{ base } C(\mathfrak{g}), \text{ and cofiber } K(M, 1) \} / \sim \\ \cong \quad \text{Prop. 5.16} \\
& [K(M, 2), C(\mathfrak{g})]
\end{array}
\]

Where the relations $\sim$ are given by equivalence of central extensions and equivalence of principal cofiber sequences.

**Proof.** The proof of this statement follows immediately from the construction of the bijections between these sets.
6 Future Work

The immediate next step will be to generalize Thm. 5.21 to cohomology with nontrivial coefficients and more general extensions of Lie algebras. After this connection is solidified, this homotopical framework can be utilized to study cohomology with coefficients in non-abelian Lie algebras.
A Appendix

The following results are well known to experts, but are difficult to find in literature at the appropriate level of generality needed in this thesis. The proofs given here are due to Rogers [15].

A.1 Acyclic cofibrations in CEA ∗ admit left inverses

Proposition A.1. Let \( F: (S(V), \delta) \to (S(V'), \delta') \) be an acyclic cofibration in CEA ∗. Then there exists a cdga morphism \( G: (S(V'), \delta') \to (S(V), \delta) \) such that \( GF = \text{id}_{S(V)} \).

Proof. By definition, the chain map \( F^1_1: (V, \delta^1_1) \to (V', \delta'^1_1) \) is an acyclic cofibration of complexes (i.e., a weak equivalence that is injective in all degrees). Therefore there exists a chain map \( \sigma: (V', \delta'^1_1) \to (V, \delta^1_1) \) such that \( \sigma F^1_1 = \text{id}_V \), and a chain homotopy \( h: V' \to V'[1] \) such that

\[
\text{id}_{V'} - F^1_1 \sigma = \delta'^1_1 h + h\delta^1_1.
\]

Thus we have an isomorphism of complexes

\[
F^1_1 \oplus j: V \oplus \ker \sigma \xrightarrow{\cong} V',
\]

where \( j: \ker \sigma \to V' \) is the inclusion. We will extend \( j \) to a cdga map

\[
J: (S(\ker \sigma), \tilde{\delta}) \to (S(V'), \delta')
\]

where \( \tilde{\delta}_1 := \delta^1_1 |_{\ker \sigma} \). Indeed, let \( J^1_1 := j \) and for \( k \geq 2 \) define

\[
J^k_1: \ker \sigma \to S^k(V'), \quad J^k_1 := \delta'^k \circ h \circ j.
\]

By using the fact that \( h \) is a chain homotopy, and that \( \delta' \circ \delta' = 0 \), a direct calculation shows that \( \delta' J = J \tilde{\delta} \). Since the chain map (14) is an isomorphism, it follows from Prop. 4.8 that

\[
F \otimes J: S(V) \otimes S(\ker \sigma) \to S(V')
\]

is a isomorphism of cdgas. Note that \((F \otimes J) \circ i_{S(V)} = F\), where \( i_{S(V)}: S(V) \to S(V) \otimes S(\ker \sigma) \) is the inclusion.
Next, observe that the projection $\Pr: S(V) \otimes S(\ker \sigma) \to S(V)$ defined as

$$\Pr(x \otimes y) := \begin{cases} x, & \text{if } y = 1_k \\ 0, & \text{otherwise} \end{cases}$$

is a morphism of cdgas. Finally, let $G: S(V') \to S(V)$ be the composition $G := \Pr \circ (F \otimes J)^{-1}$. Then we have

$$GF = (\Pr \circ (F \otimes J)^{-1})(F \otimes J) \circ \iota_{S(V)} = \Pr \iota_{S(V)} = \id_{S(V)},$$

and this completes the proof.

\[\square\]

**A.2 Every weak equivalence in $\text{CEAlg}$ is a quasi-isomorphism of cdgas**

We first prove two simple lemmas that will allow us to avoid using calculations involving spectral sequences.

Let $(A, d)$ be a $\mathbb{Z}$-graded cochain complex equipped with a decreasing filtration of subcomplexes

$$\cdots \supseteq \mathcal{F}_n A \supseteq \mathcal{F}_{n+1} A \supseteq \cdots$$

satisfying the following two conditions:

1. The decreasing filtration is **locally bounded on the left** i.e. for all $p \in \mathbb{Z}$ there exists an integer $\ell_p$ such that


2. The filtration is **locally bounded on the right** i.e. for all $p \in \mathbb{Z}$ there exists an integer $r_p$ such that

   $$\mathcal{F}_{r_p} A^p = \{0\}.$$

Let $\Gr(A)$ denote the **associated graded** cochain complex, i.e. the direct sum of quotient complexes:

$$\Gr(A) := \bigoplus_n \mathcal{F}_n A / \mathcal{F}_{n+1} A.$$
Recall that if $A$ and $B$ are filtered cochain complexes, then a chain map $f: A \to B$ preserves the filtration iff for all $n \in \mathbb{Z}$

$$f(\mathcal{F}_n A) \subseteq \mathcal{F}_n B.$$ 

A filtration preserving chain map induces a chain map on the associated graded complexes

$$\text{Gr}(f): \text{Gr}(A) \to \text{Gr}(B).$$

in the obvious way.

**Lemma A.2.** Let $(A, d)$ be a $\mathbb{Z}$-graded cochain complex of $R$-modules equipped with a decreasing filtration that is locally bounded on the left and right. If $\text{Gr}(A)$ is acyclic, then $A$ is acyclic as well.

**Proof.** Let $a \in A^p$ be a degree $p$ cocycle in $A$. Then there exists an $\ell$ such that $a \in \mathcal{F}_\ell A$ and hence $a$ is a $p$-cocycle in the complex $\mathcal{F}_\ell A/\mathcal{F}_{\ell+1} A$ as well. Since $\text{Gr}(A)$ is acyclic, $\mathcal{F}_\ell A/\mathcal{F}_{\ell+1} A$ is acyclic, therefore there exists $b_1 \in \mathcal{F}_\ell A^{p-1}$ such that

$$a - db_1 \in \mathcal{F}_{\ell+1} A^p.$$ 

By construction $a-db_1$ is a degree $p$ cocycle in $\mathcal{F}_{\ell+1} A$ and hence a cocycle in the quotient $\mathcal{F}_{\ell+1} A/\mathcal{F}_{\ell+2} A$. As before, since $\text{Gr}(C)$ is acyclic, there exists $b_2 \in \mathcal{F}_{\ell+1} A^{p-1}$ such that

$$a - db_1 - db_2 \in \mathcal{F}_{\ell+2} A^p.$$ 

Continuing this way, we obtain a sequence $b_1, b_2, \ldots$ of elements in $A$ such that for any $n \geq 0$ we have $a - (db_1 + db_2 + \cdots + db_n) \in \mathcal{F}_{\ell+n} A^p$. Since the filtration on $A$ is locally bounded on the right, the sequence is finite i.e. there exists an $r$ such that

$$a - (db_1 + db_2 + \cdots + db_r) \in \mathcal{F}_r A^p = \{0\}$$

Hence, $a = \sum_{i=1}^r db_i \in A$, and so $a$ is a coboundary. 

**Lemma A.3.** Let $(A, d_A)$ and $(B, d_B)$ be a $\mathbb{Z}$-graded cochain complex of $R$-modules equipped with a decreasing filtration that is locally bounded on the left and right. Let $f: A \to B$ be a filtration
preserving chain map. If the induced chain map on the associated graded cochain complexes

\[ \text{Gr}(f) : \text{Gr}(A) \to \text{Gr}(B) \]

is a quasi-isomorphism, then \( f \) is a quasi-isomorphism as well.

**Proof.** Let \((\text{cone}(f), \partial)\) denote the cochain complex corresponding to the mapping cone of \( f \), i.e.,

\[ \text{cone}(f)^p := A[-1]^p \oplus B^p = A^{p+1} \oplus B^p, \quad \partial(a, b) := (-d_A a, d_B b - f(a)). \]

The locally bounded filtrations on \( A \) and \( B \) induce a decreasing filtration on \( \text{cone}(f) \):

\[ F_n \text{cone}(f) := F_n A[-1] \oplus F_n B. \]

Note that each filtered piece \( F_n \text{cone}(f) \) is a subcomplex since \( f \) preserves the filtrations. Moreover, a simple check shows that the induced filtration on \( \text{cone}(f) \) is locally bounded on the left and right.

Recall that the a chain map between complexes of \( R \)-modules is a quasi-isomorphism if and only if its mapping cone is acyclic [19, Cor. 1.5.4]. Hence, by hypothesis the cochain complex \( \text{cone}(\text{Gr}(f)) \) is acyclic. On the other hand, since quotients of complexes commute with direct sums, we have a natural isomorphism

\[ \text{cone}(\text{Gr}(f)) \cong \text{Gr}(\text{cone}(f)). \]

Hence, the associated graded complex \( \text{Gr}(\text{cone}(f)) \) is acyclic, and so Lemma A.2 implies that \( \text{cone}(f) \) is acyclic. Therefore, \( f \) is a quasi-isomorphism. \( \square \)

Next, recall that there is a functor

\[ S : \text{Ch}_R^\ast \to \text{cdga}, \quad (15) \]

which sends a chain map \( f : (V, d) \to (V', d') \) to a morphism of cdgas

\[ F := S(f) : (S(V), d_S) \to (S(V'), d'_S) \]
where

\[ F(x_1 \lor x_2 \lor \cdots \lor x_n) := f(x_1) \lor f(x_2) \lor \cdots \lor f(x_n) \]

\[ d_S(x_1 \lor x_2 \lor \cdots \lor x_n) := \sum_i \pm dx_i \lor x_1 \lor \cdots \lor x_n \]

\[ d'_S(x'_1 \lor x'_2 \lor \cdots \lor x'_n) := \sum_i \pm d'x'_i \lor x'_1 \lor \cdots \lor x'_n. \]

\[ (16) \]

**Proposition A.4.** Let \((V, d)\) be a non-negatively graded cochain complex of vector spaces over a field \(\mathbb{k}\) of characteristic zero. There exists a natural isomorphism of graded commutative algebras

\[ H(S(V), d_S) \cong S(H(V, d)). \]

**Proof.** Since we are working over a field, the Künneth theorem implies that \(H(T(V), d_T) \cong T(H(V, d))\), where \(T(V)\) is the tensor algebra generated by \(V\). Since we are working over a field of characteristic zero, taking co-invariants commutes with taking cohomology. Hence, \(H(S(V), d_S) \cong S(H(V, d))\). See the proof of Prop. 2.1 in [12, Appendix B] for the complete details. \qed

**Corollary A.5.** Let \(\mathbb{k}\) be a field with \(\text{char} \mathbb{k} = 0\). The functor \((15) S: \text{Ch}^*_{\mathbb{k}} \to \text{cdga}\) preserves quasi-isomorphisms.

**Proof.** Follows from the fact that the isomorphism in Prop. A.4 is natural. \qed

**Theorem A.6.** If \(F: (S(V), \delta) \to (S(V'), \delta')\) is a weak equivalence in the category CEAlg, then it is also a quasi-isomorphism of cdgas. That is, \(F\) induces an isomorphism of graded vector spaces:

\[ H(S(V), \delta) \xrightarrow{\cong} H(S(V'), \delta') \]

**Proof.** For any CE algebra \((S(V), \delta)\), there is a canonical descending filtration by linear subspaces

\[ S(V) = \mathcal{F}_0S(V) \supseteq \mathcal{F}_1S(V) \supseteq \mathcal{F}_2S(V) \supseteq \cdots \]

defined by “word-length”, i.e.,

\[ \mathcal{F}_nS(V) := \bigoplus_{n \leq k} S^k(V). \]

\[ (17) \]
If \( y = x_1 \lor \cdots \lor x_m \in \mathcal{F}_n S(V) \), then by definition
\[
\delta y = \sum_i \pm \delta_1^a (x_i) \lor x_1 \lor \cdots \lor \hat{x}_i \lor \cdots \lor x_m.
\]

By definition of a CE algebra, \( V \) is concentrated in degrees \( \geq 1 \). Hence, for any \( x \in V \), \( \delta_1^a (x) \in S(V) \) is a finite sum \( \delta x = \sum_{k \geq 1} \delta_k (x) \), given by linear maps \( \delta_k^a : V \to S^k(V) \). It then follows that the word length of \( \delta y \) is greater than or equal to the word length of \( y \). Hence, \( \mathcal{F}_n S(V) \) is a sub-cochain complex of \( S(V) \).

By construction, the word-length filtration (17) is clearly locally bounded on the left. Now suppose \( y \in S(V)^p \) is an element of degree \( p > 0 \), homogeneous with respect to word length. Since all generators (i.e., elements of \( V \)) have degree \( \geq 1 \), it follows that the word-length of \( y \) is less than or equal to \( p \). Hence,
\[
\mathcal{F}_{p+1} S(V)^p = 0,
\]
and therefore the filtration is locally bounded on the right.

Next, we analyze the associate graded complex \( \text{Gr}(S(V)) \). Observe that for each \( n \geq 0 \), we have an isomorphism of graded vector spaces
\[
\mathcal{F}_n S(V) / \mathcal{F}_{n+1} S(V) \cong S^n(V),
\]
and therefore \( \text{Gr}(S(V)) \cong S(V) \) as graded vector spaces. Now consider the induced differential \( \delta_{\text{Gr}} \) on \( \mathcal{F}_n S(V) / \mathcal{F}_{n+1} S(V) \). If \( y = x_1 \lor \cdots \lor x_m \in \mathcal{F}_n S(V) \), then we have
\[
\delta y = \sum_{i \geq 1} \sum_{k \geq 1} \pm \delta_1^a (x_i) \lor x_1 \lor \cdots \lor \hat{x}_i \lor \cdots \lor x_m.
\]
And since \( \text{im} \delta_1^a \subseteq S^k(V) \), we deduce that
\[
\delta_{\text{Gr}} (y + \mathcal{F}_{n+1} S(V)) = \sum_{i \geq 1} + \delta_1^a (x_i) \lor x_1 \lor \cdots \lor \hat{x}_i \lor \cdots \lor x_m + \mathcal{F}_{n+1} S(V).
\]
Hence, the differential \( \delta_{\text{Gr}} \) only depends on the differential \( d := \delta_1^a \) on the cochain complex \( (V, d) \).

Therefore, we have an isomorphism of cochain complexes
\[
\left( \text{Gr}(S(V)), \delta_{\text{Gr}} \right) \cong \left( S(V), d_S \right),
\]
where \( d_S \) is the differential (16).

Finally, suppose \( F: (S(V), \delta) \to (S(V'), \delta') \) is a weak equivalence. For any generator \( x \in V \) we have \( F(x) = F^1(x) + \sum_{k \geq 2} F^k(x) \), where the finite sum on the right hand side is given by linear maps \( F^k: V \to S^k(V') \). It follows that \( F \) respects the word length filtrations on \( S(V) \) and \( S(V') \). Moreover, this observation also implies that the following diagram of cochain complexes commutes

\[
\begin{array}{ccc}
\left( \text{Gr}(S(V)), \delta_{Gr} \right) & \xrightarrow{\text{Gr}(F)} & \left( \text{Gr}(S(V')), \delta'_{Gr} \right) \\
\cong & & \cong \\
(S(V), d_S) & \xrightarrow{S(F^1)} & (S(V'), d'_S)
\end{array}
\]  

(18)

where \( S(F^1) \) is the morphism obtained by applying the functor (15) to the chain map

\[ F^1: (V, d) \to (V', d'). \]

Since \( F \) is a weak equivalence, \( F^1 \) is a quasi-isomorphism, by definition. Hence, \( S(F^1) \) is quasi-isomorphism by Cor. A.5, and hence Gr \((F)\) is as well, by the commutativity of diagram (18). Therefore, Lemma A.3 implies that \( F \) is a quasi-isomorphism, and this completes the proof.

A.3 Obstruction theory for semi-free cdga morphisms

Let \( (S(V), \delta) \) and \( (S(V'), \delta') \) be CE algebras. For each \( k \geq 1 \) let \( \text{Hom}_k(V, S^k(V')) \) denote the cochain complex whose underlining graded vector space consists of linear maps \( f^k: V \to S^k(V') \) of arbitrary degree, with differential

\[ \partial(f^k) := \delta^k f - (-1)^{|f^k|} f^k \circ \delta^1. \]

Let \( (A, \partial_A) \) denote the direct product of these cochain complexes i.e.

\[ A := \prod_{k \leq 1} \text{Hom}_k(V, S^k(V')) \cong \text{Hom}_k(V, \oplus_{k \geq 1} S^k(V')). \]

Given a map \( f: V \to \oplus_{i \geq 1} S^i(V') \) in \( A \), we denote by \( f^k \) the composition with the projection

\[ f^k := \pi_k f, \quad \pi_k: \oplus_{i \geq 1} S^i(V') \to S^k(V'), \]
and when convenient we identify $f \in A$ with its corresponding infinite sequence $(f^k)_{k \geq 1}$.

We observe that $A$ is equipped with a canonical decreasing filtration of sub cochain complexes

$$A = F_1 A \supseteq F_2 A \supseteq \cdots$$

where

$$F_n A := \{ f \in A \mid f^k = 0 \quad \forall k \leq n - 1 \}.$$  

A simple calculation shows that for each $n \geq 1$ we have isomorphisms of complexes

$$A/F_n A \cong \bigoplus_{i=1}^{n-1} \text{Hom}_k(V, S^i(V')),$$

and that $(A, \partial_A)$ is complete with respect to this filtration i.e.,

$$(A, \partial_A) \cong \text{proj lim}_{n \geq 0} A/F_n A$$

We now introduce an auxiliary collection of degree 1 symmetric multi-linear maps

$$L_{(m)}: A^\otimes m \to A, \quad \forall m \geq 1.$$  

For $m = 1$, for all $f \in A$ and for each $k \geq 1$ we define

$$L_{(1)}(f)^k: V \to S^k(V')$$

$$L_{(1)}(f)^k := \sum_{i=1}^{k} \delta_i^k f^i - (-1)^{\deg f} f^k \delta_1^1.$$  

For $m \geq 2$, for all $f_{(1)}, f_{(2)}, \ldots, f_{(m)} \in A$ and for each $k \geq 1$ we define

$$L_{(m)}(f_{(1)}, f_{(2)}, \ldots, f_{(m)})^k: V \to S^k(V')$$

$$L_{(m)}(f_{(1)}, f_{(2)}, \ldots, f_{(m)})^k := - \sum_{i_1+i_2+\ldots+i_m=k} (f_{(1)}^{i_1} \lor f_{(2)}^{i_2} \lor \ldots \lor f_{(m)}^{i_m}) \delta_i^m.$$  

We note that the above maps are compatible with the filtration on $A$ in the following sense: For all $m \geq 1$, and $i_1, i_2, \ldots, i_m \geq 1$ we have an inclusion

$$L_{(m)}(F_{i_1} A \otimes F_{i_2} A \otimes \cdots \otimes F_{i_m} A) \subseteq F_{i_1+i_2+\ldots+i_m} A.$$  

(19)

The multi-linear maps $L_{(m)}$ allow us to define the following polynomial function on the degree
zero elements of $A$:

$$\kappa: A^0 \to A^1$$

$$\kappa(f) := \sum_{m \geq 1} L_{(m)}(f, f, \ldots, f).$$

(20)

The infinite series in the definition of $\kappa$ is well defined, since: (19) implies that $\kappa$ is compatible with the filtration, $A = F_1 A$, and since $A$ is complete with respect to the filtration.

**Lemma A.7.** Let $Z(\kappa) \subseteq A^0$ denote the zero locus of the polynomial $\kappa$. There is a bijection of sets

$$\text{hom}_{\text{CEAlg}}(S(V), S(V')) \cong Z(\kappa)$$

**Proof.** Let $f \in A^0$. A simple unraveling the definition (20) of $\kappa$ shows that for each $n \geq 1$, the degree 1 linear map $\kappa(f)^n: V \to S^n(V')$ is

$$\kappa(f)^n = \sum_{k=1}^n \delta_k^n f^k - \sum_{k=1}^n \sum_{i_1 + i_2 + \cdots + i_k = n} (f^{i_1} \vee f^{i_2} \vee \cdots \vee f^{i_k}) \delta_1^k.$$

Hence, for every linear map $f \in A^0$ satisfying $\kappa(f) = 0$ there is a unique algebra map $F: S(V) \to S(V')$ satisfying $\delta' F = F \delta$ with $F_1 := F|_V = f$. \hfill $\square$

It follows from the compatibility (19) of the multi-linear maps $L_{(m)}$ with the filtration that, for each $n \geq 0$, the function $\kappa$ induces a well defined map to the quotients

$$\tilde{\kappa}_n: A^0 / F_n A^0 \to A^1 / F_n A^1$$

$$\tilde{\kappa}_n(f + F_n A^0) := \kappa(f) + F_n A^1$$

**Lemma A.8.** For all $f \in A^0$ the following identity holds

$$L_{(1)} \kappa(f) - \sum_{m \geq 2} L_{(m+1)}(f, f, \ldots, f, \kappa(f)) = 0.$$

**Proof.** One inductively “climbs up the tower” $\cdots \to A^0 / F_{n+1} A^0 \to A^0 / F_n A^0 \to \cdots$ and verifies that for each $n \geq 0$, the analogous equality involving $\tilde{\kappa}_n$ is satisfied for all elements of $A^0 / F_n A^0$. The proof then follows since $A^0$ is the projective limit of the quotients $A^0 / F_n A^0$. \hfill $\square$

**Proposition A.9.** Let $n > 1$ and suppose a collection of degree 0 linear maps $\{F^k_1: V \to S^k(V)\}_{k=1}^{n-1}$
satisfy the equations
\[ \sum_{k=1}^{m} \delta_k^{m} F_1^k = \sum_{k=1}^{m} F_k^{m} \delta_1^k \] (21)
for all \( 1 \leq m \leq n - 1 \). Then the degree 1 linear map \( q_n : V \rightarrow S^n(V') \) defined as
\[ q_n := \sum_{k=1}^{n} \delta_k^{n} F_1^k - \sum_{k=2}^{n} F_k^{n} \delta_1^k \]
is a 1-cocycle in the cochain complex \((A, \partial_A)\).

Proof. The linear maps \( \{ F_1^k : V \rightarrow S^k(V) \}_{k=1}^{n-1} \) represent a class \( \tilde{f} = \sum_{k=1}^{n-1} F_1^k + \mathcal{F}_n A^0 \) in the quotient
\[ A^0 / \mathcal{F}_n A^0 \cong \bigoplus_{k=1}^{n-1} \text{Hom}_k(V, S^k(V'))^0. \]
The equations (21) are satisfied if and only if
\[ \tilde{\kappa}_n(\tilde{f}) = 0 \quad \Leftrightarrow \quad \kappa(f) \in \mathcal{F}_n A^1. \]
On the other hand, the identity Lemma A.8 implies that
\[ (L_{(1)} \kappa(f))^n - \sum_{m \geq 2} L_{(m+1)}(f, f, \ldots, m, f, \kappa(f))^n = 0 \]
But since \( \kappa(f) \in \mathcal{F}_n A^1 \), the above equality reduces to
\[ (L_{(1)} \kappa(f))^n = 0. \]
A direct calculation then shows that
\[ \kappa(f)^n = q_n, \]
and it follows from the definition of \( L_{(1)} \) that
\[ (L_{(1)} \kappa(f))^n = \partial_A q_n. \]
Hence, \( \partial_A q_n = 0. \)
References


