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University of Nevada, Reno

**Khovanov Homology on Symmetric Unions of Certain 2-Bridge Knots**

A thesis submitted in partial fulfillment of  
requirements for the degrees of  
Bachelor of Science in Mathematics and the Honors Program

by

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May, 2011

**UNIVERSITY  
OF NEVADA,  
RENO**

**THE HONORS PROGRAM**

We recommend that the thesis  
prepared under our supervision by

**T.J. GAFFNEY**

entitled

**Khovanov Homology on Symmetric Unions of Certain 2-Bridge Knots**

is accepted in partial fulfillment of  
the requirements for the degrees of

**BACHELOR OF SCIENCE**

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Stanislav Jabuka, Thesis Advisor

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May, 2011

## **Abstract**

Symmetric unions, originally introduced in 1957 by Shin'ichi Kinoshita and Hidetaka Terasaka [3], have since been studied due to their failure to be distinguished by some invariants. In his 2000 paper, Mikhail Khovanov introduced a powerful new invariant, Khovanov Homology, which gives more information (an entire homology) than some more classic invariants. In this paper, we look at Khovanov Homology's ability to distinguish or classify symmetric unions of particular knots. We are able to show that Khovanov Homology can, in fact, classify symmetric unions of  $(2,m)$ -torus knots and certain other 2-bridge knots.

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## 1. INTRODUCTION

In an historical 1957 paper, *On unions of knots* [3], by Shin'ichi Kinoshita and Hidetaka Terasaka, a new type of knot operation was introduced, called the knot union. This union is obtained by aligning two knots with two pairs of adjacent arcs (in each pair, an arc from each knot): Along the first pair of arcs, perform a usual connect sum, and along the second intertwine the two arcs with some number of twists. Figure 1 illustrates this. The number of twists used in the construction is called the winding number.

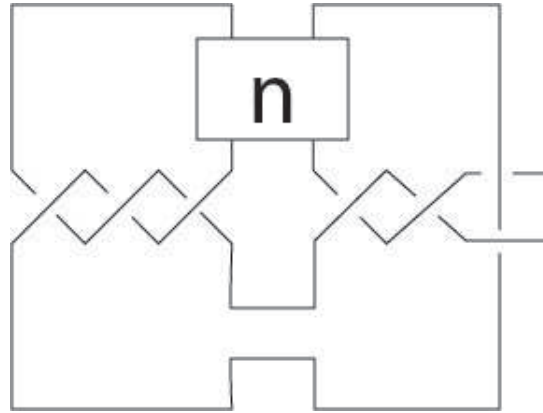


FIGURE 1. Kinoshita and Terasaka's knot union.

If this knot operation is performed on a knot and its mirror image, the resulting knot is called a *symmetric union*. In their paper, Kinoshita and Terasaka showed that the Alexander polynomial could distinguish only the parity of the winding number used in the construction of a symmetric union. Since Kinoshita and Terasaka's paper, more mathematicians have explored knot unions and symmetric unions. Some have even generalized the notion of a knot union allowing for more than one set of connecting crossings. (Notably, Christopher Lamm provides a nice generalization in his 2000 paper, *Knot unions and ribbon knots* [4].) Usually symmetric unions have been of interest due to their general indistinguishability. One notable example (for its similarity to this paper) is a presentation given by Toshifumi Tanaka at the 2009 Joint Meeting of the KMS and the AMS, titled *Symmetric unions indistinguishable by knot*

*Floer and Khovanov homology* [5], wherein (among other things) a family of symmetric unions of general 2-bridge knots is constructed that cannot be distinguished by Khovanov Homology.

In our paper, we will only look at Kinoshita and Terasaka's classic definition of the symmetric union. (We note that Tanaka, in order to construct his family of knots, needed to employ Lamm's more general definition.) We examine the ability of Khovanov Homology to distinguish symmetric unions with more detail than the classic Alexander polynomial.

Khovanov Homology is a fairly new knot invariant, first introduced in Mikhail Khovanov's 2000 paper, *A categorification of the Jones polynomial* [2]. Khovanov Homology is innately more powerful because it gives an entire graded homology for each knot, which contains more information than just a polynomial. In fact, Khovanov Homology, being a categorification of the Jones polynomial, is strictly more powerful than the Jones polynomial; that is, any two knots that can be distinguished by the Jones polynomial can be distinguished by Khovanov Homology, but the converse is not true. Though the computation of Khovanov's Homology is lengthy (details of this are provided in the next section), we are already optimistic that this invariant can, in fact, distinguish symmetric unions better than the Alexander polynomial. However, there is room for additional optimism. In his paper, Khovanov outlines a "shortcut" for computing his homology on knot diagrams with a sequence of left-handed twists. (We discuss this shortcut in detail in Section 3.) This is exactly what appears in the construction of a symmetric union!<sup>1</sup> So it seems reasonable that Khovanov Homology will change as we increase the number of twists.

We found that, in fact, the shortcut is useful because in some cases it reduces parts of the Khovanov Homology into that of the unlink. This fact makes the computation far less complex, but there is still some complication in characterizing the homology

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<sup>1</sup>Or right-handed twists, but Khovanov homology relates nicely to its mirror image. It suffices to show that the number of left-handed twists can be distinguished.

of an entire family of knots. Nonetheless, we were able to prove that Khovanov Homology can completely classify the families of symmetric unions of 2-torus knots and certain 2-bridge knots.

## 2. KHOVANOV HOMOLOGY

In this section, we will discuss Khovanov Homology and how to compute it. Our discussion draws primarily from Khovanov's landmark paper, *A categorification of the Jones polynomial* [2], in which he defines his eponymous homology invariant. Some details and terminology are withheld; this section should be thought of a literature review only. In addition to Khovanov's paper, we will also briefly discuss Dror Bar-Natan's paper, *On Khovanov's categorification of the Jones polynomial* [1], which presented Khovanov Homology very clearly when this invariant was still fairly new.

We begin the computation of Khovanov Homology by fixing a knot diagram. Khovanov, in his paper, proved that his homology is invariant under choice of diagram. Given a crossing in a knot diagram, we can define the 0-resolution and the 1-resolution of this crossing as in Figure 2.

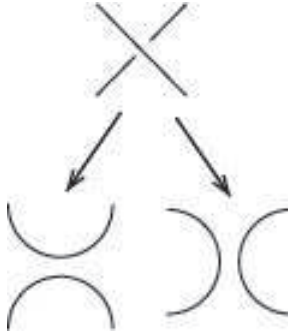


FIGURE 2. 0-resolution (left) and 1-resolution (right).

We'll call it *resolving a crossing* when we replace that crossing with one of pairs of uncrossed arc drawn in Figure 2. And we'll call the resulting diagram a *resolution* when we resolve all of the crossings at once. Since we have two choices for each crossing, it follows that an  $n$  crossing knot diagram has  $2^n$  associated resolutions. Given a knot diagram,  $D$ , we define the set of all crossings on  $D$  as  $\mathcal{I}$ . If  $\mathcal{L} \subseteq \mathcal{I}$ ,

then define  $D(\mathcal{L})$  as the resolution in which the crossings in  $\mathcal{L}$  have been given a 1-resolution, and the rest (the crossings belonging to  $\mathcal{I} \setminus \mathcal{L}$ ) have been given a 0-resolution.

**Example 2.1.** *As we describe the computation of Khovanov Homology, it will be useful to discuss an example, so we will examine the Hopf link, as seen in Figure 3. We show all of the possible resolutions in Figure 4.*

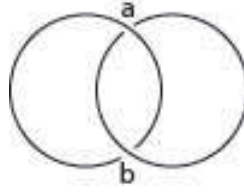


FIGURE 3. Hopf link;  $\mathcal{I} = a, b$ .

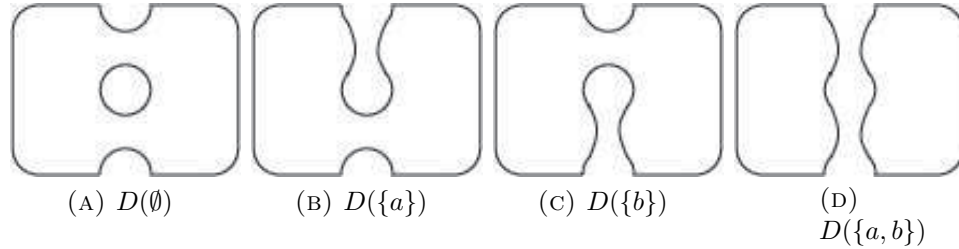


FIGURE 4. Hopf link resolutions.

As we put an algebraic structure on this, we'll need to define a graded algebra,  $\mathcal{A}$ , with a coalgebra structure, generated over  $\mathbb{Z}$  by elements  $\mathbf{1}$  and  $X$ , of degree 1 and  $-1$ , respectively. As we change our choice of uncrossed arcs at any crossing, we change from one resolution to another. This change induces a map between algebras. We will discuss these maps briefly here. On  $\mathcal{A}$ , we'll define comultiplication,  $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ , given by:

$$\begin{aligned}\Delta(\mathbf{1}) &= \mathbf{1} \otimes X + X \otimes \mathbf{1} \\ \Delta(X) &= X \otimes X\end{aligned}$$

and multiplication,  $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ , given by:

$$\begin{aligned} m(\mathbf{1} \otimes \mathbf{1}) &= \mathbf{1} \\ m(X \otimes \mathbf{1}) &= X \\ m(\mathbf{1} \otimes X) &= X \\ m(X \otimes X) &= 0 \end{aligned}$$

Now, to each resolution,  $D(\mathcal{L})$ , we assign an associated graded ring,  $V(\mathcal{L}) = \mathcal{A}^{\otimes n}$ , where  $n$  is the number of circles in  $D(\mathcal{L})$ . For each  $\alpha \in \mathcal{I} \setminus \mathcal{L}$ , we define a map  $\xi_\alpha^\mathcal{L} : V(\mathcal{L}) \rightarrow V(\mathcal{L} \cup \{\alpha\})$ . This map  $\xi_\alpha^\mathcal{L}$  is either comultiplication or multiplication, depending upon whether changing the resolution of crossing  $\alpha$  from a 0-resolution to a 1-resolution will combine or split apart (respectively) circles in the resolutions.

Following Example 2.1, we can analyze the maps:

$$\xi_{\{a\}}^\emptyset = m, \quad \xi_{\{b\}}^\emptyset = m, \quad \xi_{\{b\}}^{\{a\}} = \Delta, \quad \xi_{\{a\}}^{\{b\}} = \Delta$$

Here it is understood if when we are talking about a multiplication or comultiplication map. In general, though the maps are determined by the crossing that changes, we want a nice way to write our multiplication and comultiplication maps so that we can immediately tell which circles are being combined or split. So we'll number the circles in each resolution, and modify Khovanov's notation so that  ${}_{ij}m_k$  is the multiplication that combines circles  $i$  and  $j$  and puts the result into the  $k$ -th circle in the image resolution, keeping the order of the remaining circles. (Define  ${}_k\Delta_{ij}$  in a similar manner.)

We've almost defined all of the maps between the resolutions, except that Khovanov Homology requires that these maps anti-commute, which requires us to sprinkle negative signs about. Bar-Natan<sup>2</sup>, in his paper [1], gives an algorithm for doing this. We first choose a preferred ordering of our crossings. (Continuing Example 2.1, these

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<sup>2</sup>Khovanov, in his paper [2], also builds an algorithm. Bar-Natan's paper is clearer in some areas, and we use his algorithm here.

would be  $(a, b)$ .) Our resolutions can then be named by simply listing the crossings as either 0 or 1 in order. (In our example, this gives us that  $D(\emptyset) = 00$ ,  $D(\{a\}) = 10$ ,  $D(\{b\}) = 01$ , and  $D(\{a, b\}) = 11$ .) Since only a crossing gets changed at a time, we can now write any map between algebras by naming the crossing that remain the same, and placing a  $*$  in place of the crossing that has change. (With our example,  $\xi_{\{a\}}^\emptyset = *0$ ,  $\xi_{\{b\}}^\emptyset = 0*$ ,  $\xi_{\{b\}}^{\{a\}} = 1*$ , and  $\xi_{\{a\}}^{\{b\}} = *1$ .) Finally, the benefit of naming our maps this way is that we can now simply count the number of 1's before the  $*$ , and if this number is odd, then we make that map negative. The reader can verify that this algorithm will produce an anti-commutative, or *skew-commutative*, diagram.

We can now define the chain complex,  $\{\bar{C}^i(D), \bar{d}^i\}_i$ , with groups  $\bar{C}^i(D) = \bigoplus_{\mathcal{L} \subseteq \mathcal{I}, |\mathcal{L}|=i} V(\mathcal{L})\{-|\mathcal{L}|\}$ , and  $\bar{d}^i = \bigoplus_{\mathcal{L} \subseteq \mathcal{I}, |\mathcal{L}|=i} \left( \bigoplus_{\alpha \in \mathcal{L}} \xi_\alpha^\mathcal{L} \right)$ , or simply all possible maps between the resolutions. Where understood, we may leave the  $(D)$  off of  $C^i(D)$  and just write  $C^i$ .

Finishing Example 2.1, we have:

$$\begin{aligned} \bar{C}^0 &= \mathcal{A} \otimes \mathcal{A} & \bar{C}^1 &= \mathcal{A} \oplus \mathcal{A} & \bar{C}^2 &= \mathcal{A} \otimes \mathcal{A} \\ \bar{d}^0 &= m \oplus m & \bar{d}^1 &= \Delta \oplus 0 - 0 \oplus \Delta \end{aligned}$$

From this, we can compute the homology:

$$\bar{H}^0 = \langle X \otimes X \rangle \quad \bar{H}^1 = 0 \quad \bar{H}^2 = \langle \mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes X \rangle$$

However,  $\bar{H}^i$  is not a knot invariant. Rather, we need to perform sufficient *internal* and *external grading shifts*, per Khovanov:

$$H^i = \bar{H}^i \{2x(D) - y(D)\}[x(D)]$$

where  $x(D)$  is the number of *right-handed crossings*, and  $y(D)$  is the number of *left-handed crossings*. (These types of crossings are illustrated in Figure ??, if the reader is unfamiliar. The number,  $2x(D) - y(D)$ , inside the braces, tells us by how much to shift the grading of  $H^i$  that is inherited from the algebra  $\mathcal{A}$ . In future

instances, we shall refer to this grading as the *internal grading*. And the number,  $x(D)$ , inside the brackets, tells us by how much to lower the cohomological grading, which we shall refer to as the *external grading*.

We conclude this section with a classic example. Here, to save ink, we will omit the tensor product sign, where understood.

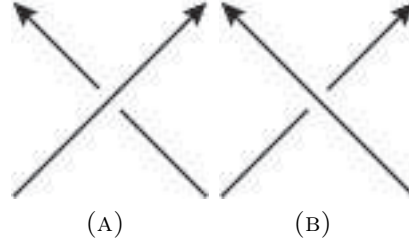


FIGURE 5. Left-handed (a) and Right-handed (b) crossings.

**Example 2.2.** We will compute the Khovanov Homology of the trefoil as drawn in Figure 6.

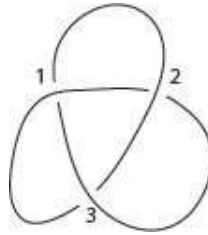


FIGURE 6. Trefoil knot.

We draw out all possible resolutions in Figure 7, which yield the maps:

$$d^0(\mathbf{1} \otimes \mathbf{1}) = (\mathbf{1}, \mathbf{1}, \mathbf{1})$$

$$d^0(\mathbf{1} \otimes X) = (X, X, X)$$

$$d^0(X \otimes \mathbf{1}) = (X, X, X)$$

$$d^0(X \otimes X) = (0, 0, 0)$$

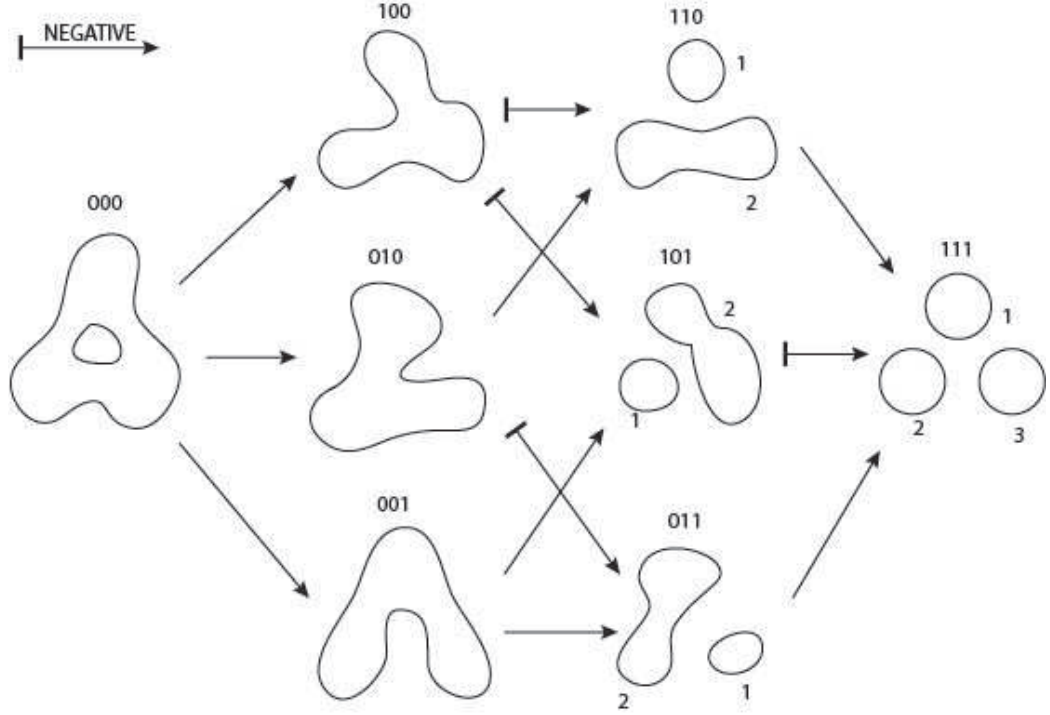


FIGURE 7. Trefoil resolutions.

$$d^1((\mathbf{1}, 0, 0)) = (-\mathbf{1} \otimes X - X \otimes \mathbf{1}, -\mathbf{1} \otimes X - X \otimes \mathbf{1}, 0)$$

$$d^1((X, 0, 0)) = (-X \otimes X, -X \otimes X, 0)$$

$$d^1((0, \mathbf{1}, 0)) = (\mathbf{1} \otimes X + X \otimes \mathbf{1}, 0, -\mathbf{1} \otimes X - X \otimes \mathbf{1})$$

$$d^1((0, X, 0)) = (X \otimes X, 0, -X \otimes X)$$

$$d^1((0, 0, \mathbf{1})) = (0, \mathbf{1} \otimes X + X \otimes \mathbf{1}, \mathbf{1} \otimes X + X \otimes \mathbf{1})$$

$$d^1((0, 0, X)) = (0, X \otimes X, X \otimes X)$$

$$d^2(\mathbf{1} \otimes \mathbf{1}, 0, 0) = \mathbf{1} \otimes X \otimes \mathbf{1} + X \otimes \mathbf{1} \otimes \mathbf{1}$$

$$d^2(\mathbf{1} \otimes X, 0, 0) = \mathbf{1} \otimes X \otimes X + X \otimes \mathbf{1} \otimes X$$

$$d^2(X \otimes \mathbf{1}, 0, 0) = X \otimes X \otimes \mathbf{1}$$



$$d^2(X \otimes X, 0, 0) = X \otimes X \otimes X$$

$$d^2(0, \mathbf{1} \otimes \mathbf{1}, 0) = -\mathbf{1} \otimes X \otimes \mathbf{1} - X \otimes \mathbf{1} \otimes \mathbf{1}$$

$$d^2(0, \mathbf{1} \otimes X, 0) = -\mathbf{1} \otimes X \otimes X - X \otimes \mathbf{1} \otimes X$$

$$d^2(0, X \otimes \mathbf{1}, 0) = -X \otimes X \otimes \mathbf{1}$$

$$d^2(0, X \otimes X, 0) = -X \otimes X \otimes X$$

$$d^2(0, 0, \mathbf{1} \otimes \mathbf{1}) = \mathbf{1} \otimes X \otimes \mathbf{1} + X \otimes \mathbf{1} \otimes \mathbf{1}$$

$$d^2(0, 0, \mathbf{1} \otimes X) = \mathbf{1} \otimes X \otimes X + X \otimes \mathbf{1} \otimes X$$

$$d^2(0, 0, X \otimes \mathbf{1}) = X \otimes X \otimes \mathbf{1}$$

$$d^2(0, 0, X \otimes X) = X \otimes X \otimes X$$

Where our choice of ordering on the  $\overline{C}^2$  resolutions always puts the bigger circle first. (This problem contains a certain simplicity that allows us to reorder the circles on the fly. In general, computations will be much more rigid.)

$$\overline{H}^0 = \langle X \otimes X \rangle$$

$$\overline{H}^1 = \frac{\langle (\mathbf{1}, \mathbf{1}, \mathbf{1}), (X, X, X) \rangle}{\langle (\mathbf{1}, \mathbf{1}, \mathbf{1}), (X, X, X) \rangle} = 0$$

$$\begin{aligned} \overline{H}^2 &= \frac{\langle (XX, XX, 0), (0, XX, XX), (\mathbf{1}X + X\mathbf{1}, \mathbf{1}X + X\mathbf{1}, 0), (0, \mathbf{1}X + X\mathbf{1}, \mathbf{1}X + X\mathbf{1}), (\mathbf{1}X, X\mathbf{1}, \mathbf{1}X) \rangle}{\langle (XX, XX, 0), (0, XX, XX), (\mathbf{1}X + X\mathbf{1}, \mathbf{1}X + X\mathbf{1}, 0), (0, \mathbf{1}X + X\mathbf{1}, \mathbf{1}X + X\mathbf{1}) \rangle} \\ &= \langle (\mathbf{1}X, X\mathbf{1}, \mathbf{1}X) \rangle \cong \mathbb{Z}_{(0)} \end{aligned}$$

$$\begin{aligned} \overline{H}^3 &= \frac{\langle a=\mathbf{1}\mathbf{1}\mathbf{1}, b=\mathbf{1}\mathbf{1}X, c=\mathbf{1}X\mathbf{1}, d=\mathbf{1}XX, e=X\mathbf{1}\mathbf{1}, f=X\mathbf{1}X, g=XX\mathbf{1}, h=XXX \rangle}{\langle b+c, -c-d, b+d, e, f, g, h \rangle} \\ &= \langle a, b | 2b \rangle \cong \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(1)} / 2\mathbb{Z}_{(1)} \end{aligned}$$

## 3. KHOVANOV'S SHORTCUT

In the paper by Khovanov that we've been discussing, he outlines a computational shortcut. This shortcut not only makes direct computations shorter, but more importantly it also allows us to make general statements about knots, without completely computing their homology, giving us a way to reduce a chain of two or more left-handed crossings, as drawn in Figure 8a. This will become clearer in the next section.

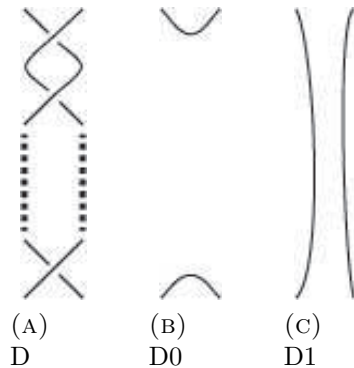


FIGURE 8. Resolutions from Khovanov's shortcut.

This shortcut produces two *partial resolutions*; that is, knot diagrams that come from resolving some, but not all, of the crossings. We'll call these particular partial resolutions  $D_0$  and  $D_1$ , when the original diagram is named  $D$ . These are illustrated in Figure 8b and Figure 8c respectively. Khovanov then gives maps between these, written in Equation 1. (For now, we'll say that the chain we're resolving has  $l$  crossings.)

$$(1) \quad \begin{aligned} H^*(D_0) \otimes X^{\otimes l}[-l-1] &\longrightarrow \dots \xrightarrow{g} H^*(D_0) \otimes X^{\otimes 3}[-4] \xrightarrow{f} H^*(D_0) \otimes X^{\otimes 2}[-3] \\ &\xrightarrow{g} H^*(D_0) \otimes X[-2] \xrightarrow{f} H^*(D_0)[-1] \xrightarrow{b} H^*(D_1) \end{aligned}$$

Here the maps  $f$ ,  $g$ , and  $b$  are also defined by Khovanov, although he does not use these names. The map  $b$  is simply the map that is induced from changing the

resolution on the crossings, either multiplication or comultiplication. We don't work with this map in this paper. The maps  $f$  and  $g$ , however, are integral to our paper. Khovanov calls these  $(u_X - v_X)$  and  $(u_X + v_X)$  respectively. Here  $u_X$  and  $v_X$  are multiplication of  $X$  with those top and bottom arcs of the  $D_0$  resolution.

The homology of the sequence in Equation 1 is the Khovanov Homology of the knot. However, each of these resolutions is itself a knot diagram (because we haven't considered the other knot crossings yet). So the chain complex in Equation 1 really has each element itself a sequence, so that it is actually a commutative diagram. We draw this diagram as in Figure 9, with the rows and columns drawn diagonally, so that any two groups on the same horizontal level have the same external grading. This allows us to read the homology more easily.

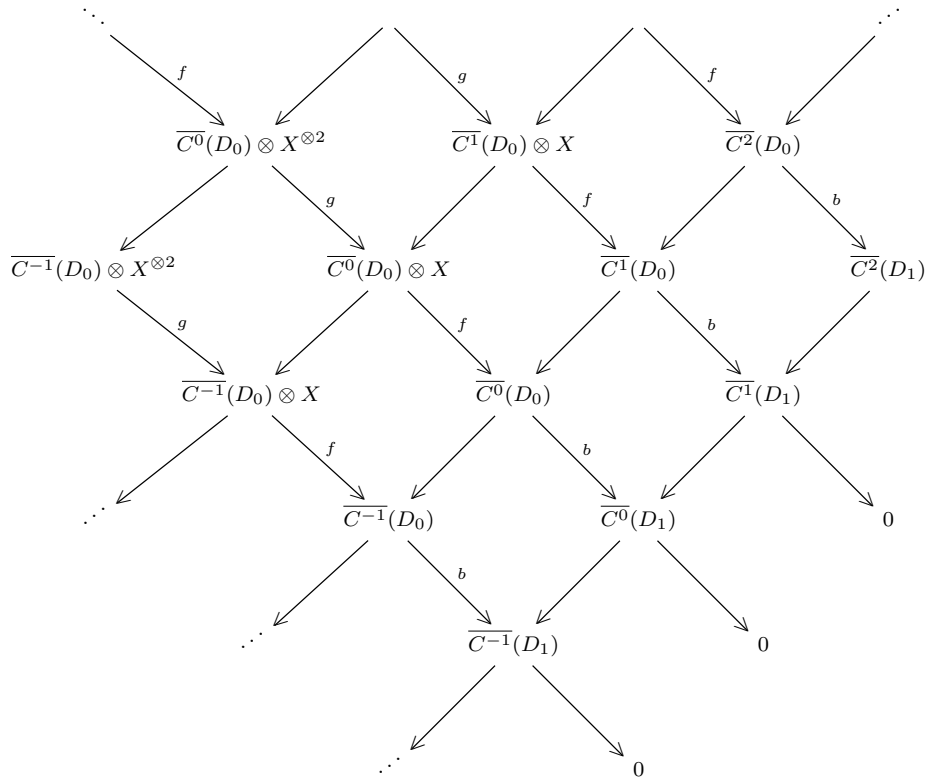


FIGURE 9. Khovanov's short cut commutative diagram.

In practice, we will replace the chain complex for each diagram (the diagonals) with the homology of that knot diagram for computational purposes.

## 4. 2-TORUS KNOTS

We investigate  $(2,m)$ -torus knots. These knots have a general form as drawn in Figure 10. The symmetric union of these looks like the knot illustrated in Figure 11.

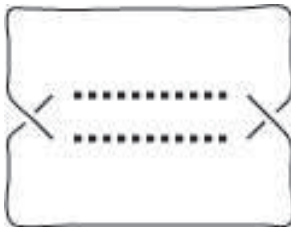


FIGURE 10.  $(2,m)$ -torus knot

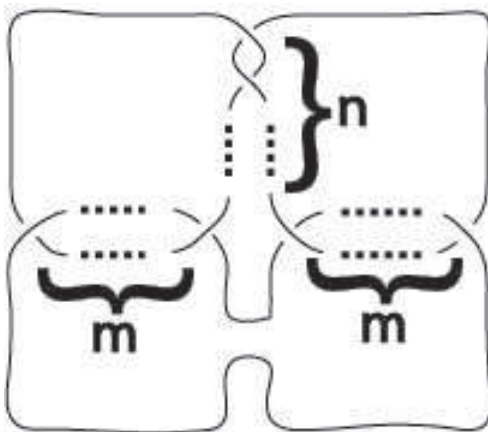


FIGURE 11.  $(2,m)$ -torus symmetric union.

When we do this, the construction of the symmetric union gives us some choice of the number of crossings at the top. We'll investigate the symmetric union with winding number  $n$ . Although  $n$  can be any natural number, we'll demand that  $n \geq 2$ . We want to show that Khovanov Homology is a powerful enough invariant to detect  $n$ . It will not matter that we're working with left-handed crossings because the case of right-handed crossings is the mirror image (since everything else is symmetric), and Khovanov gives a way to obtain the homology for the mirror image of a knot from the homology of the original; it will suffice to show that Khovanov's homology can detect the number,  $n$ , on left-handed crossings in order to make the same statement about right-handed crossings.

We will use Khovanov's shortcut on the crossings used in the construction of symmetric union. This will yield two resolutions,  $D_0$  (which comes from resolving the chain of crossings as in Figure 8b) and  $D_1$  (which comes from resolving the chain of crossings as in Figure 8c). This yields two resolutions, as drawn in Figure 12 and Figure 13. As we can see, the knot associated to the first of these diagrams comes completely undone to a 2-component unlink, via a sequence of third Reidmeister moves. Further we know the Khovanov Homology of the unlink:

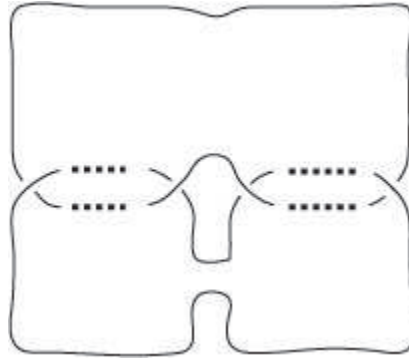


FIGURE 12.  $D_0$ = Two-component unlink.

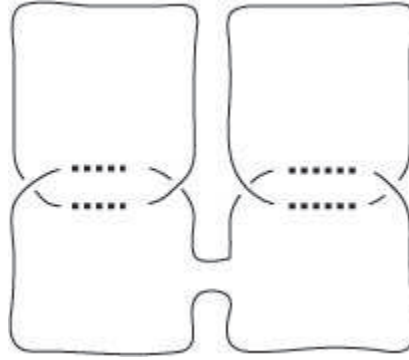


FIGURE 13.  $D_1$ = Connected sum of torus knot and mirror.

$$\overline{H}^i \cong \begin{cases} \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(0)}^2 \oplus \mathbb{Z}_{(-2)} & ; i = 0 \\ 0 & ; otherwise \end{cases}$$

Since this is generated by 4 elements, we will name them  $\overline{H}^0(D_0) \cong \langle A, B, C, D \rangle$  for convenience. Here  $A$  is the homology element that is equivalent (via this sequence

of Reidmeister moves) to  $\mathbf{1} \otimes \mathbf{1}$  on the 2-component unlink. As well  $B$ ,  $C$ , and  $D$  are the homology elements equivalent to  $\mathbf{1} \otimes X$ ,  $X \otimes \mathbf{1}$ , and  $X \otimes X$ , respectively. These elements though are not the same as their equivalent elements on the canonical unlink diagram (that is two separated circles). Of course, they cannot be; with different crossings, they must exist on different resolutions. Fortunately, in his proof of invariance under choice of diagram, Khovanov outlines a way to carry to carry homology elements across diagrams differing by the third Reidmeister move. We will outline this method briefly.

The third Reidmeister move on knots, as drawn in Figure 15a, corresponds to a map (call it  $\alpha$ ) on resolutions in Khovanov homology, as drawn in Figure 15b. Let  $\phi$  be the map associated to the transformation in Figure 14. That is, changing the way we resolve these two crossings, will induce some map, either multiplication or comultiplication. We can then define  $\alpha := id \oplus (\phi \otimes \mathbf{1})$ , where the identity part maps the resolution to itself, and the  $(\phi \otimes \mathbf{1})$  maps to the other resolution in 15b, by first following the map  $\phi$  to split, then attaching the new circle as  $\mathbf{1}$ .

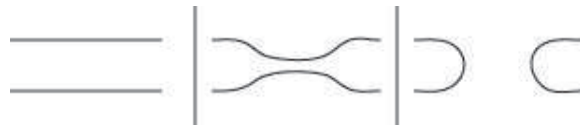


FIGURE 14. Splitting resolutions from third Reidmeister move.

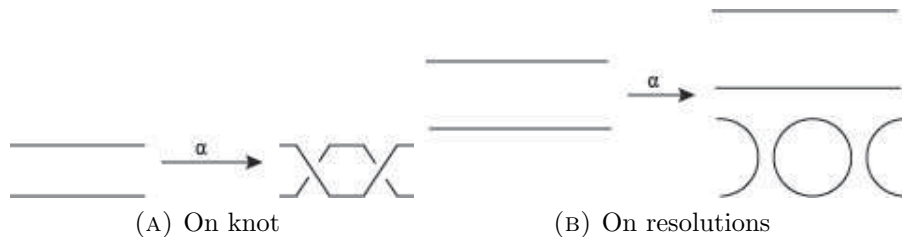


FIGURE 15. 2nd Reidmeister move.

So we now have a way to start with the homology generators on the 2-component unlink, drawn canonically with two separated circles (see Figure 16), and obtain the

homology generators of the unlink drawn as  $D_0$ , in Figure 12. We do so by following the maps between diagrams drawn in Figure 17, which induce the maps on resolutions, as drawn in Figure 18.



FIGURE 16. Canonical diagram for the 2-component unlink.

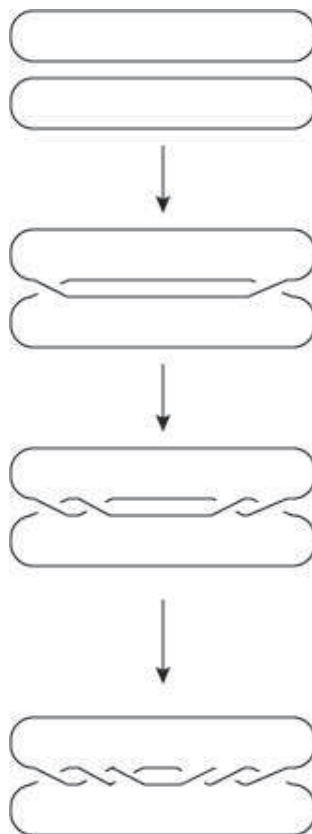


FIGURE 17. Sequence of Reidemeister moves on symmetric union.

Let's index the maps on resolutions  $(\alpha_1, \alpha_2, \alpha_3, \dots)$ . And likewise, let's label the homology generators of each corresponding diagram,  $A_i, B_i, C_i$ , and  $D_i$  ( $i = 0, 1, 2, \dots$ ), where when  $i$  is zero, these will refer to precisely the generators of the canonical diagram. We can then compute these generators.

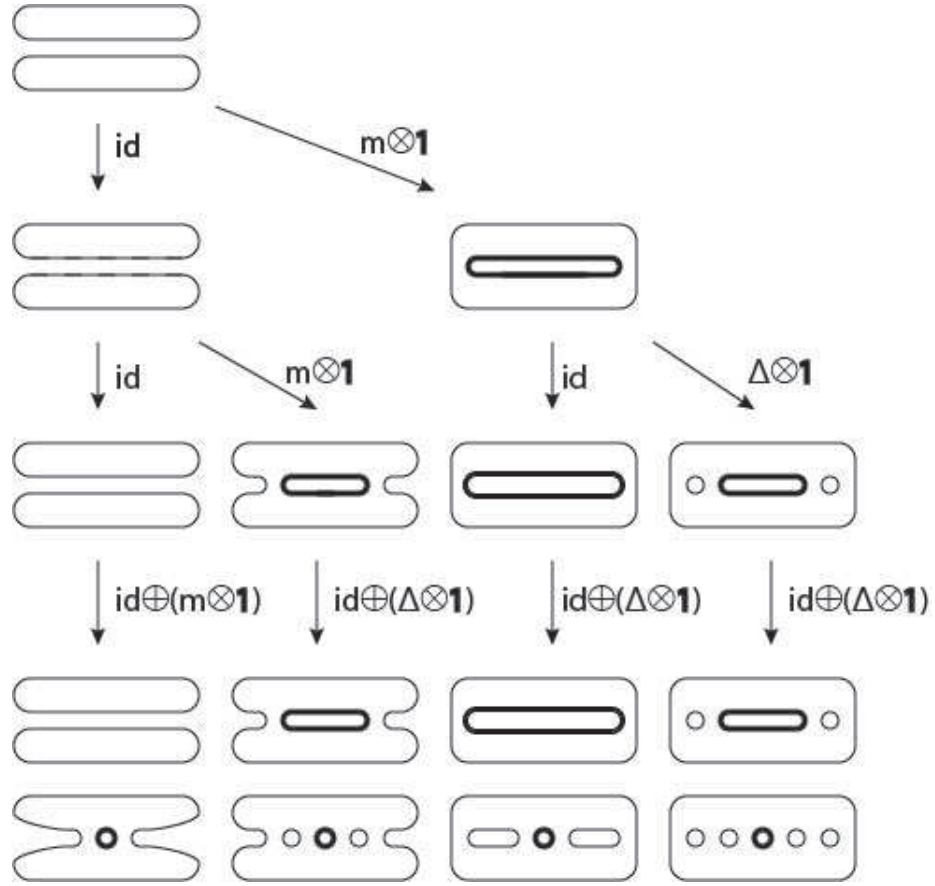


FIGURE 18. Sequence of  $\alpha$  maps on symmetric union.

$$\alpha_1 = id \oplus [{}_{12}m_1 \otimes \mathbf{1}_2]$$

$$\alpha_2 = (id \oplus [{}_{12}m_1 \otimes \mathbf{1}_2], id \oplus [{}_2\Delta_{24} \otimes \mathbf{1}_3])$$

$$\alpha_3 = (id \oplus [{}_{12}m_1 \otimes \mathbf{1}_2], id \oplus [{}_2\Delta_{24} \otimes \mathbf{1}_3], id \oplus [{}_2\Delta_{24} \otimes \mathbf{1}_3], id \oplus [{}_3\Delta_{35} \otimes \mathbf{1}_4])$$

$$A_0 = \mathbf{1} \otimes \mathbf{1}$$

$$B_0 = \mathbf{1} \otimes X$$

$$C_0 = X \otimes \mathbf{1}$$

$$D_0 = X \otimes X$$



$$A_1 = \alpha_1(X_0) = (\mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{1})$$

$$B_1 = \alpha_1(Y_0) = (\mathbf{1} \otimes X, X \otimes \mathbf{1})$$

$$C_1 = \alpha_1(Z_0) = (X \otimes \mathbf{1}, X \otimes \mathbf{1})$$

$$D_1 = \alpha_1(W_0) = (X \otimes X, 0)$$

$$A_2 = \alpha_2(X_0) = (\mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{1},$$

$$\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes X + \mathbf{1} \otimes X \otimes \mathbf{1} \otimes \mathbf{1})$$

$$B_2 = \alpha_2(Y_0) = (\mathbf{1} \otimes X, X \otimes \mathbf{1}, X \otimes \mathbf{1},$$

$$X \otimes \mathbf{1} \otimes \mathbf{1} \otimes X + X \otimes X \otimes \mathbf{1} \otimes \mathbf{1})$$

$$C_2 = \alpha_2(Z_0) = (X \otimes \mathbf{1}, X \otimes \mathbf{1}, X \otimes \mathbf{1},$$

$$X \otimes \mathbf{1} \otimes \mathbf{1} \otimes X + X \otimes X \otimes \mathbf{1} \otimes \mathbf{1})$$

$$D_2 = \alpha_2(W_0) = (X \otimes X, 0, 0, 0)$$

$$A_3 = \alpha_3(X_0) = (\mathbf{1}\mathbf{1}, \mathbf{1}\mathbf{1}, \mathbf{1}\mathbf{1}, \mathbf{1}\mathbf{1}\mathbf{1}X + \mathbf{1}X\mathbf{1}\mathbf{1}, \mathbf{1}\mathbf{1}, \mathbf{1}\mathbf{1}\mathbf{1}X + \mathbf{1}X\mathbf{1}\mathbf{1},$$

$$\mathbf{1}\mathbf{1}\mathbf{1}X + \mathbf{1}X\mathbf{1}\mathbf{1}, \mathbf{1}\mathbf{1}\mathbf{1}\mathbf{1}XX + \mathbf{1}\mathbf{1}X\mathbf{1}\mathbf{1}X + \mathbf{1}X\mathbf{1}\mathbf{1}X\mathbf{1}$$

$$+ \mathbf{1}XX\mathbf{1}\mathbf{1}\mathbf{1})$$

$$B_3 = \alpha_3(Y_0) = (\mathbf{1}X, X\mathbf{1}, X\mathbf{1}, X\mathbf{1}\mathbf{1}X + XX\mathbf{1}\mathbf{1}, X\mathbf{1}, X\mathbf{1}\mathbf{1}X + XX\mathbf{1}\mathbf{1},$$

$$X\mathbf{1}\mathbf{1}X + XX\mathbf{1}\mathbf{1}, X\mathbf{1}\mathbf{1}\mathbf{1}XX + X\mathbf{1}X\mathbf{1}\mathbf{1}X + XX\mathbf{1}\mathbf{1}X\mathbf{1}$$

$$+ XXX\mathbf{1}\mathbf{1}\mathbf{1})$$

$$C_3 = \alpha_3(Z_0) = (X\mathbf{1}, X\mathbf{1}, X\mathbf{1}, X\mathbf{1}\mathbf{1}X + XX\mathbf{1}\mathbf{1}, X\mathbf{1}, X\mathbf{1}\mathbf{1}X + XX\mathbf{1}\mathbf{1},$$

$$X\mathbf{1}\mathbf{1}X + XX\mathbf{1}\mathbf{1}, X\mathbf{1}\mathbf{1}\mathbf{1}XX + X\mathbf{1}X\mathbf{1}\mathbf{1}X + XX\mathbf{1}\mathbf{1}X\mathbf{1}$$

$$+ XXX111)$$

$$D_3 = \alpha_3(W_0) = (XX, 0, 0, 0, 0, 0, 0, 0)$$

We notice that there's a distinct difference between the first resolution (the one with two parallel circles) and the remaining resolutions. Firstly the increasing number of circles in the other resolutions follow a nice pattern. Secondly the maps,  $f$  and  $g$ , for the rest of the resolutions act on two different circles, but in this first, special resolution, the maps act on the same circle twice. These distinctions motivate us to define *classes*. We'll call the first resolution  $\mathcal{X}^*$  and the remaining classes  $\mathcal{X}$ , as shown in Figures 19 and 20, respectively.



FIGURE 19. Class  $\mathcal{X}^*$ .

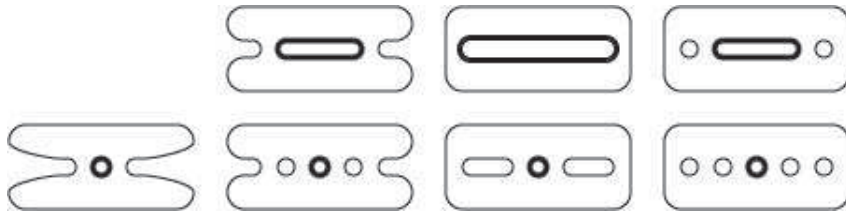


FIGURE 20. Class  $\mathcal{X}$ .

As mentioned, we notice that the increasingly complex sequence of elements follow a pattern. This is because the innermost  $\mathbf{1}$  repeatedly gets acted upon in the same way, due to how we defined this family of knots. We use  $\underline{\mathbf{1}}$  to mean a finite sequence of  $(\mathbf{1}, \mathbf{11X} + \mathbf{X11}, \mathbf{111XX} + \mathbf{1X11X} + \mathbf{X11X1} + \mathbf{XX111}, \dots)$  obtained by repeatedly replacing the innermost  $\mathbf{1}$  with  $\mathbf{11X} + \mathbf{X11}$  (the image of  $\mathbf{1}$  under  $\alpha$ ) to match the correct number of circles in the resolution. Since it will come up, we define  $\underline{X}$  to be  $\underline{\mathbf{1}}$  with the middle inside circle replaced with an  $X$ . That is,  $\underline{X} = (X, \mathbf{1XX} + \mathbf{XX1}, \mathbf{11XXX} + \mathbf{1XX1X} + \mathbf{X1XX1} + \mathbf{XXX11}, \dots)$ . When necessary to distinguish,

we'll use  $\underline{\mathbf{1}}_m$  to mean the  $m$ -th term of this expansion, or the resolution with  $2m + 1$   $\mathbf{1}$  circles. And further still we may call the collection of resolutions considered by  $\underline{\mathbf{1}}_m$  itself a class, called  $\mathcal{X}_m$ . Even though  $\mathbf{1}$  is defined in this way, the reader might notice that  $\underline{\mathbf{1}}_m$  is also the sum of  $2^m$  terms, one for each combination of  $m$   $\mathbf{1}$  and  $X$ 's appearing before the center  $\mathbf{1}$ . This follows from an easy induction proof and will be useful later.

This gives us the language to talk about the entire homology generators, without reference to how many twists were used in building our torus knot. Let's agree to list the homology elements as  $(\mathcal{X}^*, \mathcal{X})$ . Then we have:

$$\begin{aligned} A &= (\mathbf{1}\mathbf{1}, \underline{\mathbf{1}}\mathbf{1}) \\ B &= (\mathbf{1}X, X\underline{\mathbf{1}}) \\ C &= (X\underline{\mathbf{1}}, X\underline{\mathbf{1}}) \\ D &= (XX, 0) \end{aligned}$$

With these in place, and using Khovanov's shortcut, we can now convert the general commutative diagram in Figure 9 to a cleaner, problem-specific diagram in Figure 21.

We can now state how the maps  $f$  and  $g$ , discussed in the previous section, act on the homology generators.

$$\begin{aligned} f(A) &= (0, X\underline{\mathbf{1}} - \mathbf{1}\underline{X}) \\ f(B) &= (0, -X\underline{X}) \\ f(C) &= (0, -X\underline{X}) \\ f(D) &= (0, 0) \end{aligned}$$

$$\begin{aligned} g(A) &= (2X\underline{\mathbf{1}}, X\underline{\mathbf{1}} + \mathbf{1}\underline{X}) \\ g(B) &= (2XX, X\underline{X}) \\ g(C) &= (0, X\underline{X}) \\ g(D) &= (0, 0) \end{aligned}$$

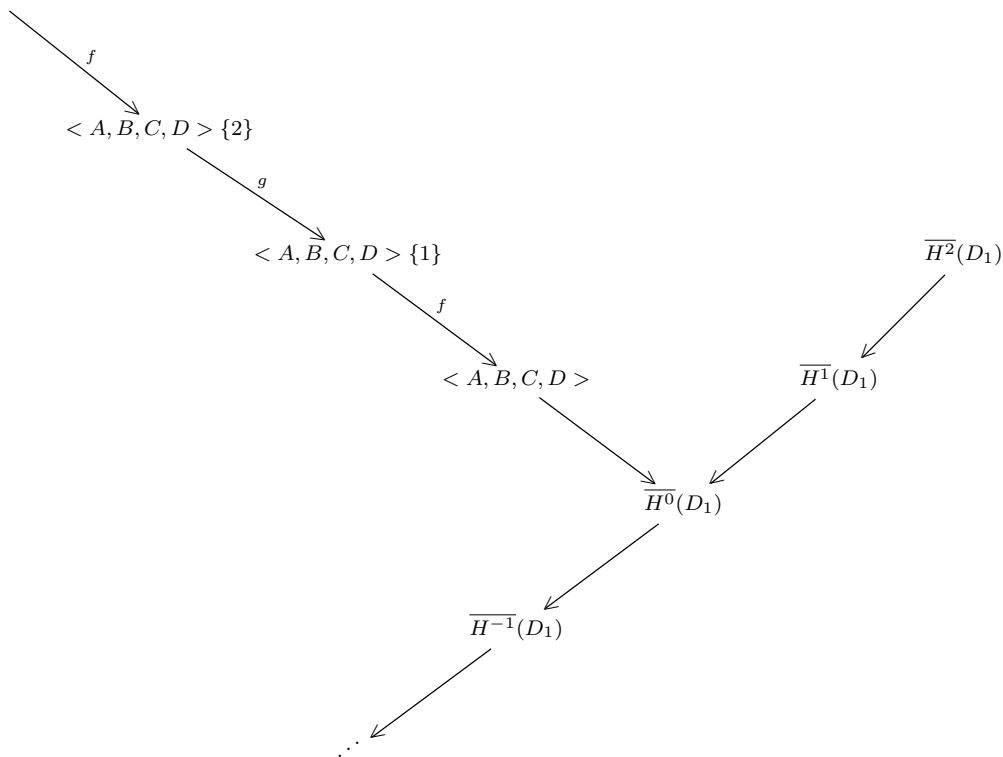


FIGURE 21. New diagram

This gives us four immediate relationships:

$$\begin{aligned}
 (2) \quad & f(A) + g(A) = 2C \\
 & f(B) + g(B) = 2D \\
 & f(C) + g(C) = 0 \\
 & f(D) + g(D) = 0
 \end{aligned}$$

We should point out that the maps  $f$  and  $g$ , clearly, do not map exactly to some linear combination of the homology generators  $A, B, C$ , and  $D$ , but rather map to their homology classes. Of course, the groups,  $\langle A, B, C, D \rangle$  in the commutative diagram drawn in Figure 21, are the 0-th (unshifted) homology group of the knot diagram  $D_0$ ,  $\overline{H^0}(D_0)$ . To distinguish, we write  $\equiv$  when referring to equality on the level of homological equivalence. However when, as in (2), we get exact equality (written with the usual  $=$ ), this pins down which equivalence class we're working

with. Here this is illustrated by the fact that  $f(D) = g(D) = 0$ . It will be convenient to name the maps in the diagonal chain that yields this homology, so we will call  $\overline{H^0}(D_0) \cong \frac{Ker\zeta}{Im\eta}$ .

Our goal is to determine into what equivalence class  $f$  and  $g$  map the homology generators. If we knew the entirety of  $Im(\eta)$ , we would be able to determine these maps. However, this is a question that quickly becomes too computationally complex to get our hands on. As an attempt to understand these maps we look at single resolution. If we consider the images of all maps that go into a particular fixed resolution, we will have considered all possible identifications on that resolution. With this, it will suffice to look at how  $f$  and  $g$  act on this resolution; some images may be impossible under any possible identifications. Similarly, we may look at a collection of resolutions.

In this symmetric union, we choose the single resolution in the class  $\mathcal{X}^*$ , which has a nice property that all maps going into it are comultiplication. This restricts our images, whereas a single multiplication map will allow for any image. Specifically, we have  $Im(\eta|_{\eta^{-1}(\mathcal{X}^*)}) = \langle B + C, D \rangle$ . This tells us that  $f$  and  $g$  are determined by how they map the resolution  $\mathcal{X}^*$ , upto linear combinations of  $B + C$  and  $D$ . Since  $Ker(\zeta^*)/Im(\eta|_{\eta^{-1}(\mathcal{X}^*)})$  may only differ (on this resolution) by  $B + C$  or  $D$ . These linear combinations are limited even more internal grading.<sup>3</sup> So we have:

$$\begin{aligned} f(A) &= a(B + C) & g(A) &= 2C + b(B + C) \\ f(B) &= cW & g(B) &= dW \\ f(C) &= eW & g(C) &= fW \\ f(D) &= 0 & g(D) &= 0 \end{aligned}$$

For some  $a, b, c, d, e, f \in \mathbb{Z}$ . As we'll see, the relationships in (2) hold in a more general setting. These relationships are very natural and occur because of the sign

<sup>3</sup>This may be an over-simplification because it is possible that, for example,  $f(B) \equiv D$  is equivalent in homology to  $f(B) \equiv D - D + (B + C)$ , which still has a consistent internal grading, simply because the  $D$ s cancelled. However, a thorough argument about internal grading can be made and mirrors the argument made in the 2-bridge knot computation given later.

differences in  $f$  and  $g$ . However, due to the simplicity of this first computation, additional (and convenient) relationships hold:

$$\begin{aligned} f(B) - f(C) &= 0 \\ g(B) - g(C) &= 2D \end{aligned}$$

As well, by nature of a chain complex, we know that:

$$(3) \quad \begin{aligned} f \circ g(A) &= 0 \\ g \circ f(A) &= 0 \end{aligned}$$

By taking all of these equations together, it is a simple algebra to pin down what  $a, b, c, d, e, f$  must be. We do, however, get two possibilities:

$$(4) \quad \begin{array}{ll} f(A) \equiv 2C & g(A) \equiv 0 \\ f(B) \equiv 2D & g(B) \equiv 0 \\ f(C) \equiv 0 & g(C) \equiv 0 \\ f(D) \equiv 0 & g(D) \equiv 0 \end{array}$$

$$(5) \quad \begin{array}{ll} f(A) \equiv C - B & g(A) \equiv C + B \\ f(B) \equiv D & g(B) \equiv D \\ f(C) \equiv -D & g(C) \equiv D \\ f(D) \equiv 0 & g(B) \equiv 0 \end{array}$$

In fact, either of these can occur, and which occurs depends on the parity of  $m$  in the  $(2,m)$ -torus knots. To show this, we'll use a technique that we will use in the next proof. We look at the class  $\mathcal{X}$ , more specifically the resolution with the maximal number of circles, or  $\mathcal{X}_m$ , using our notation from earlier. In this resolution, like with the  $\mathcal{X}^*$  resolution, all of the maps going into it are comultiplication. Things get a

little less concrete here, because there are  $m$  maps, so we'll need to find a way to characterize all of their images.

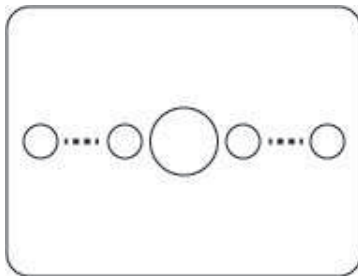


FIGURE 22. Resolution  $\mathcal{X}_m$ .

The comultiplication maps all come from changing one of circles in the left half of the resolution. The pre-image of this resolution is a number of resolutions where, in any one, two of the circles on the left side are put together (to be split up via comultiplication when going forward) with the rest of the circles left alone (to be mapped via the identity). This yields the images:

$$\begin{aligned} & \dots X \mathbf{1} \dots + \dots \mathbf{1} X \dots \\ & \dots X X \dots \end{aligned}$$

These tell us that the maps  $f$  and  $g$  are determined by their action on this resolution, up to the homological equivalences:

$$\begin{aligned} & \dots X \mathbf{1} \dots \equiv - \dots \mathbf{1} X \dots \\ & \dots X X \dots \equiv 0 \end{aligned}$$

So the idea here is that any  $X$  that shows up in the first  $m+1$  circles can be moved, one circle at a time, by trading the adjacent terms and changing sign. If, at any point, we have two adjacent  $X$ 's, we can cancel the entire term. These equivalences consider all images, so if we choose a consistent representative of these equivalence classes, we will always know if a term cancels. So, let's agree that if there is an  $X$  in the first  $m+1$  circles, to move it to the outside (or first) circle. As well, if there are multiple  $X$ 's in these circles, let's agree to cancel the whole term. This equivalence has the

property that these  $\underline{\mathbf{1}}_m$  terms will mostly cancel out. (This is where it is useful to use our characterization that these  $\underline{\mathbf{1}}$  terms contain every combination of  $X$  and  $\mathbf{1}$  before the middle circle.) We can write our generators as:

$$\begin{aligned} A &\equiv \mathbf{1}\mathbf{1}\dots\mathbf{1} + \dots \\ B &\equiv X\mathbf{1}\dots\mathbf{1} \\ C &\equiv X\mathbf{1}\dots\mathbf{1} \\ D &\equiv 0 \end{aligned}$$

Although we haven't written out all of  $A$  (which is complicated), what's important is that this cannot combine with the other generators to give us zero. Our images are determined up to a linear combinations of these  $B - C$  and  $D$ . Let's examine  $f(A)$ :

$$\begin{aligned} f(A) &= X\underline{\mathbf{1}} - \underline{\mathbf{1}}X \equiv X\mathbf{1}\dots\mathbf{1} - (-1)^m X\mathbf{1}\dots\mathbf{1} \\ &= \begin{cases} 0 & ; m \text{ even} \\ 2B & ; m \text{ odd} \end{cases} \end{aligned}$$

Returning now to our two possible cases (in Equations (4) and (5)), we see that if  $m$  is even then only (4) is possible, and otherwise only (5) is possible. Knowing these maps allows us to make a general statement about the Khovanov Homology in terms of the  $D_1$  resolutions. We will carry our next calculation out to an explicit formula.

## 5. 2-BRIDGE KNOTS

We now use the same techniques to compute the Khovanov Homology on symmetric unions of 2-bridge knots of the general form illustrated in Figure 23.

We will use Khovanov's shortcut on the crossings used in the construction of the symmetric union, as indicated at the top. This again yields the two resolutions, as drawn in Figure 24 and Figure 25. Again we notice that the first of these diagrams comes completely undone, motivating our exploration of this family of knots.

We again name the homology generators, but this time we'll give them new names:  $\overline{H^0} \cong \langle X, Y, Z, W \rangle$ . We repeat the computation of the previous example, wherein



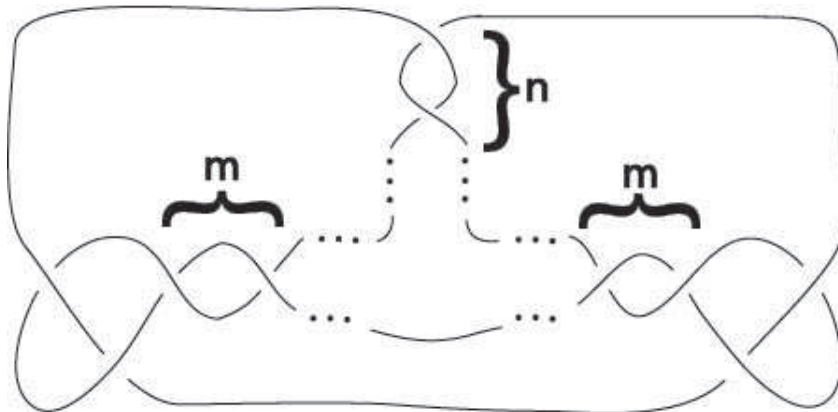


FIGURE 23. 2-Bridge symmetric union.

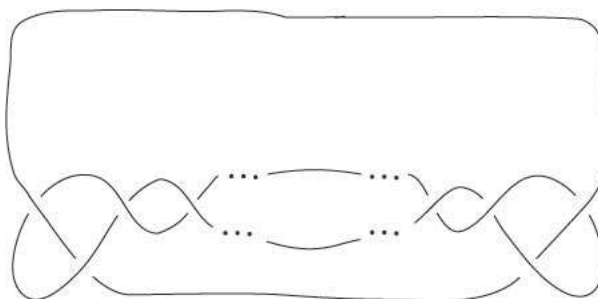


FIGURE 24.  $D_0$ = Two-component unlink.

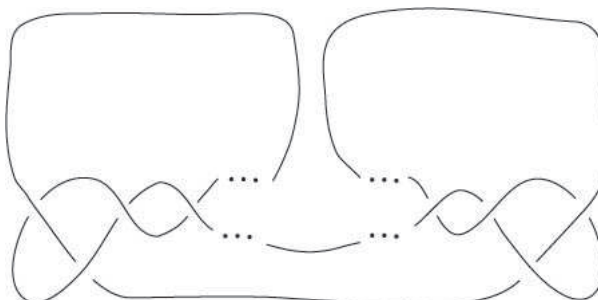


FIGURE 25.  $D_1$ = Connected sum of this bridge knot and mirror.

we follow the homology generators on the canonical 2-component unlink diagram through the sequence of  $\alpha$  maps. Because of the change of structure, these maps do follow a different path.

$$\begin{aligned}
\alpha_1 &= id \oplus [{}_{12}m_1 \otimes \mathbf{1}_2] \\
\alpha_2 &= (id \oplus [{}_2\Delta_{24} \otimes \mathbf{1}_3], id \oplus [{}_{12}m_1 \otimes \mathbf{1}_2]) \\
\alpha_3 &= (id \oplus [{}_{12}m_1 \otimes \mathbf{1}_2], id \oplus [{}_{13}m_1 \otimes \mathbf{1}_3], id \oplus [{}_2\Delta_{24} \otimes \mathbf{1}_3], \\
&\quad id \oplus [{}_{12}m_1 \otimes \mathbf{1}_2])
\end{aligned}$$

$$X_0 = \mathbf{1} \otimes \mathbf{1}$$

$$Y_0 = \mathbf{1} \otimes X$$

$$Z_0 = X \otimes \mathbf{1}$$

$$W_0 = X \otimes X$$

$$X_1 = \alpha_1(X_0) = (\mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{1})$$

$$Y_1 = \alpha_1(Y_0) = (\mathbf{1} \otimes X, X \otimes \mathbf{1})$$

$$Z_1 = \alpha_1(Z_0) = (X \otimes \mathbf{1}, X \otimes \mathbf{1})$$

$$W_1 = \alpha_1(W_0) = (X \otimes X, 0)$$

$$X_2 = \alpha_2(X_1) = (\mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes X + \mathbf{1} \otimes X \otimes \mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{1})$$

$$Y_2 = \alpha_2(Y_1) = (\mathbf{1} \otimes X, \mathbf{1} \otimes X \otimes \mathbf{1} \otimes X, X \otimes \mathbf{1}, X \otimes \mathbf{1})$$

$$\begin{aligned}
Z_2 = \alpha_2(Z_1) &= (X \otimes \mathbf{1}, X \otimes \mathbf{1} \otimes \mathbf{1} \otimes X + X \otimes X \otimes \mathbf{1} \otimes \mathbf{1}, X \otimes \mathbf{1}, \\
&\quad X \otimes \mathbf{1})
\end{aligned}$$

$$W_2 = \alpha_2(W_1) = (X \otimes X, X \otimes X \otimes \mathbf{1} \otimes X, 0, 0)$$

$$X_3 = \alpha_3(X_2) = (\mathbf{11}, \mathbf{11}, \mathbf{111}X + \mathbf{1}X\mathbf{11}, \mathbf{111}X + \mathbf{1}X\mathbf{11}, \mathbf{11},$$

$$\mathbf{111X} + \mathbf{1X11}, \mathbf{11}, \mathbf{11})$$

$$Y_3 = \alpha_3(Y_2) = (\mathbf{1X}, \mathbf{X1}, \mathbf{1X1X}, \mathbf{1X1X}, \mathbf{X1}, \mathbf{X1}, \mathbf{X1}, \mathbf{X1})$$

$$Z_3 = \alpha_3(Z_2) = (\mathbf{X1}, \mathbf{X1}, \mathbf{X11X} + \mathbf{XX11}, \mathbf{X11X} + \mathbf{XX11}, \mathbf{X1}, \mathbf{X1}, \\ \mathbf{X1}, \mathbf{X1})$$

$$W_3 = \alpha_3(W_2) = (\mathbf{XX}, \mathbf{0}, \mathbf{XX1X}, \mathbf{XX1X}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$$

We will stop listing the groups explicitly at this point because the sequences stabilize. What we mean by this is that with each new  $\alpha$  map, the new resolutions will either map to an already existing combination or will map an  $\mathbf{1}$  to  $\mathbf{11X} + \mathbf{X11}$ . Recognizing this allows us to make general statements about these generators without regard to how many twists we have in this strand. In fact we see that all of the diagrams fit into one of the following classes.

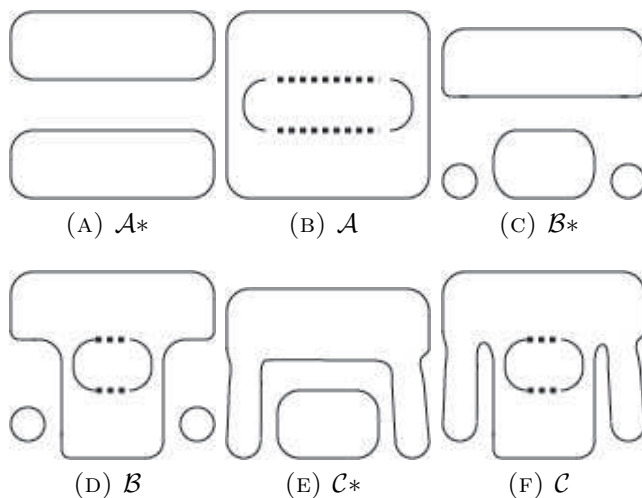


FIGURE 26. Limiting resolutions.

This allows us to write the homology generators, which we'll agree to write as  $(\mathcal{A}^*, \mathcal{A}, \mathcal{B}^*, \mathcal{B}, \mathcal{C}^*, \mathcal{C})$ .

$$\begin{aligned}
 X &= (\mathbf{11}, \quad \mathbf{1}\underline{\mathbf{1}}, \quad \mathbf{111}X + \mathbf{1}X\mathbf{11}, \quad \mathbf{11}\underline{\mathbf{1}}X + \mathbf{1}X\underline{\mathbf{1}}\mathbf{1}, \quad \mathbf{11}, \quad \mathbf{1}\underline{\mathbf{1}}) \\
 Y &= (\mathbf{1}X, \quad X\underline{\mathbf{1}}, \quad \mathbf{1}X\mathbf{1}X, \quad \mathbf{1}X\underline{\mathbf{1}}X, \quad X\mathbf{1}, \quad X\underline{\mathbf{1}}) \\
 Z &= (X\mathbf{1}, \quad X\underline{\mathbf{1}}, \quad X\mathbf{11}X + XX\mathbf{11}, \quad X\underline{\mathbf{1}}\mathbf{1}X + XX\underline{\mathbf{1}}\mathbf{1}, \quad X\mathbf{1}, \quad X\underline{\mathbf{1}}) \\
 W &= (XX, \quad 0, \quad XX\mathbf{1}X, \quad XX\underline{\mathbf{1}}X, \quad 0, \quad 0)
 \end{aligned}$$

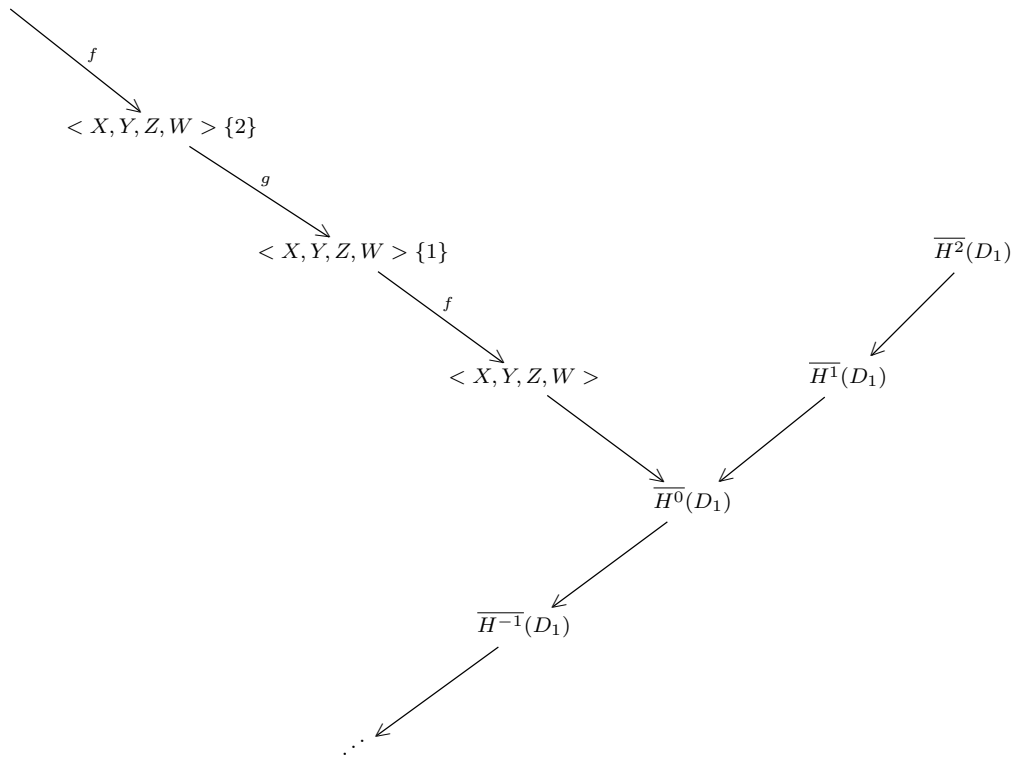


FIGURE 27. New diagram.

We again have a commutative diagram, this time drawn in Figure 31. Within classes  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$ , the maps  $f$  and  $g$  act on the same arcs. That is,  $u_X$  always multiplies  $X$  to the outermost circle of the resolution, and  $v_X$  always multiplies  $X$  to the middle inside circle of the resolution. So we compute:

$$\begin{aligned}
f(X) &= (0, X\underline{1} - \underline{1X}, 0, X\underline{11X} + XX\underline{11} - \underline{11XX} - \underline{1XX1}, 0, \\
&\quad X\underline{1} - \underline{1X}) \\
f(Y) &= (0, -XX, 0, XX\underline{1X} - \underline{1XXX}, 0, -XX) \\
f(Z) &= (0, -XX, 0, -X\underline{1XX} - XX\underline{X1}, 0, -XX) \\
f(W) &= (0, 0, 0, -XXX\underline{X}, 0, 0) \\
g(X) &= (2X\underline{1}, X\underline{1} + \underline{1X}, 2(X\underline{11X} + XX\underline{11}), X\underline{11X} + XX\underline{11} + \\
&\quad \underline{11XX} + \underline{1XX1}, 2X\underline{1}, X\underline{1} + \underline{1X}) \\
g(Y) &= (2XX, X\underline{X}, 2XX\underline{1X}, XX\underline{1X} + \underline{1XXX}, 0, X\underline{X}) \\
g(Z) &= (0, X\underline{X}, 0, X\underline{1XX} + XX\underline{X1}, 0, X\underline{X}) \\
g(W) &= (0, 0, 0, XX\underline{XX}, 0, 0)
\end{aligned}$$

This gives us four immediate relationships:

$$\begin{aligned}
(6) \quad f(X) + g(X) &= 2Z \\
f(Y) + g(Y) &= 2W \\
f(Z) + g(Z) &= 0 \\
f(W) + g(W) &= 0
\end{aligned}$$

To narrow this down, we again use the technique of restricting our image elements to the preimage of only a single resolution or group of resolutions. We look first at the resolution  $\mathcal{B}_m$ . Like before, we choose this one because  $\alpha|_{\alpha^{-1}(\mathcal{B}_m)}$  is a direct sum of comultiplication maps, which is restrictive. These comultiplication maps, specifically are  $\Delta_{14}$ ,  $\Delta_{13}$ , and  $\Delta_3$ . Drawn below, Figure 28.

So by  $\Delta_{13}$ , we mean comultiplication going to the outside circle and the first (left-most) of the group of circles labelled 3. And by  $\Delta_3$ , we mean a family of maps within the middle group of circles. Specifically, these come from splitting into adjacent circles among the first  $m + 1$  of the  $2m + 1$  middle circles. These relationships again

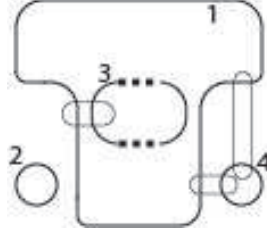


FIGURE 28. Comultiplication maps drawn.

allow us to move  $X$ 's over a circle at a time, alternating signs as we go, and delete the entire term if we ever get two adjacent  $X$ 's inside the  $\underline{1}_m$  or  $\underline{X}$  terms.

With these relationships, and the two from above we immediately get the reductions on the  $\mathcal{B}_m$  resolutions of the homology generators ( $X$  doesn't reduce nicely):

$$\begin{aligned}
 Y &= \mathbf{1}X\underline{\mathbf{1}}X && \equiv -XX\underline{\mathbf{1}}\mathbf{1} && \equiv -XX(\mathbf{1}\dots\underline{\mathbf{1}}X\dots X)\mathbf{1} \\
 Z &= X\underline{\mathbf{1}}\mathbf{1}X + XX\underline{\mathbf{1}}\mathbf{1} && \equiv XX\underline{\mathbf{1}}\mathbf{1} && \equiv XX(\mathbf{1}\dots\underline{\mathbf{1}}X\dots X)\mathbf{1} \\
 W &= XX\underline{\mathbf{1}}X && \equiv 0
 \end{aligned}$$

We see that in  $Y$  and  $Z$  the  $\underline{\mathbf{1}}$  gets reduced to the only part of the summation that doesn't cancel; that is the part that has all  $\mathbf{1}$ 's before the center  $\mathbf{1}$ . And  $W$ , in all cases, is the image of  $\Delta_{14}$ .  $X$  does not have such a reduction; since there are so many  $\mathbf{1}$ 's, expansions of  $\underline{\mathbf{1}}_m$  with one or no  $\mathbf{1}$ 's do not go away. These relationships tell us that images of  $f$  and  $g$  are determined by their images on the resolution  $\mathcal{B}_m$  up to a linear combinations of  $(Y + Z)$  and  $W$ . For short-hand, we'll call any such linear combination  $\mathcal{L}$ . These resolutions are precisely:

$$\begin{aligned}
 f(X) &= X\underline{\mathbf{1}}\mathbf{1}X + XX\underline{\mathbf{1}}\mathbf{1} - \mathbf{1}\mathbf{1}\underline{X}X - \mathbf{1}X\underline{X}\mathbf{1} \\
 &\equiv XX\underline{\mathbf{1}}\mathbf{1} - \mathbf{1}X\underline{X}\mathbf{1} \\
 &\equiv XX(\mathbf{1}\dots\underline{\mathbf{1}}X\dots X)\mathbf{1} - (-1)^m XX(\mathbf{1}\dots\underline{\mathbf{1}}X\dots X)\mathbf{1} \\
 f(Y) &\equiv XX(\mathbf{1}\dots\underline{\mathbf{1}}X\dots X)\mathbf{1} \equiv 0 \\
 f(Z) &\equiv -XX(\mathbf{1}\dots\underline{\mathbf{1}}X\dots X)\mathbf{1} \equiv 0 \\
 f(W) &\equiv 0
 \end{aligned}$$

$$\begin{aligned}
f(X) &= X\mathbf{1}\underline{\mathbf{1}}X + XX\underline{\mathbf{1}}\mathbf{1} + \mathbf{1}\underline{\mathbf{1}}\underline{X}X + \mathbf{1}X\underline{X}\mathbf{1} \\
&\equiv XX\underline{\mathbf{1}}\mathbf{1} + \mathbf{1}X\underline{X}\mathbf{1} \\
&\equiv XX(\mathbf{1}\dots\mathbf{1}\underline{\mathbf{1}}X\dots X)\mathbf{1} + (-1)^m XX(\mathbf{1}\dots\mathbf{1}\underline{\mathbf{1}}X\dots X)\mathbf{1} \\
f(Y) &\equiv -XX(\mathbf{1}\dots\mathbf{1}X\underline{X}\dots X)\mathbf{1} \equiv 0 \\
f(Z) &\equiv XX(\mathbf{1}\dots\mathbf{1}X\underline{X}\dots X)\mathbf{1} \equiv 0 \\
f(W) &\equiv 0
\end{aligned}$$

Where the  $(-1)^m$  comes from moving the inner-most  $X$  a term at a time to the leading term to make it match the other piece of  $f(X)$ . So finally with these relationships, we have that:

$$\begin{aligned}
(7) \quad f(X) &\equiv -Y + (-1)^m Y + \mathcal{L} & g(X) &\equiv -Y - (-1)^m Y + \mathcal{L} \\
f(Y) &\equiv \mathcal{L} & g(Y) &\equiv \mathcal{L} \\
f(Z) &\equiv \mathcal{L} & g(Z) &\equiv \mathcal{L} \\
f(W) &\equiv \mathcal{L} & g(W) &\equiv \mathcal{L}
\end{aligned}$$

We can be yet more specific about what these linear combinations,  $\mathcal{L}$ , are. The equations (8) tell us that either  $f(X)$  or  $g(X)$  must have a  $Y$  term. This fixes the internal grading for that image. And since  $f(X) + g(X) = 2Z$ , the internal grading is fixed for both. Similarly, since  $f(Y) + g(Y) = 2W$ , this fixes the internal grading for both of these. (This doesn't necessarily disallow for one of these to be exactly zero.) We next argue that  $f(Z)$  (and therefore  $g(Z) = -f(Z)$ ) must either be zero, or have internal grading equal to that of  $W$ . Suppose, to the contrary that it does not. Then  $f(Z) = \lambda(Y + Z)$  for some  $\lambda \neq 0$ . And by nature of the chain complex we again know that:

$$\begin{aligned}
(8) \quad f \circ g(Z) &= 0 \\
g \circ f(Z) &= 0
\end{aligned}$$

So, it would follow that

$$0 = g \circ f(Z) = \lambda g(Y + Z) = \lambda[g(Y) - \lambda(Y + Z)]$$

Which is a contradiction, since  $g(Y)$  is either zero or a multiple of  $W$ . Finally, we argue that  $f(W)$  and  $g(W)$  are both exactly zero. We do so by first claiming that these are either zero or a linear combination of  $(Y + Z)$ , which follows by exactly the same argument as above. We now have the stronger relationships:

$$\begin{aligned} f(X) &\equiv -Y + (-1)^m Y + a_1(Y + Z) \\ f(Y) &\equiv bW \\ f(Z) &\equiv cW \\ f(W) &\equiv d(X + Y) \end{aligned} \tag{9}$$

$$\begin{aligned} g(X) &\equiv -Y - (-1)^m Y + a_2(Y + Z) \\ g(Y) &\equiv (2 - b)W \\ g(Z) &\equiv -cW \\ g(W) &\equiv -d(X + Y) \end{aligned}$$

Where  $a_1, a_2, b, c, d \in \mathbb{Z}$ . Again composing  $f$  with  $g$  and  $g$  with  $f$ , we get the relationships:

$$\begin{aligned} d(2 - b - c) &= 0 \\ -d(b + c) &= 0 \end{aligned}$$

which can only be consistent if  $d = 0$ . Therefore, we have the relationships:



$$\begin{aligned}
f(X) &\equiv -Y + (-1)^m Y + a_1(Y + Z) \\
f(Y) &\equiv bW \\
f(Z) &\equiv cW \\
f(W) &\equiv 0
\end{aligned}
\tag{10}$$

$$\begin{aligned}
g(X) &\equiv -Y - (-1)^m Y + a_2(Y + Z) \\
g(Y) &\equiv (2 - b)W \\
g(Z) &\equiv -cW \\
g(W) &\equiv 0
\end{aligned}$$

Suppose, without loss of generality, that  $m$  is odd.<sup>4</sup> Using the equations from (6) and exactness arguments, we can narrow down, algebraically, the possibilities. We have the 2 possibilities:

$$\begin{aligned}
f(X) &\equiv 2Z & g(X) &\equiv 0 \\
f(Y) &\equiv bW & g(Y) &\equiv (2 - b)W \\
f(Z) &\equiv 0 & g(Z) &\equiv 0 \\
f(W) &\equiv 0 & g(W) &\equiv 0
\end{aligned}
\tag{11}$$

$$\begin{aligned}
f(X) &\equiv 2Z + a(Y + Z) & g(X) &\equiv (2 - a)(Y + Z) \\
f(Y) &\equiv (2 - a)W & g(Y) &\equiv aW \\
f(Z) &\equiv (a - 2)W & g(Z) &\equiv (2 - a)W \\
f(W) &\equiv 0 & g(W) &\equiv 0
\end{aligned}
\tag{12}$$

---

<sup>4</sup>This abuse of language should bother the reader, since the parity of  $m$  was hugely important in the previous computation. We will verify later in the proof that the choice does not actually matter here, but this isn't yet obvious.

We claim that (11) cannot ever happen. The reason for this is that (11) has  $g(X)$  (or possibly  $f(X)$ , in general) equivalent to zero. We can show that under some restriction of resolutions, the two images are strictly non-zero, which would sufficiently show that (11) is impossible. In order to choose the correct resolutions, we must first make a more fine distinction between classes. We mentioned that classes have multiple resolutions for each number of circles. These different resolutions come from repeatedly combining the outermost circles from the middle group to the very outside circle. This process gives us many different ways to realize all  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$ . However, we don't usually worry about this distinction because it has so little effect on the computation. Yet, we notice something between two of the resolutions of  $\mathcal{A}$ , drawn in Figure ??.

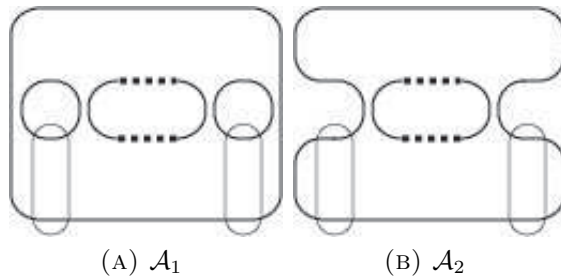


FIGURE 29. Different resolutions of  $\mathcal{A}$ .

In Figure 29a, we notice that by changing the crossings on the bottom of the resolution, we would be combining some of the middle circles with the outside circle. We'll call this *sub-class*  $\mathcal{A}_1$ . On the remaining resolutions, however, changing this crossing would split the outside circle into two circles. We'll call this sub-class  $\mathcal{A}_2$ . This is a marked difference that only exist here; hence the motivation for defining these sub-classes. We now considered, like before the classes with maximum number of middle circles. Except now, we consider all at once. That is, we will compute the image on the resolutions  $((\mathcal{A}_1)_{m-1}, (\mathcal{A}_2)_m, \mathcal{B}_m, \mathcal{C}_m)$ . The advantage of looking at multiple resolutions at once is that image elements come from more than one map acting on more than one of the restricted resolutions. The idea is that if we looked at

all resolutions we would have precisely the entire image, so looking at multiple maps usually gets us closer. Figure 30 shows some of the maps acting on these resolutions:

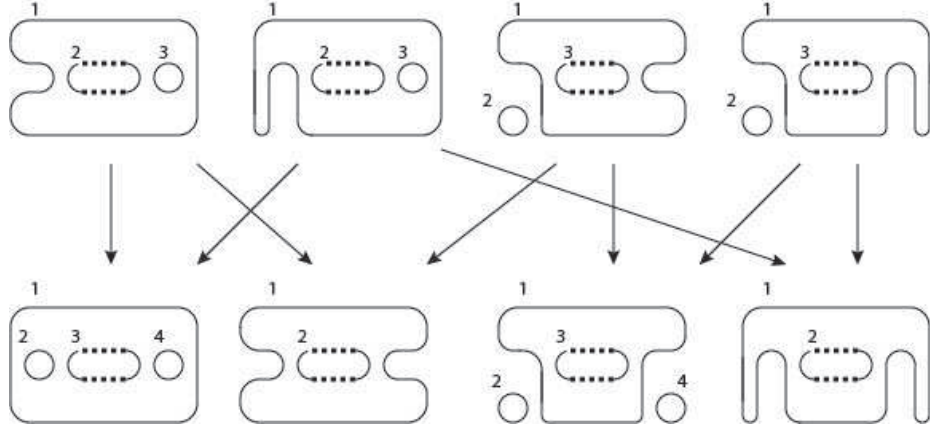


FIGURE 30. Some maps going into  $((\mathcal{A}_1)_{m-1}, (\mathcal{A}_2)_m, \mathcal{B}_m, \mathcal{C}_m)$ .

The remaining maps are comultiplication among the inside circles and a comultiplication with of the outside circle with the left-most of the inside circles. Doing the cokernel computation (used here to capture equivalences), we get:

$$\begin{aligned}
 \text{Coker}(\alpha|_{\alpha^{-1}((\mathcal{A}_1)_{m-1}, (\mathcal{A}_2)_m, \mathcal{B}_m, \mathcal{C}_m)}) &= \langle (\mathbf{1}\mathbf{1}\mathbf{1}\mathbf{1}, 0, 0, 0), (\mathbf{1}\mathbf{1}\mathbf{X}\mathbf{1}, 0, 0, 0), \\
 & \quad (\mathbf{X}\mathbf{1}\mathbf{1}\mathbf{1}, 0, 0, 0), (\mathbf{X}\mathbf{1}\mathbf{X}\mathbf{1}, 0, 0, 0), \\
 & \quad (\mathbf{1}\mathbf{1}\mathbf{1}\mathbf{X}, 0, 0, 0), (\mathbf{1}\mathbf{1}\mathbf{X}\mathbf{X}, 0, 0, 0), \\
 & \quad (0, 0, \mathbf{1}\mathbf{1}\mathbf{1}\mathbf{1}, 0), (0, 0, \mathbf{1}\mathbf{1}\mathbf{X}\mathbf{1}, 0), \\
 & \quad (0, 0, \mathbf{1}\mathbf{X}\mathbf{1}\mathbf{1}, 0), (0, 0, \mathbf{1}\mathbf{X}\mathbf{X}\mathbf{1}, 0) | \\
 & \quad 2(\mathbf{X}\mathbf{1}\mathbf{X}\mathbf{1}, 0, 0, 0), \\
 & \quad 2[(\mathbf{1}\mathbf{1}\mathbf{X}\mathbf{1} - \mathbf{X}\mathbf{1}\mathbf{1}\mathbf{1}, 0, 0, 0)] \rangle
 \end{aligned}
 \tag{13}$$

Of course, this is not the only representation, and there are associated equivalences that are important to the proof. A full detail of the computation is included in the appendix.

This reduces the homology groups and their images under  $f$  and  $g$ . We don't list  $X$  here because it is complicated; all that matters is that there is no way to cancel it.

$$\begin{aligned}
Y &\equiv 2(X\underline{1X\underline{1}}, 0, 0, 0) \\
Z &\equiv 2(X\underline{1X\underline{1}}, 0, 0, 0) \\
W &\equiv 0 \\
\\
f(X) &\equiv -2(0, 0, \underline{1X\underline{X1}}, 0) \\
f(Y) &\equiv 0 \\
f(Z) &\equiv 0 \\
f(W) &\equiv 0
\end{aligned}
\tag{14}$$

$$\begin{aligned}
g(X) &\equiv 2(\underline{11\underline{X\underline{X}}}, 0, 0, 0) \\
g(Y) &\equiv 0 \\
g(Z) &\equiv 0 \\
g(W) &\equiv 0
\end{aligned}$$

What's important here is that neither  $f(X)$  nor  $g(X)$  can be zero. So the first possibility, (11), cannot occur.

So we look at the second possibility, (12). Again, we've assumed that  $m$  is odd. There are two special cases (Equations (15) and (16)) and a more general case (Equation (18)).

$$\begin{aligned}
f(X) &\equiv -2Y & g(X) &\equiv 2(Y + Z) \\
f(Y) &\equiv 2W & g(Y) &\equiv 0 \\
f(Z) &\equiv -2W & g(Z) &\equiv 2W \\
f(W) &\equiv 0 & g(W) &\equiv 0
\end{aligned}
\tag{15}$$

$$(16) \quad \begin{array}{l} f(X) \equiv 2Z \quad g(X) \equiv 0 \\ (a=2) \quad f(Y) \equiv 0 \quad g(Y) \equiv 2W \\ f(Z) \equiv 0 \quad g(Z) \equiv 0 \\ f(W) \equiv 0 \quad g(W) \equiv 0 \end{array}$$

The equations in (16) cannot work because again, it sets  $g(X) = 0$ . From the equations in (15), we get the following kernels and images:

$$\begin{array}{l} Ker(f) = \langle Y + Z, W \rangle \quad Im(f) = \langle 2Y, 2W \rangle \\ Ker(g) = \langle Y, W \rangle \quad Im(g) = \langle 2(Y + Z), 2W \rangle \end{array}$$

Which yields a homology of

$$(17) \quad \overline{H}^i(X) \equiv \begin{cases} \frac{\langle Y+Z, W \rangle}{\langle 2(Y+Z), 2W \rangle} & ; \quad i \text{ odd} \\ \frac{\langle Y, W \rangle}{\langle 2Y, 2W \rangle} & ; \quad i \text{ even} \end{cases} \cong \mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)}$$

We'll now discuss the more general case:

$$(18) \quad \begin{array}{l} f(X) \equiv -2Y + a(Y + Z) \quad g(X) \equiv (2 - a)(Y + Z) \\ f(Y) \equiv (2 - a)W \quad g(Y) \equiv aW \\ f(Z) \equiv (a - 2)W \quad g(Z) \equiv (2 - a)W \\ f(W) \equiv 0 \quad g(W) \equiv 0 \end{array}$$

In this set of equations, we have:

$$\begin{array}{l} Ker(f) = \langle Y + Z, W \rangle \\ Im(f) = \langle (2 - a)Y + aZ, (a - 2)W \rangle \\ Ker(g) = \langle (2 - a)Y + aZ, W \rangle \\ Im(g) = \langle (a - 2)(Y + Z), aW, (a - 2)W \rangle \end{array}$$

This yields

$$(19) \quad \overline{H}^i(X) \equiv \begin{cases} \frac{\langle Y+Z, W \rangle}{\langle (a-2)(X+Y), aW, (a-2)W \rangle} & ; i \text{ odd} \\ \frac{\langle (2-a)Y+aZ, W \rangle}{\langle (2-a)Y+aZ, (a-2)W \rangle} & ; i \text{ even} \end{cases}$$

We notice that these homology groups, (19) and (17) have something special in common: They are both entirely torsion. Before we go forward, we should point out that our assumption that  $m$  is odd can be safely made. We see now that homology groups are the same (at least in  $\mathbb{Q}$  coefficients) regardless. If  $m$  is even, we have an identical argument, renaming  $f$  and  $g$ . Therefore, we can make the general statement about these homology groups, by taking coefficients in the rationals.

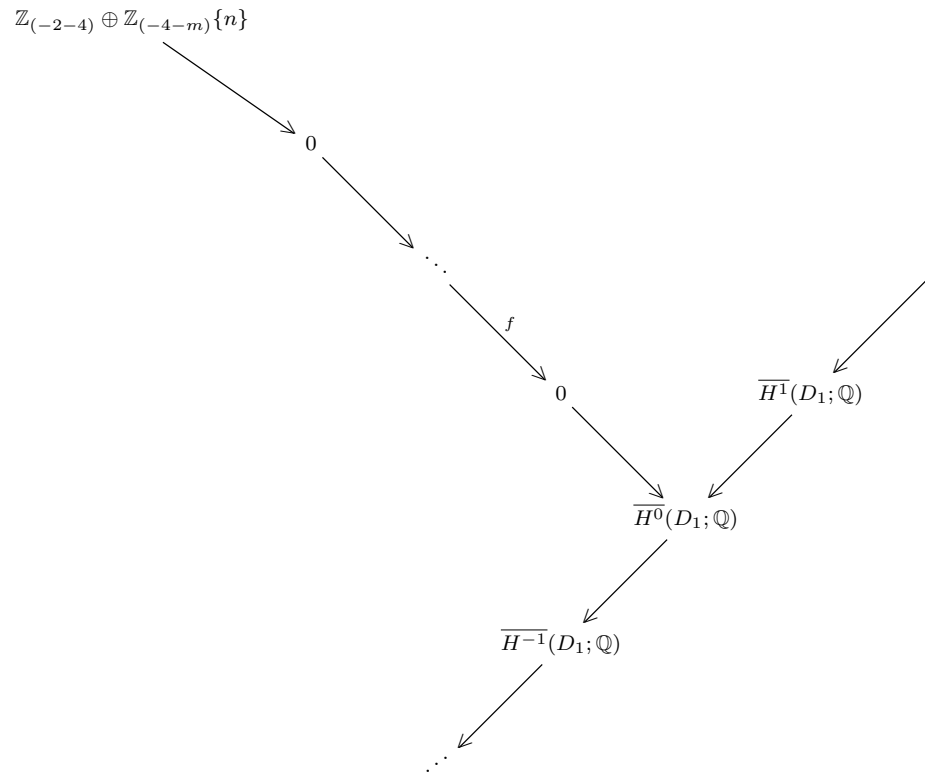


FIGURE 31. Commutative diagram

The top homology group corresponds to the kernel of some map ( $f$  or  $g$ , which have the same kernel), quotiented out by nothing. Where the  $n$  is the number of twists used in the construction of the symmetric union. This is the first time we've mentioned it, but as we see, it is important. As we increase  $n$ , we decrease the internal and external grading on this top homology group. (This is shown in the diagram of Figure 31.) However, since the number of crossings also increases, the shifts will change, so that the external grading doesn't change (though the homology groups around it do), and the internal grading actually decreases. It is this decrease by  $n$  that characterizes these symmetric unions for us. In general, we have:

$$H^i(K(m, n), \mathbb{Q}) \cong \begin{cases} H^{i+n}(K(m, 0); \mathbb{Q}) \oplus \mathbb{Q}_{(-1-m-2n)} \oplus \mathbb{Q}_{(-3-m-2n)} & : i = -m - n - 2 \\ H^{i+2}(K(m, 2); \mathbb{Q}) & : i = -n - 4, -n - 5 \\ H^i(K(m, 0); \mathbb{Q}) & : i \neq -m - n - 2, -n - 4, -n - 5 \end{cases}$$

## 6. FURTHER RESEARCH

So we were able to show that Khovanov Homology can determine the winding number used in the construction of the symmetric union, but only for some knots. It is a point of further research to make a much more general statement. In the two families of knots investigated, there was a very nice property that changing the crossings in the union to a connect sum produced the unknot. This will happen with a great deal of symmetric unions. Because the two families had this property, they shared many properties that should hold for any knot with this property. A nice generalization would be to determine what additional properties (if any) must hold for Khovanov Homology to be able to determine the winding number used in the construction of the symmetric union. With these maps  $f$  and  $g$ , mere distinguishability (sparing some of this paper's detail) can occur under relatively relaxed conditions.

## REFERENCES

- [1] Dror Bar Natan. On khovanov's categorification of the jones polynomial. *Algebraic and Geometric Topology*, 2:337–370, 2002.
- [2] Mikhail Khovanov. A categorification of the jones polynomial. *Duke Mathematics Journal*, 101(3):359–426, 2000.
- [3] Shin'ichi Kinoshita and Hidetaka Terasaka. On unions of knots. *Osaka Mathematics Journal*, 9:131–153, 1957.
- [4] Christopher Lamm. On unions of knots. *Osaka Mathematics Journal*, 37:537–550, 2002.
- [5] Toshifumi Tanaka. Symmetric unions indistinguishable by knot floer and khovanov homology. In *Joint Meeting of the KMS and the AMS*, Special Session on Knot Theory and Related Topics, Ewha Womans University, Seoul, Korea, 2009.



## 7. APPENDIX

We show the aforementioned computation on the resolutions shown in Figure 32. We want to understand the image elements that come from these resolutions, the relationships that those induce on the homology groups, and the restriction that that places on our homology groups.

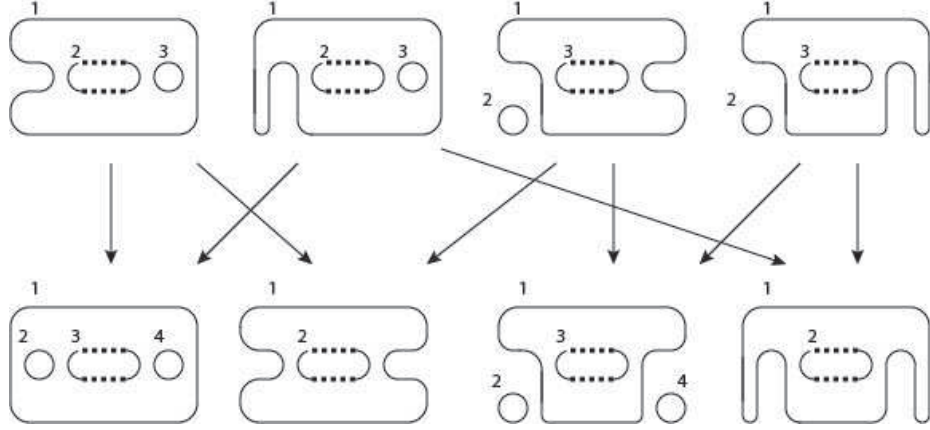


FIGURE 32. Some maps going into  $((\mathcal{A}_1)_{m-1}, (\mathcal{A}_2)_m, \mathcal{B}_m, \mathcal{C}_m)$ .

We first name the resolution elements. For these, we will agree to list the elements as  $((\mathcal{A}_1)_{m-1}, (\mathcal{A}_2)_m, \mathcal{B}_m, \mathcal{C}_m)$ , or going from left to right in Figure 32.

$$\begin{array}{ll}
 a = (\mathbf{1111}, 0, 0, 0) & b = (\mathbf{111X}, 0, 0, 0) \\
 c = (\mathbf{11X1}, 0, 0, 0) & d = (\mathbf{11XX}, 0, 0, 0) \\
 e = (\mathbf{1X11}, 0, 0, 0) & f = (\mathbf{1X1X}, 0, 0, 0) \\
 g = (\mathbf{1XX1}, 0, 0, 0) & h = (\mathbf{1XXX}, 0, 0, 0) \\
 i = (\mathbf{X111}, 0, 0, 0) & j = (\mathbf{X11X}, 0, 0, 0) \\
 k = (\mathbf{X1X1}, 0, 0, 0) & l = (\mathbf{X1XX}, 0, 0, 0) \\
 m = (\mathbf{XX11}, 0, 0, 0) & n = (\mathbf{XX1X}, 0, 0, 0) \\
 o = (\mathbf{XXX1}, 0, 0, 0) & p = (\mathbf{XXXX}, 0, 0, 0)
 \end{array}$$

$$\begin{array}{ll}
q = (0, \mathbf{11}, 0, 0) & r = (0, \mathbf{1X}, 0, 0) \\
s = (0, X\mathbf{1}, 0, 0) & t = (0, XX, 0, 0) \\
A = (0, 0, \mathbf{1111}, 0) & B = (0, 0, \mathbf{111X}, 0) \\
C = (0, 0, \mathbf{11X1}, 0) & D = (0, 0, \mathbf{11XX}, 0) \\
E = (0, 0, \mathbf{1X11}, 0) & F = (0, 0, \mathbf{1X1X}, 0) \\
G = (0, 0, \mathbf{1XX1}, 0) & H = (0, 0, \mathbf{1XXX}, 0) \\
I = (0, 0, X\mathbf{111}, 0) & J = (0, 0, X\mathbf{11X}, 0) \\
K = (0, 0, X\mathbf{1X1}, 0) & L = (0, 0, X\mathbf{1XX}, 0) \\
M = (0, 0, XX\mathbf{11}, 0) & N = (0, 0, XX\mathbf{1X}, 0) \\
O = (0, 0, XXX\mathbf{1}, 0) & P = (0, 0, XXXX, 0) \\
Q = (0, 0, 0, \mathbf{11}) & R = (0, 0, 0, \mathbf{1X}) \\
S = (0, 0, 0, X\mathbf{1}) & T = (0, 0, 0, XX)
\end{array}$$

The image elements are:

$$\begin{array}{lll}
j & l & n \\
p & b + i & d + k \\
f + m & n + o & r + s \\
t & B + C & D \\
F + G & H & J + K \\
L & N + O & P \\
R + S & T & c + i - q
\end{array}$$

$$\begin{array}{ccc}
d + j - r & g + m - s & h + n - t \\
k - s & l - t & c + i - Q \\
d + j - R & g + m - S & h + n - T \\
k - S & h - T & C + E + q \\
D + F + r & K + M + s & L + N + t \\
G + s & H + t & C + E - Q \\
B + F - R & K + M - S & L + N - T \\
G - S & H - T & o \\
p & O & P
\end{array}$$

This yields the following variable identifications:

$$\begin{array}{ccc}
g \rightarrow f + d & h \rightarrow 0 & i \rightarrow -b \\
j \rightarrow 0 & k \rightarrow -d & l \rightarrow 0 \\
m \rightarrow -f & n \rightarrow 0 & o \rightarrow 0 \\
p \rightarrow 0 & q \rightarrow c - b & r \rightarrow d \\
s \rightarrow k & t \rightarrow 0 & C \rightarrow -B \\
D \rightarrow 0 & E \rightarrow B - c + b & F \rightarrow -d \\
G \rightarrow -F & H \rightarrow 0 & K \rightarrow -J \\
L \rightarrow 0 & M \rightarrow J + D & N \rightarrow 0 \\
O \rightarrow 0 & P \rightarrow 0 & Q \rightarrow c - b \\
R \rightarrow d & S \rightarrow k & T \rightarrow 0
\end{array}$$

This exhausts all of the image elements, leaving only  $2d$  and  $2(b-c)$ , after identifications. So any homology element can be written in terms of only  $a, b, c, d, e, f, A, B, I$ , and  $J$ . With these identifications we can write:

$$X \equiv e + i + q + E + I + Q$$

$$Y \equiv g + k + s + M + S$$

$$Z \equiv g + k + s + G + K + S$$

$$W \equiv O$$

$$f(X) \equiv g + k - f - j + s - r + G + K - F - J + S - R \equiv -2J$$

$$f(Y) \equiv -h - l - t + O - N - T \equiv 0$$

$$f(Z) \equiv -h - l - t - H - L - T \equiv 0$$

$$f(W) \equiv -P \equiv 0$$

$$f(X) \equiv g + k + f + j + s + r + G + K + F + J + S - R \equiv 2f$$

$$f(Y) \equiv h + l + t + O + N + T \equiv 0$$

$$f(Z) \equiv h + l + t + H + L + T \equiv 0$$

$$f(W) \equiv P \equiv 0$$

Which suffices to show that  $f(X)$  and  $g(X)$  must be nonzero.