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Income Inequality, Externalities, and Housing Segregation

A thesis submitted in partial fulfillment of the
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By

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Abstract

The relationship between household income and housing segregation is examined, with a focus on the idea that higher income households disproportionately generate positive externalities that increase housing prices. A review of existing literature is provided. A series of models are also developed to capture and examine theory. The primary finding in the model is that, if higher income households do disproportionately generate positive externalities relative to lower income households, then the externalities can significantly exacerbate housing segregation that is naturally caused by income inequality.

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Introduction

The aim of this thesis is to examine the idea that income inequality, housing prices, and residential housing segregation are linked. It is apparent that higher income families can more readily afford to locate in neighborhoods with higher housing prices. The more subtle link examined here is based up on the proposition that higher income individuals also tend to disproportionately emit positive externalities rather than negative externalities. If higher incomes are disproportionately linked to positive externalities, and if lower income families cannot afford to locate into higher priced housing, then it is sensible to think higher income families disproportionately benefit from the positive externalities and segregation will be exacerbated by the positive externalities.

This thesis reviews literature related to this idea of interest, and various versions of a relatively simple dynamic model are presented to capture and examine the theory.

Literature Review

This section reviews research in economics and sociology on income inequality and externalities, particularly as to how they relate to residential segregation.

Scarpa (2015) examined the potential causality from income inequality to residential segregation. He examined the city of Malmö, Sweden from the years of 1991-2010. He chose this city due to the popular notion that the expanding gap of living conditions between neighborhoods was being widely caused by immigrant inflow. He found that there were many other factors that needed to be taken into account in order to understand residential segregation.

To test the immigration hypothesis, Scarpa divided the population into two groups: those born in Sweden and those born in countries outside of Western Europe and the Nordic countries. For the purposes of this analysis, no other country of origins were taken into account. Scarpa's data set contained information on income, employment status, demographic and household characteristics, and neighborhood of residence for all agents permanently living in Sweden. He found that changes in immigration only accounted for a small amount of the observed income inequality. The major driver of the increase in household income inequality was labor market earnings which were becoming more unequally distributed over time. He noted that the bottom of the income distribution accounted for most of the increase in income inequality, which suggests that poorer neighborhoods especially did not experience increases in incomes during the observed time period.

Wheeler and La Jeunesse (2008) looked at income inequality within residential neighborhoods in the United States from 1980-2000. They seperated their data into two geographic units: block group and a tract group, where the block group represented on average 526.5 households with an approximate median land area of 0.33 square miles as compared to a tract group which was comprised of on average 1,648.8 households with an approximate median land area of 1.31 square miles. Because the block group was smaller, it was easier to use for the analysis. They used three measures, between-neighborhood variation, within-neighborhood varation, and an overall income variance, to understand the overall trends of income inequality. All three of these measures grew, indicating more inequality.

These increases in variation correspond to the overall rise in income disparity in the U.S. during this time. Table 1 summarizes their findings. The mean of the within-block components shows an increase as well with the largest increase taking place from 1980 to 1990. The between block group components showed very sharp increase between 1980 to 1990 as well but this decreased by 0.01 the next decade. The most interesting finding was that within neighborhood inequality accounted for more than three quarters of the overall variance observed. This showed that within a small array of housing units, considerable income and wealth disparateness exists.

Table 1: Block Group Income Inequality

Year	Variable	Mean	Standard Deviation	Minimum	Maximum
1980	Variance	0.55	0.06	0.43	0.75
	Within Component	0.47	0.05	0.37	0.64
	Between Component	0.07	0.04	0.003	0.24
	Within Share of Variance	0.87	0.06	0.68	0.99
	Between Share of Variance	0.13	0.06	0.006	0.32
1990	Variance	0.64	0.07	0.48	0.94
	Within Component	0.5	0.05	0.39	0.65
	Between Component	0.14	0.05	0.04	0.31
	Within Share of Variance	0.79	0.06	0.61	0.92
	Between Share of Variance	0.21	0.06	0.08	0.39
2000	Variance	0.65	0.08	0.48	1.05
	Within Component	0.52	0.05	0.41	0.7
	Between Component	0.13	0.05	0.02	0.38

	Within Share of Variance	0.8	0.06	0.64	0.95
	Between Share of Variance	0.2	0.06	0.05	0.36

Wheeler and La Jeunesse (2008) found that a great portion of a sampled metropolitan areas' househould income inequality was associated with variation amidst housholds residing in the same block group. They studied several metropolitan cities and found that the correlation between metropolitan area-level income variance and both the between-neighborhood inequality and within-neighborhood inequality was large. This suggests that income inequality was prevalent between neighborhoods as well as within neighborhoods. However, it was noted that metropolitan areas with high levels of within-neighborhood income variation did not necessarily mean they sustained high levels of between-neighborhood variation. They also noted that a great extent of the overall income variation could be linked to within-tract income variation rather than between-tract variation. This suggests that the disparities in income were more prevalent within larger geographic areas compared to between larger geographic areas. Table 2 presents a summary of their findings.

Wheeler and La Jeunesse (2008) concluded that using block groups as opposed to tract groups helped better understand income segregation and how it perpetuates income inequality. Their block group results show that the degree of income segregation across neighborhoods has increased. That is residental segregation and rising income inequality are correlated.

Table 2: Metropolitan Areas with Highest and Lowest Within-Block Group Inequality

Top 15 Metro Areas	Average Inequality 1980-2000	Bottom 15 Metro Areas	Average Inequality 1980-2000
Santa Cruz—Watsonville, CA	0.619	Springfield, OH.	0.436
College Station—Bryan, TX.	0.618	Sheboygan, WI.	0.436
McAllen—Edinburg—Mission, TX.	0.614	Fond du Lac, WI.	0.434
Bridgeport—Stanford—Norwalk, CT.	0.613	Muskegon—Northern Shores, MI.	0.433
Morgantown, WV.	0.608	Colorado Springs, CO.	0.432
Lafayette, LA.	0.594	Salt Lake City, UT.	0.432
Athens – Clarke County, GA.	0.591	Virginia Beach—Norfolk—Newport News, VA.	04.31
Laredo, TX.	0.591	Vallejo—Fairfield, CA.	0.431
New York – Northern New Jersey – Long Island NY—NJ—PA.	0.591	Columbus, OH.	0.431
Brownsville—Harlingen, TX.	0.589	York—Hanover, PA.	0.428
Ithaca, NY.	0.588	Provo—Orem, UT.	0.425
Santa Fe, NM.	0.588	Fort Wayne, IN.	0.423
Greenville, NC.	0.587	Jacksonville, NC.	0.417
Hattiesburg, MS.	0.579	Warner Robins, GA.	0.402

Gainesville, FL.	0.578	Ogden—Clearfield, UT.	0.392

Frank (2013) explains that the increased spending from those at the top of the income distribution is largely affecting the individuals below them. He asserts that the increasing price of the median home initiates a problem for middle-class families due to the established fact that the quality of schools in the U.S. are connected to the area's property taxes, which is dependent upon the area's real estate prices. He assumes that middle-class families want their children to go to higher quality schools in order to better their lives. Due to the fact that this tends to be the case, he argues that this leads to psyhcological and economic stress put on to middle-class families who are forced to purchase a more expensive and larger home than they would have ever needed in order to send their children to the approriate schools. He goes into detail explaining the trend that individuals living in neighborhoods will be pressured into keeping up with their neighbors in order to sustain the reputation of the community or neighborhood. This in turn, adds more undo stress to middle-class families, needing to purchase certain cars, wear certain clothes, and be in certain professions. For the purpose of this thesis, it is significant that Frank links income, neighborhood location, but then also the positive effects of quality schooling in the more expensive neighborhoods.

Benabou and Ok (2001) established the Prospect of Upward Mobility Hypothesis (POUMH) which states that individuals who are poorer than average will not support income redistribution measures because the individual rationally expects to be richer than the average in the future. The results hold when three major assumptions are considered;

the mobility process is concave in expectations, the assumed redistribution policies are anticipated to last for a rather long period, and individuals do not display high risk aversion. They suggested that income inequality would be present because of these factors. After theorizing this hypothesis, they went on to test it using US mobility data. They found evidence supporting their hypothesis. They also discovered that those in favor of income redistribution, or a higher tax rate, valued it as a social insurance more than anything else.

Checchi and Filippin (2004) took the POUMH and simulated it using 95 undergraduates at the University of Milan. They found that when faced with the decision to support or reject a tax, those with below average income declined the tax expecting to earn a higher than average income in the next period. This result supports the theoretical POUMH as Benabou and Ok predicted.

Particularly relevant to this thesis, Solari (2012) evaluated the wealth and poverty in neighborhoods as a factor maintaining social and economic inequality. She found that those in affluent neighborhoods shaped their neighborhood positively in a variety of ways, including the provision of quality of public schools, safety, and residential privacy. She found that the reputation of more affluent neighborhoods compounded the positive, with reputation playing a role of maintaining the stability of the neighborhood. She found that reputation creates a hieracrchy in which those at the top of the hierarchy distance themselves from the disadvantaged groups both spatially and socially.

Solari (2012) also reviewed data of the US population at the city level and divided them up based upon several characteristics. Table 3 shows several characteristics found in both wealthy and poor neighborhoods.

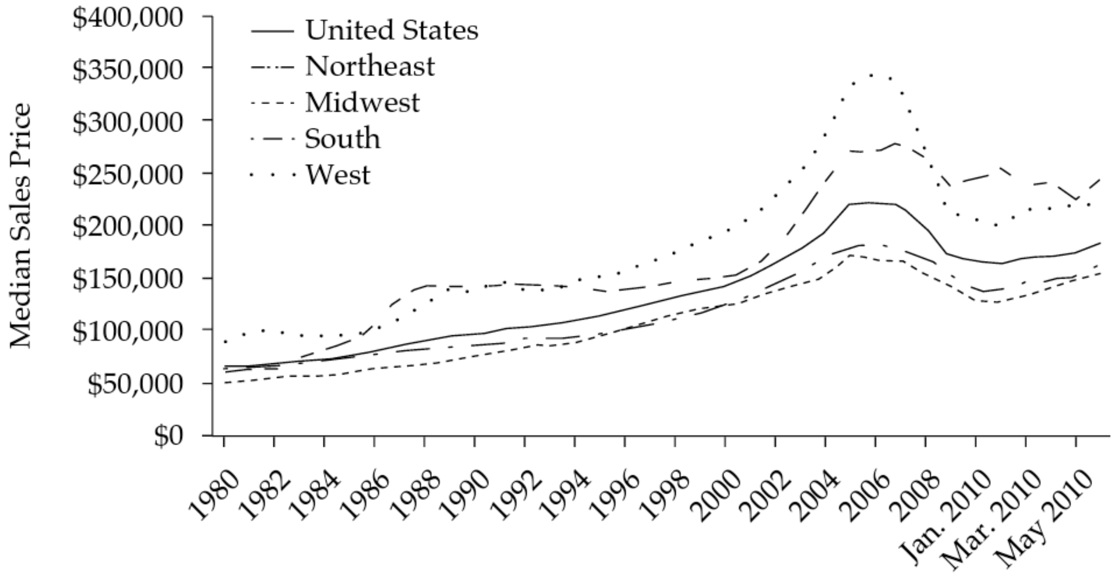
Table 3: Affluent and Poor Median Neighborhood Charateristics, MSAs 1970-2010							
All MSAs	Income ($)	Percent Unemployed	Percent White	Percent Black	Percent College+	Percent Professional	Percent Female Headed
Affluent 1970	102,962	2.5	99.2	0.3	29.2	46	5.9
Poor 1970	38,186	6.5	78.2	18.6	3.4	11	25.9
Difference	64,776	-4	21	-18.3	25.8	34.9	-20
Affluent in 1980	105,993	3.3	96.9	0.7	37.7	44.8	8.2
Poor in 1980	34,101	10.5	57.5	19.6	6.1	14.7	39.9
Difference	71,891	-7.1	39.5	-18.9	31.6	30.1	-31.7
Percent Change 1970-1980	**11**	**80**	**88**	**3**	**23**	**-14**	**58**
Affluent in 1990	127,118	3	95.1	1.1	46.4	50.1	9.3
Poor in 1990	34,664	13.4	40.2	29.4	6.5	16.2	48.6
Difference	92,453	-10.4	54.9	-28.3	39.9	33.9	-39.3
Percent Change 1980-1990	**29**	**46**	**39**	**50**	**26**	**13**	**24**
Affluent 2000	139,150	2.6	92.5	1.7	54.6	56	9.6
Poor 2000	38,209	12.4	38.8	30.5	7.9	17.3	48.3
Difference	100,942	-9.8	53.7	-28.8	46.6	38.7	-39.7
Percent	**9**	**-5**	**-2**	**2**	**17**	**14**	**-1**

Change 1990-2000							
Affluent 2010	145,073	4.5	85.3	1.5	57.3	56.3	4.5
Poor 2010	32,099	14.2	16.3	16.3	10.3	18.3	31.1
Difference	112,974	-9.7	69	-24.6	47	37.9	-26.6
Percent Change 2000-2010	**12**	**-1**	**29**	**-15**	**1**	**-2**	**-31**
Percent Change 1970-2010	**74**	**146**	**228**	**34**	**82**	**9**	**33**

Solari (2012) found that the income gap between wealthy and poor neighborhoods increased by 74 percent from 1970 to 2010. She attributes this increase in the gap to rising median incomes in wealthy neighborhoods, but notes that between the years of 2000 to 2010 this was due to both the rising median income in wealthy neighborhoods as well as decreasing median income in poor neighborhoods.

Wolff, Owens, and Burak (2011) looked at how the Great Recession affected neighborhoods and income. They noted that housing assets makes up a total of 30 percent of total assets for all Americans and 65 percent of total assets for the middle class; this immediately shows how vulnerable the middle class is during economics downturn. Figure 1 shows how dramatically home values depreciated following the Great Recession. The collapse in the value of homes led to a vast increase in the number of Americans facing a negative home equity. They found that those with less than twelve years of schooling has the smallest number of negative home equity occurances.

Figure 1: Median Home Sales Price by Region in the U.S.



Wolff, Owens, and Burak (2011) found an inverted U-shaped correlation between income and negative home equity occurances. That is, those with incomes between $50,000 to $75,000 displayed the greatest amount of negative home equity occurances. They attribute this to this particular income group having more available resources to take on more motgage debt than the poor. When compared to the rich, the rich did not need as much mortgage debt to afford their housing. It was noted that those at the top of the income distribution were hit during the Great Recession not through the housing bubble directly but by having their average net worth decrease. They reported that the top 50 of Forbes 400 list saw their average net worth drop by 17 percent between 2008 and 2009.

Smeeding, Thompson, Levanon, and Burak (2011) attribute the increase in poverty during the Great Recession directly to the labor market. They note that long-term unemployement rose during this time and in turn created less income for individuals leading them to an extended state of poverty. They do recognize that the recession did not increase poverty rate proportionally across all vulnerable groups, just that there was an overall increase in poverty for Americans during this time. They examined data from 1983 to 2010 and highlighted that changes in income inequality was strictly induced by increases in the top quantile of income shares.

Smeeding, Thompson, Levanon, and Burak (2011) highlight the point that the measurement of wealth is important in the assessment of income inequality. Housing is a primary asset for the typical American household, so housing is a factor impacting the distribution of wealth. A large source of income from wealth increases for the median household leading up to the Great Recession was from home ownership. During the Great Recession there was a drastic decrease in wealth since home values decreased immensely. They note that during the first two years after the Great Recession that income was quickly recovering but home values were still 30 percent below their 2006 values, which suggested a large lag in the recovery time.

Wissink, Schwanen, and Van Kempen (2016) argue that two developments have influenced residential segregation. First, numerous developments have been built using premium infrastructure available at only selective sites. These are usually in areas where the elite work, live, and spend their money. Second, people in general are increasingly mobile, but mobilitiy is unevenly available to different portions of the population.

In the literature that examines the relationship between income inequality and residential segregation, the Schelling model is continually referenced. Schelling (1971) analyzed individual incentives along with perceptions of differences that result into segregation collectively. His main motivation for this model was to better understand racial segregation, but this model can be used to look at several different types of segregation as well. In his model, he found that individuals do not mind being in the same neighborhood of those with a different race but that when given a choice, they would choose to segregate themselves from those of a different race over time. This finding is dependent upon the assumption that each race wishes to not be the minority group. He asserts that the only exception for this would be a mixture that is 50-50, but in every other case a minority group exists and complete segregation will occur. In the case that lower limits to the minority status that either color can accept exists, and if complete segregation occurs initially, then no agent will move to a neighborhood where they would be in the minority group. This then leads to the assertion that complete segregation is a stable equilibrium.

Zhang (2011) examines the Schelling Model further and uses a multi-neighborhood model to show that segregation persists in a world with individuals wanting to live in mixed-race neighborhoods. He suggests that segregation occurs not because of prejudicial preferences but because it is in fact a steady state. His model shows that this is a steady state due to the fact that the process of moving away from segregation includes the first step of one race, blacks, moving into a neighborhood filled with another race, whites. As survey data imply, individuals dislike feeling isolated in a neighborhood filled with those that are not of their race, this shows that individuals would end up moving back to the segregated state, even though they prefer being in a mixed neighborhood, showing that this indeed is a steady state.

Schelling (1971) also introduced the concept of neighborhood tipping, or a drastic change in which a new minority group enters a neighborhood in ample amounts to induce the previous residents to leave the neighborhood. An example of this would be a neighborhood changing from high income individuals to low income individuals, or white to black.

Anas (1980) used the Schelling model and showed that neighborhood tipping was caused by economic factors and not prejudicial preferences. Emphasizing the demand and supply interactions instead of prejudicial preferences, he creates a model that explains why a neighborhood tips to exogenous changes in the city; this model is heavily dependent on characteristics from the demand functions of the neighborhoods. This model shows that neighborhood tipping could be due to pure rational profit maximization. He illustrates that through a gradual exogenous change, such as rents increasing, it can lead to a discontinuous change in prices of homes and thus the neighborhood composition. He notes that the main results do not ignore the discrimination and prejudice but just shows that even in the absence of prejudicial externalities, neighborhood tipping still occurs due to continuous changes in residential preferences, population growth, and in income.

To summarize, the literature reviewed here does indicate there is a link between income inequality, housing prices, and residential segregation. With what follows, we seek to provide some modeling depth to this literature. In particular, we seek to provide a dynamic model which characterizes how the degree of neighborhood segregation may change as higher income households with more ability to move into higher priced neighborhoods disproportionately emit positive externalities which further reinforce the income inequality and segregation.

Modeling Neighborhood Dynamics

We seek a dynamic model that reasonably captures the theory and observations in the literature. We ended up finding a simple model that is reasonable, but not until trying another approach that seemed reasonable at first. For this thesis, we chose to present the evolution of the modeling we pursued, rather than just presenting the final product because there is learning that comes from understanding why certain modeling assumptions do not end up providing a reasonable model.

The first modeling approach we pursued assumed that the externality experienced by living in a neighborhood is the sum of the externalities emitted by individual homeowners in the neighborhood. The second approach assumed that the externality experienced depended upon the income threshold needed to live in the wealthier neighborhood. It turns out that the second approach provides a more reasonable model.

The problem with the first modeling approach is that it generates a dynamic pattern of two period cycles, where a neighborhood is higher priced and more exclusive one period but then lower priced and less exclusive in the next period. As noted above, this cycling is not observed; rather, a more affluent neighborhood tends to become exclusive and stay exclusive. Part of what is presented below are our efforts to modify the model so to eliminate the cycling. Unsatisfied with these efforts, we moved on to the second approach, which turned out to provide a simpler model that generates results more consistent with theory and observed housing dynamics.

An Initial Model

Consider a city with two neighborhoods indexed by j, where $j = A$ is the poorer neighborhood and $j = B$ is the wealthier neighborhood. The population is a continuum of agents, and income is uniformly distributed over these agents, ranging from y_{min} to y_{max}. We are interested in endogenously determining, for each time period t, the share of the continuum S_t^A living in the poorer neighborhood versus the share of S_t^A living in the wealthier neighborhood. In this section, we present evolution of modeling developed to show how neighborhood externalities affect housing prices and consequently housing segregation through housing prices.

Assume each agent emits an externality $z_{i,t+1}$ in period t+1 that is proportionate to the agent's period t income y_{it}, so

(1) $z_{i,t+1} = \theta y_{it}$,

where $0 < \theta < 1$ measures the sensitivity of externality to income. An externality emitted by an individual, $z_{i,t+1}$, can be thought of as a homeowner mowing their lawn, or keep their home aesthetically nice. An agent emits an externality in either neighborhood A or neighborhood B. Let Z_t^j denote the aggregate externality emitted in neighborhood $j = \{A, B\}$ during time period t.

Let P_t^j denote the price of housing in neighborhood j during time period t. Assume housing prices are proportionate to the neighborhood aggregate externality, so

(2) $P_t^A = \gamma Z_t^A$

and

(3) $P_t^B = \gamma Z_t^B$.

where γ is the fraction of the positive externality contributing to the price of the housing payment. An example of γ could be the government's effort to control the price of housing as to not make it unaffordable to poorer income individuals.

An agent's ability to afford a housing depends upon both the agent's income and the fraction ϕ of income an agent is able to spend on housing. To be able to afford to live in the wealthier neighborhood, the agent's income must satisfy,

(4) $P_t^B \leq \phi y_{it}$.

In the United States, to obtain a mortgage, a home buyer is usually not allowed to spend more than 35% of their income on the purchase price of the home, so in that case $\phi = .35$ would be representative. In essence, the lower ϕ is, the wealthier the agent is.

Because experiencing a positive externality enhances well-being, each agent prefers to live in the wealthier neighborhood, but not all agents can afford to do so. Substituting condition (3) into condition (4) we find that the income level $\hat{y}_t = \frac{\gamma}{\phi} Z_t^B$ is an important "threshold" income. In particular if,

$y_{it} \geq \hat{y}_t = \frac{\gamma}{\phi} Z_t^B$, then agent i can afford to live in neighborhood $j = B$;

$y_{it} \leq \hat{y}_t = \frac{\gamma}{\phi} Z_t^B$, then agent i can only afford to live in neighborhood $j = A$.

The externality Z_{t+1}^j experienced by any agent in neighborhood j in period $t + 1$ is the sum of all the individual externalities emitted. For neighborhood A, it is shown in the Appendix that the externality is:

(5) $Z_{t+1}^{A} = \int_{y_{min}}^{\hat{y}_t} [z_t^i] dy_t^i = \frac{\theta}{2}\left[\left[\frac{\gamma Z_t^B}{\phi}\right]^2 - [y_{min}]^2\right]$

(6) $Z_{t+1}^{B} = \int_{\hat{y}_t}^{y_{max}} [z_t^i] dy_t^i = \frac{\theta}{2}\left[[y_{max}]^2 - \left[\frac{\gamma Z_t^B}{\phi}\right]^2\right]$

The shares of agents living in the two neighborhoods are then given by

(7) $S_t^A = \frac{\hat{y}_t - y_{min}}{y_{max} - y_{min}}$

and

(8) $S_t^B = \frac{y_{max} - \hat{y}_t}{y_{max} - y_{min}}$.

Entering period t, equations (1)-(8) determine the endogenous variables P_t^A, P_t^B, $\hat{y}_t$, S_t^A, S_t^B, z_t^i, Z_{t+1}^A, and Z_{t+1}^B. The variables Z_t^A, and Z_t^B are predetermined. The variables $\phi, \gamma, \theta, y_{min}$, and y_{max} are exogenous.

A first step in the analysis is finding the steady state(s). Setting $Z_{t+1}^B = Z_t^B = Z$, we show in the Appendix that the steady state(s) are given by

(9) $Z = \frac{\phi^2}{2\gamma^2} \pm \frac{\sqrt{1+\frac{4\gamma^2\theta[y_{max}]^2}{\phi^2 \quad 2}}}{\frac{2\gamma^2}{\phi^2}}$.

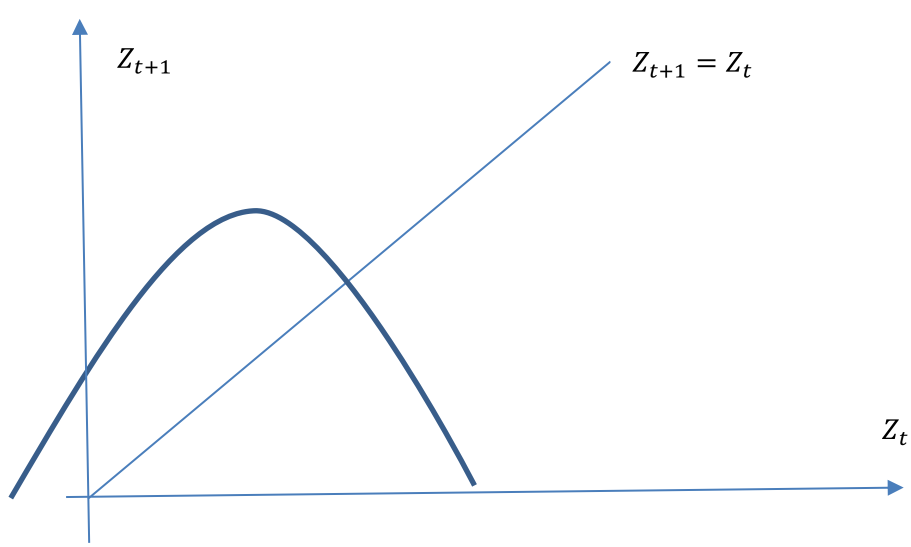
Since a negative value for the steady state is not meaningful, we will only consider the positive value of (9).

Condition (9) indicates the steady state value Z of the positive externality experienced in the wealthier neighborhood depends upon θ, ϕ, γ, and y_{max}. We also know the threshold income $\hat{y}_t$ increases as the wealthy neighborhood externality increases, and we know the share of households in the wealthy neighborhood decreases as the threshold income increases. Thus, we find that the following exogenous changes make the wealthy neighborhood more exclusive in the steady state:

- An increase in income increases the positive externality by a larger degree (higher θ);
- A larger share of income must be spent on housing is larger (higher ϕ);
- The externality has a smaller impact on the housing price (smaller γ);
- The degree of income inequality is larger (larger y_{max}).

A phase diagram analysis can be used to consider how the model behaves outside the steady state. Plotting the dynamic equation (6) along with the steady state line $Z_{t+1} = Z_t$, we obtain Figure 1. Since equation (6) is quadratic, we know it is symmetric. Since the dynamic equation has a negative when it strikes the steady state line, the phase diagram indicates the system will experience cycles. In the Appendix, it is shown that the slope must actually be less than negative 1. Therefore, we also know that the steady state is locally unstable, meaning the externalities experienced will move away from the steady state over time. Because this does not seem to describe what we observe in the real world, we will modify the model.

Figure 2: Dynamics (For model with $z_{i,t+1} = \theta y_{it}$)



Adding Diminishing Returns to Income

It is reasonable to think that the positive externality generated increases with income, as we assume with equation (1), but it is also reasonable to think there may be diminishing returns. To add a diminishing returns assumption, consider the model of the previous section with equation (1) replaced with

(10) $$z_t^i = \theta\sqrt{y_t^i}.$$

Our primary interest is to see whether adding diminishing returns eliminates the cycling indicated by the phase diagram of Figure 2.

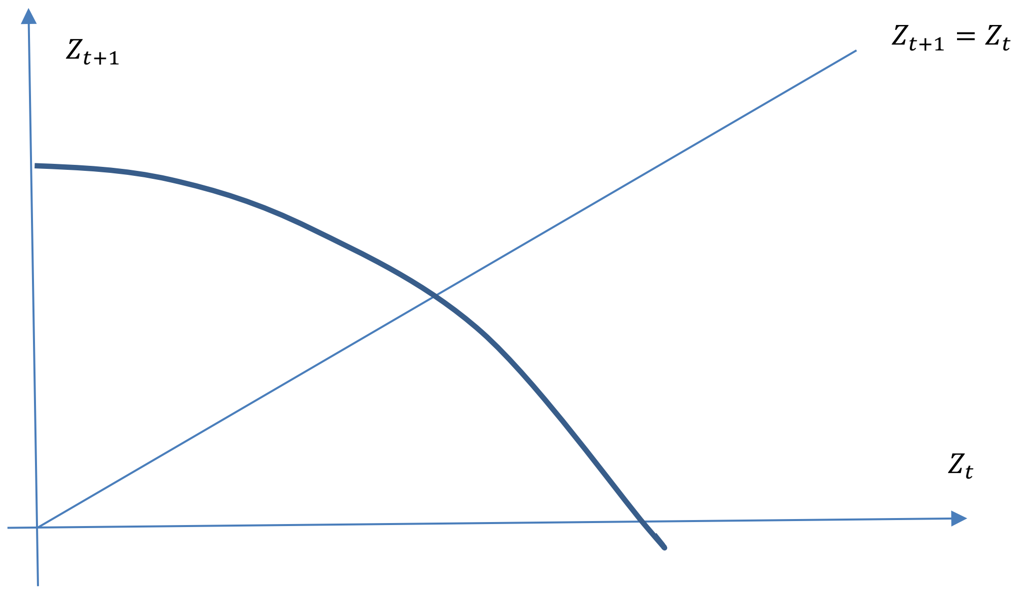
Using the new externality condition (10), we show in the Appendix that equations (5) and (6) become

(11) $$Z_{t+1}^A = \frac{2\theta}{3}\left[\left[\frac{\gamma Z_t^B}{\phi}\right]^{\frac{3}{2}} - [y_{min}]^{\frac{3}{2}}\right]$$

(12) $$Z_{t+1}^B = \frac{2\theta}{3}\left[[y_{max}]^{\frac{3}{2}} - \left[\frac{\gamma Z_t^B}{\phi}\right]^{\frac{3}{2}}\right]$$

It follows that $\frac{\partial}{\partial Z_t}[Z_{t+1}] = -\frac{\theta\gamma^{\frac{3}{2}}}{\phi^{\frac{3}{2}}}\sqrt{Z_t} < 0$. From this calculation, we learn that the primary modeling impact of introducing diminishing returns is the slope of the dynamic equation is now always negative. Also, the slope at the steady state may be greater than -1, so the steady state may be stable. However, the negative slope indicates two period cycles would occur. Because such cycling is not typically observed, it is reasonable to ask whether there is some assumption that will produce a locally stable steady state without cycling.

Figure 3: Dynamics (For model with $z_t^i = \theta \sqrt{y_t^i}$)



An Attempt to Eliminate Cycling

Cycling occurs because a high total externality level indicates a high price of housing, which reduces the share of the population living in the wealthier area. However, the reduction in population share living in the wealthy area reduces the externality emitted in the wealthy area, which reduces the price of housing in the wealthy area, which then increases the share of the population living in the wealthy area. It is conceivable that this cycling can be ameliorated if there is a threshold income beyond which an increase in income actually reduces the externality emitted. To examine this possibility, we explored replacing equation (1) with

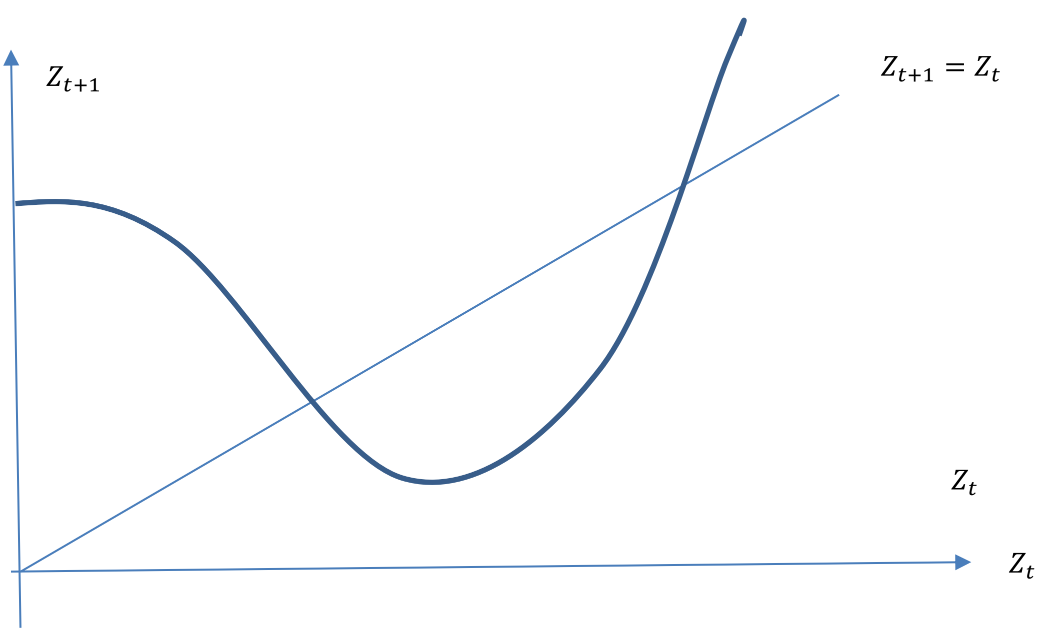
(13) $$z_t^i = \theta\left[\bar{y}^2 - \left[\bar{y} - y_t^i\right]^2\right]$$

This condition indicates that income has a positive impact on the externality up until $y_t^i = \bar{y}$, but then further increases in the income level y_t^i reduce the externality. In the Appendix, it is shown that this individual externality dynamic generates an externality in the wealthier neighborhood equal to

$$(14) \qquad Z_{t+1}^B = \frac{\theta}{3}\left[\frac{\gamma Z_t}{\phi}\right]^3 - \theta\bar{y}\left[\frac{\gamma Z_t}{\phi}\right]^2 + \left[\bar{y} - \frac{y_{max}}{3}\right]\theta[y_{max}]^2.$$

If we plot condition (14) in the standard phase diagram space, we find that the phase line will have a positive slope when $Z_t > \frac{\phi\bar{y}}{\gamma}$, so there is hope that the system could have a locally stable steady state with no cycling. However, using much time to examine the phase diagram and system using simulation with different parameters, we find that the dynamic (14) is very sensitive to small changes in the parameters. While we could find a set of parameters that generated a locally stable steady state, the parameter values needed were extreme, meaning they were not values one would expect in any real-world context.

Figure 4: Dynamics (For model with $z_t^i = \theta\left[\bar{y}^2 - \left[\bar{y} - y_t^i\right]^2\right]$)



A Second Modeling Approach: Threshold Income Externality

The previous model, which proposed that the positive externality experienced in a neighborhood is the sum of individual externalities emitted, is not able to produce a path for the aggregate externality nor a path for housing prices that fits what is observed. In particular, two period cycling is not observed in reality but characteristically appears with the first modeling approach. After some experimentation, we found that we can obtain a model that more reasonably characterizes what is observed if we assume the externality experienced in a neighborhood depends upon the threshold income level needed to live in the neighborhood. In essence, we are stating that the individual already lives in either neighborhood A or B, and now experiences the externalities present in the neighborhood in which they reside in.

Because we work only with the aggregate externality, we no longer use equation (1) in the model. We continue to relate housing prices to the neighborhood aggregate externality, but because it helps create a more reasonable model we now introduce the idea that some factors independent of the externality influence the housing price, so we adjust equations (2) and (3) to become

(15) $P_t^A = \bar{P}^A + \gamma Z_t^A$

and

(16) $P_t^B = \bar{P}^B + \gamma Z_t^B$.

Both $\bar{P}^A$ and $\bar{P}^B$ could be viewed as differences in locations, views, distance to a busy roadway, or the city's airport having a flight path above the neighborhood or not.

Keeping condition (4) on an agent's ability to afford a housing, we find that the income threshold level for living in the wealthier neighborhood becomes $\hat{y}_t = \frac{\bar{P}^B}{\phi} + \frac{\gamma}{\phi} Z_t^B$. The primary change in the model is that the externality Z_{t+1}^j experienced by any agent in neighborhood j in period $t+1$ is no longer the sum of individual externalities emitted, but rather depends only upon the income threshold. We implement this new assumption by replacing equations (5) and (6) with

(17) $Z_{t+1}^A = \theta^A \hat{y}_t$

and

$$Z_{t+1}^B = \theta^B \hat{y}_t.$$

We now have a new, fully specified model. Entering period t, equations (4), (7), (8), (15),(16), (17), 0, and (18) determine the endogenous variables $\hat{y}_t$, P_t^A, P_t^B, S_t^A, S_t^B, Z_{t+1}^A, and Z_{t+1}^B. The variables Z_t^A, and Z_t^B are predetermined. The variables ϕ, γ, θ^A, θ^B, y_{min}, and y_{max} are exogenous.

Given that the threshold level of income is $\hat{y}_t = \frac{\bar{P}^B}{\phi} + \frac{\gamma}{\phi} Z_t^B$, condition 0 becomes

(18) $Z_{t+1}^B = \frac{\theta^B \bar{P}^B}{\phi} + \frac{\theta^B \gamma}{\phi} Z_t^B$.

Condition (18) is the basic dynamic equation for the model.

The steady state for the new model is the state where $Z_{t+1}^B = Z_t^B = Z^B$. Using condition (18), we find

(19) $Z^B = \frac{\theta^B \bar{P}^B}{\phi - \theta^B \gamma}$.

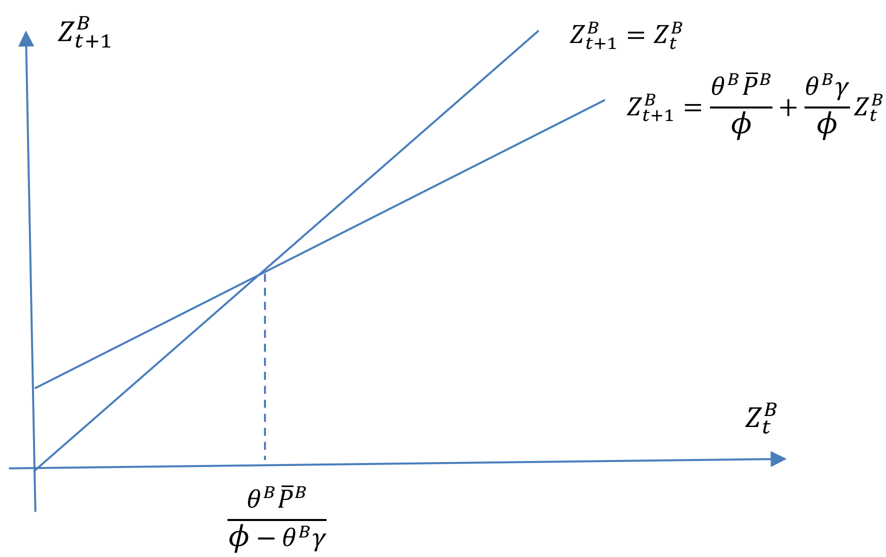
We can only have a positive steady state, then, if we maintain the restriction

(20) $\phi > \theta^B \gamma$.

From condition (20), we learn that the share of income that must be allocated toward housing must be large relative to the effects of income on externality and externality on price. If this condition does not hold, then the positive externality experienced in the wealthier neighborhood will only increase over time.

When condition (20) holds, we obtain a phase diagram like that shown in Figure 5. Because the slope of the phase line at the steady state is $0 < \frac{\theta^B \gamma}{\phi} < 1$, we know the steady state is stable. If the wealthy neighborhood is not experiencing the positive externality given by the steady state (19), then it is headed toward that condition.

Figure 5: Dynamics (For model with $Z^B_{t+1} = \theta^B \hat{y}_t$)



Using (7) from the initial model along with (8), the shares of agents in each neighborhood can be found. Assuming that we are in a steady state in the fundamentally different model, we will use $\hat{y}_t = \frac{\bar{P}^B}{\phi} + \frac{\gamma}{\phi} Z^B_t$ to substitute in to both (7) and (8). We see that

$Z_{t+1}^B = Z_t^B = Z^B$ so then equation (19) can be used. The threshold income equation now transforms into

(21) $$\hat{y}_t = \frac{\bar{P}^B}{\phi}\left[1 + \frac{\gamma\theta^B}{\phi - \theta^B\gamma}\right] = \left[\frac{\bar{P}^B}{\phi - \theta^B\gamma}\right]$$

Now reducing this down, and inserting it into equations (7) and (8), we obtain

(22) $$S_t^A = \frac{\frac{\bar{P}^B}{\phi - \theta^B\gamma} - y_{min}}{y_{max} - y_{min}}$$

and

(23) $$S_t^B = \frac{y_{max} - \frac{\bar{P}^B}{\phi - \theta^B\gamma}}{y_{max} - y_{min}}.$$

Condition (21) indicates the threshold income level $\hat{y}_t$ in the steady state depends upon θ^B, ϕ, γ, and $\bar{P}^B$. Again, as the threshold income $\hat{y}_t$ increases, the wealthy neighborhood externality increases, and we know the share of households in the wealthy neighborhood decreases as the threshold income increases. Thus, we find that the following exogenous changes make the wealthy neighborhood more exclusive in the steady state:

- An increase in the positive externality associated with the wealthier neighborhood (higher θ^B);
- The externality has a larger impact on the housing price (larger γ);
- A smaller share of income must be spent on housing (smaller ϕ);
- Factors other than the externality have a greater impact that housing price (larger $\bar{P}^B$).

It is worthwhile to examine how the steady state degree of housing segregation will change as the impact of the externality changes. Since the income threshold for living in the wealthier neighborhood is $\hat{y}_t = \frac{\bar{P}^B}{\phi - \theta^B \gamma}$, the threshold would be $\hat{y}_t = \frac{\bar{P}^B}{\phi}$ if either there is no positive externality emitted (i.e., $\theta^B = 0$) or if the externality has no impact on the housing price (i.e., $\gamma = 0$). For illustration, suppose $\frac{\bar{P}^B}{\phi} = \frac{y_{max} - y_{min}}{2}$. This would imply half of households would live in the wealthier neighborhood in the steady state and half in the poorer neighborhood. If $\bar{P}^B$ increases, or exogenous factors contributing to the price of the home are valued more or if ϕ decreases, or the fraction of income agents spend on their housing payment decreases, $\hat{y}_t$ will increase leading to more housing segregation. If θ^B and γ were each positive but each very near zero, housing segregation would be very slight. However, as either θ^B or γ increases, the income threshold $\hat{y}_t$ increases and housing segregation becomes more significant. As $\hat{y}_t$ approaches y_{max}, the degree of segregation becomes extreme.

Policy Implications

If policy makers seek to mitigate housing segregation, this work indicates government might in a poorer neighborhood seek to provide what the positive externalities provide in wealthier neighborhoods. By breaking down the second modeling approach, it can be better understood how the government would be able to achieve this.

The most obvious example of government increasing positive externalities in poorer neighborhoods could be by spreading the wealth from the property taxes obtained in wealthier neighborhoods and using them in poorer neighborhoods to make those schools better. Another way the government could mitigate this problem could be by implementing more parks or community centers in poorer neighborhoods to help children during the summer as well as after school have places to go rather than going out and potentially causing trouble without adult supervision present.

It was understood that $\bar{P}^B$ affects segregation in an exogenous way. A great example of $\bar{P}^B$ would be airplanes flying over certain neighborhoods. If the government worked closely with cities to find better areas for these airplanes to fly over, it could potentially help with the problem of residential segregation.

The model also brings in the variable, ϕ, which is related to the percentage of income needed to be spent on the housing payment. If government could provide programs to help homeowners become better equipped to handle their housing payments such as loan programs, this could also help with the problem of residential segregation.

The model used γ to help explain how the aggregate externality affects the price of housing in either neighborhood A or B. A way government could help mitigate the problem of residential segregation would be to introducing rent controls. This would mitigate the problem because it would help lower income families not be priced out of neighborhoods.

Conclusion

This thesis has explored the relationship between income inequality, housing prices, and housing segregation.

The literature reviewed here is eclectic. Some does not directly link housing segregation and income inequality. In particular, the literature indicates there are causes of income inequality beyond housing segregation, and income inequality can cause housing segregation by the mere fact that higher income households can afford higher priced homes in nicer neighborhoods. However, some researchers have found evidence that housing segregation exacerbates income inequality by localizing externalities. Most notably, wealthier neighborhoods tend to provide better schools and better schools in wealthier areas logically exacerbate income inequality. However, other externalities like enhanced safety have also been associated with higher income.

The modeling task pursued in this thesis has been to capture the theory that positive externalities associated with higher income exacerbate housing segregation. The core of the theory involves three assumptions. First, higher income households tend to emit more positive externalities and fewer negative externalities than lower income households. Second, there is a threshold income needed to be able to afford a home in a higher priced neighborhood. Third, the positive externalities make the higher income neighborhood attractive, so housing prices are higher when the externality experienced in the neighborhood is higher. The dynamic model built captures how the share of the population living in the wealthier of two neighborhoods changes over time as the externality level in the wealthier neighborhood changes.

To keep the model tractable, we take the income distribution as given. That is, the model does not explain changes in the income distribution. Rather, the fact that some are richer and some are poor by assumption implies some can afford to live in the wealthier neighborhood with homes that are higher priced.

When we assume that the externality experienced in a neighborhood is the sum of individual externalities emitted by households, we obtain a two-period cycle. When the housing price is lower, a larger share of the population can afford to live in the wealthier neighborhood, and this creates a greater total positive externality experienced in the neighborhood. The high positive externality then makes the wealthy neighborhood more attractive so the housing price increases. However, the increase in the housing price reduces the share of the population that can afford to live in the wealthier neighborhood, and this decline in population share reduces the externality experienced in the wealthier neighborhood. Examining three different assumptions regarding how income generates a positive externality, the externality experienced in the wealthy neighborhood is higher in one period but then lower in the next, which implies housing prices also oscillate, which implies the population share living in each neighborhood oscillates.

Because relative stability, rather than oscillation, is observed, another modeling approach was sought. We found a reasonable model by assuming that the externality experienced in the wealthy neighborhood is directly related to the income threshold needed to live in the wealthy neighborhood. Under this assumption, the externality level, the housing price, and the population share living in the wealthy neighborhood will monotonically adjust from their initial levels and converge to steady state levels. That is, the model indicates the degree of housing segregation will tend to be stable as is roughly observed.

Even though the housing market in the model is relatively stable, the model provides insight into why the housing market can become more segregated, meaning a smaller share of the population living in the wealthier neighborhood. One factor is the degree to which there is a positive externality emitted in the wealthier neighborhood that is greater than in the poorer neighborhood. If the society where in a steady state and the only change experienced is an increase in the positive externality generated in the wealthier neighborhood, the model predicts housing prices in that neighborhood will increase and the share living in that neighborhood will decrease. Fewer wealthier people are separated from the masses. A comparable effect occurs when the positive externality has a stronger impact on the housing price, when the population cannot afford to spend as large a share of income on housing, or when some other factor changes that increases the price of housing.

To conclude, the primary effect captured by the model that is not obvious, is the impact of a positive externality on housing segregation through the housing price. Higher income alone will tend to lead to housing segregation. However, when higher incomes generate positive externalities and when the positive externalities increase housing prices, the degree of housing segregation must be greater than if no positive externalities are generated.

Works Cited

Anas, A. (1980). A Model of Residential Change and Neighborhood Tipping. *Journal of Urban Economics*, 358-370.

Benabou, R., & Ok, E. A. (2001, May). Social Mobility and the Demand for Redistribution: The Poum Hypothesis. *The Quarterly Journal of Economics, 116*(2), 447-487.

Checchi, D., & Filippin, A. (2004). An Experimental Study of the POUM Hypothesis. *Research on Economic Inequality, 11*, 115-136.

Frank, R. H. (2013). *Falling Behind: How Rising Inequality Harms the Middle Class.* Berkeley and Los Angeles: University of California Press.

Scarpa, S. (2015). The Impact of Income Inequality on Economic Residential Segregation: The case of Malmo, 1991–2010. *Urban Studies, 52(5)*, 906-922.

Schelling, T. C. (1971). Dynamic Models of Segregation. *Journal of Mathematical Sociology, 1*, 143-186.

Smeeding, T. M., Thompson, J. P., Levanon, A., & Burak, E. (2011). Poverty and Income Inequality in the Early Stages of the Great Recession. In D. B. Grusky, B. Western, & C. Wimer, *The Great Recession* (pp. 82-126). New York: Russell Sage Foundation.

Solari, C. D. (2012, December). Affluent Neighborhood Persistence and Change in U.S. Cities. *City and Community*, 370-380.

Wheeler, C. H., & La Jeunesse, E. A. (2008). Trends in Neighborhood Income Inequality In The U.S.: 1980-2000. *Journal of Regional Science, 48*(5), 879-891.

Wissink, B., Schwanen, T., & van Kempen, R. (2016, August). Beyond Residential Segregation: Introduction. *Cities: The International Journal of Urban Policy and Planning*, 126-130.

Wolff, E. N., Owens, L. A., & Burak, E. (2011). How Much Wealth Was Destroyed in the Great Recession? In D. B. Grusky, B. Western, & C. Wimer, *The Great Recession* (pp. 127-158). New York: Russell Sage Foundation.

Zhang, J. (2011). Tipping and Residential Segregation: A Unified Schelling Model. *Journal of Regional Science*, 167-193.

Appendix

Derivations for the Initial Model:

(1) Derivation of Z_{t+1}^{A}:

$$Z_{t+1}^{A} = \int_{y_{min}}^{\hat{y}_t} [z_t^i] dy_t^i$$

$$= \int_{y_{min}}^{\hat{y}_t} [\theta y_t^i] dy_t^i$$

$$= \frac{\theta}{2} [y_t^i]^2 \Big|_{y_{min}}^{\hat{y}_t}$$

$$= \frac{\theta}{2} [\hat{y}_t]^2 - \frac{\theta}{2} [y_{min}]^2$$

$$= \frac{\theta}{2} \left[\frac{\gamma Z_t^B}{\phi}\right]^2 - \frac{\theta}{2} [y_{min}]^2$$

$$= \frac{\theta}{2} [[\hat{y}_t]^2 - [y_{min}]^2]$$

$$= \frac{\theta}{2} \left[\left[\frac{\gamma Z_t^B}{\phi}\right]^2 - [y_{min}]^2\right]$$

(2) Derivation of Z_{t+1}^{B}:

$$Z_{t+1}^{B} = \int_{\hat{y}_t}^{y_{max}} [z_t^i] dy_t^i$$

$$= \int_{\hat{y}_t}^{y_{max}} [\theta y_t^i] dy_t^i$$

$$= \frac{\theta}{2} [y_t^i]^2 \Big|_{\hat{y}_t}^{y_{max}}$$

$$= \frac{\theta}{2} [[y_{max}]^2 - [\hat{y}_t]^2]$$

$$= \frac{\theta}{2}\left[[y_{max}]^2 - \left[\frac{\gamma Z_t^B}{\phi}\right]^2\right]$$

(3) Derivation of Steady State:

$$Z_{t+1}^B = \frac{\theta}{2}\left[[y_{max}]^2 - \left[\frac{\gamma Z_t^B}{\phi}\right]^2\right]$$

Letting $Z_{t+1}^B = Z_t^B = Z$

$$Z = \frac{\theta}{2}\left[[y_{max}]^2 - \left[\frac{\gamma Z}{\phi}\right]^2\right]$$

$$Z = \frac{\theta[y_{max}]^2}{2} - \frac{\gamma^2}{\phi^2}Z^2$$

$$0 = -\frac{\gamma^2}{\phi^2}Z^2 - Z + \frac{\theta[y_{max}]^2}{2}$$

$$Z = \frac{1}{\frac{2\gamma^2}{\phi^2}} \pm \frac{\sqrt{1 + \frac{4\gamma^2}{\phi^2}\frac{\theta[y_{max}]^2}{2}}}{\frac{2\gamma^2}{\phi^2}}$$

$$Z = \frac{\phi^2}{2\gamma^2} \pm \frac{\sqrt{1 + \frac{4\gamma^2}{\phi^2}\frac{\theta[y_{max}]^2}{2}}}{\frac{2\gamma^2}{\phi^2}}$$

(4) Phase Diagram Analysis

$$Z_{t+1}^B = \frac{\theta}{2}\left[[y_{max}]^2 - \left[\frac{\gamma Z_t^B}{\phi}\right]^2\right]$$

Letting $Z_t^B = Z_t$

$$Z_{t+1} = \frac{\theta}{2}\left[[y_{max}]^2 - \left[\frac{\gamma Z_t}{\phi}\right]^2\right]$$

$$Z_{t+1} = \frac{\theta}{2}[y_{max}]^2 - \frac{\theta}{2}\left[\frac{\gamma Z_t}{\phi}\right]^2$$

$$Z_{t+1} = \frac{\theta}{2}[y_{max}]^2 - \frac{\theta\gamma^2}{2\phi^2}Z_t^2$$

(5) After analyzing Z_{t+1} at the steady state, we then took the derivative with respect to Z_t and found the following:

$$\frac{\partial}{\partial Z_t}[Z_{t+1}] = \frac{\partial}{\partial Z_t}\left[\frac{\theta}{2}[y_{max}]^2\right] - \frac{\partial}{\partial Z_t}\left[\frac{\theta\gamma^2}{2\phi^2}Z_t^2\right]$$

$$\frac{\partial}{\partial Z_t}[Z_{t+1}] = -\frac{\partial}{\partial Z_t}\left[\frac{\theta\gamma^2}{2\phi^2}Z_t^2\right]$$

$$\frac{\partial}{\partial Z_t}[Z_{t+1}] = -\frac{\partial}{\partial Z_t}\left[\frac{\theta\gamma^2}{2\phi^2}Z_t^2\right]$$

$$\frac{\partial}{\partial Z_t}[Z_{t+1}] = -\frac{\theta\gamma^2}{\phi^2}Z_t < 0$$

We then examined the slope at the steady state and obtained the following result

$$\frac{\partial}{\partial Z_t}[Z_{t+1}] = -\frac{\theta\gamma^2}{\phi^2}\left[\frac{\phi^2}{2\gamma^2} + \frac{\sqrt{1 + \frac{4\gamma^2}{\phi^2}\frac{\theta[y_{max}]^2}{2}}}{\frac{2\gamma^2}{\phi^2}}\right]$$

Now checking on the restrictions sufficient to provide stability, we obtained the following result

$$\frac{\partial}{\partial Z_t}[Z_{t+1}] > -1$$

$$\frac{\theta\gamma^2}{\phi^2}\left[\frac{\phi^2}{2\gamma^2}+\frac{\sqrt{1+\frac{4\gamma^2}{\phi^2}\frac{\theta[y_{max}]^2}{2}}}{\frac{2\gamma^2}{\phi^2}}\right]<1$$

$$\frac{\phi^2}{2\gamma^2}+\frac{\sqrt{1+\frac{4\gamma^2}{\phi^2}\frac{\theta[y_{max}]^2}{2}}}{\frac{2\gamma^2}{\phi^2}}<\frac{\phi^2}{\theta\gamma^2}$$

$$\frac{\sqrt{1+\frac{4\gamma^2}{\phi^2}\frac{\theta[y_{max}]^2}{2}}}{\frac{2\gamma^2}{\phi^2}}<\frac{\phi^2}{2\theta\gamma^2}$$

$$\sqrt{1+\frac{4\gamma^2}{\phi^2}\frac{\theta[y_{max}]^2}{2}}<1$$

This tells us that it cannot be possible. Therefore this steady state cannot be stable.

<u>Derivations for Adding Diminishing Returns to Income:</u>

(6) Derivation of Z^A_{t+1}:

$$Z^A_{t+1}=\int_{y_{min}}^{\hat{y}_t}[z^i_t]dy^i_t$$

$$=\int_{y_{min}}^{\hat{y}_t}\left[\theta\sqrt{y^i_t}\quad\right]dy^i_t$$

$$=\theta\int_{y_{min}}^{\hat{y}_t}[y^i_t]^{\frac{1}{2}}dy^i_t$$

$$= \frac{2\theta}{3}\left[y_t^i\right]^{\frac{3}{2}}\Big|_{y_{min}}^{\hat{y}_t}$$

$$= \frac{2\theta}{3}\left[\left[\frac{\gamma Z_t^B}{\phi}\right]^{\frac{3}{2}} - [y_{min}]^{\frac{3}{2}}\right]$$

(7) Derivation of Z_{t+1}^B:

$$Z_{t+1}^B = \int_{\hat{y}_t}^{y_{max}} \left[z_t^i\right] dy_t^i$$

$$= \int_{\hat{y}_t}^{y_{max}} \left[\theta\sqrt{y_t^i}\right] dy_t^i$$

$$= \theta\int_{\hat{y}_t}^{y_{max}} \left[y_t^i\right]^{\frac{1}{2}} dy_t^i$$

$$= \frac{2\theta}{3}\left[y_t^i\right]^{\frac{3}{2}}\Big|_{\hat{y}_t}^{y_{max}}$$

$$= \frac{2\theta}{3}\left[[y_{max}]^{\frac{3}{2}} - [\hat{y}_t]^{\frac{3}{2}}\right]$$

$$= \frac{2\theta}{3}\left[[y_{max}]^{\frac{3}{2}} - \left[\frac{\gamma Z_t^B}{\phi}\right]^{\frac{3}{2}}\right]$$

(8) Derivation of slope at steady state:

$$Z_{t+1}^B = \frac{2\theta}{3}\left[[y_{max}]^{\frac{3}{2}} - \left[\frac{\gamma Z_t^B}{\phi}\right]^{\frac{3}{2}}\right]$$

Letting $Z_t^B = Z_t$

$$Z_{t+1} = \frac{2\theta}{3}\left[[y_{max}]^{\frac{3}{2}} - \left[\frac{\gamma Z_t}{\phi}\right]^{\frac{3}{2}}\right]$$

$$Z_{t+1} = \frac{2\theta}{3}[y_{max}]^{\frac{3}{2}} - \frac{2\theta}{3}\left[\frac{\gamma Z_t}{\phi}\right]^{\frac{3}{2}}$$

$$\frac{\partial}{\partial Z_t}[Z_{t+1}] = \frac{\partial}{\partial Z_t}\left[\frac{2\theta}{3}[y_{max}]^{\frac{3}{2}}\right] - \frac{\partial}{\partial Z_t}\left[\frac{2\theta}{3}\left[\frac{\gamma Z_t}{\phi}\right]^{\frac{3}{2}}\right]$$

$$\frac{\partial}{\partial Z_t}[Z_{t+1}] = -\frac{\theta\gamma^{\frac{3}{2}}}{\phi^{\frac{3}{2}}}\sqrt{Z_t}$$

Derivations When the Individual Externality Decreases at High Income Levels:

If $z_t^i = \theta\left[\bar{y}^2 - \left[\bar{y} - y_t^i\right]^2\right]$

then

$$Z_{t+1}^2 = \int_{\hat{y}_t}^{y_{max}} \left[z_t^i\right] dy_t^i$$

$$= \int_{\hat{y}_t}^{y_{max}} \left[\theta\left[\bar{y}^2 - \left[\bar{y} - y_t^i\right]^2\right]\right] dy_t^i$$

$$= \theta\int_{\hat{y}_t}^{y_{max}} \left[\bar{y}^2 - \left[\bar{y} - y_t^i\right]^2\right] dy_t^i$$

$$= \theta\int_{\hat{y}_t}^{y_{max}} \left[\bar{y}^2 - \left[\bar{y}^2 - 2\bar{y}y_t^i + \left[y_t^i\right]^2\right]\right] dy_t^i$$

$$= \theta\int_{\hat{y}_t}^{y_{max}} \left[\bar{y}^2 - \bar{y}^2 - \left[y_t^i\right]^2 + 2\bar{y}y_t^i\right] dy_t^i$$

$$= \theta\int_{\hat{y}_t}^{y_{max}} \left[2\bar{y}y_t^i - \left[y_t^i\right]^2\right] dy_t^i$$

$$= \left[\theta\bar{y}\left[y_t^i\right]^2 - \frac{\theta}{3}\left[y_t^i\right]^3\right]\Bigg|_{\hat{y}_t}^{y_{max}}$$

$$= \left[\theta\bar{y}[y_{max}]^2 - \frac{\theta}{3}[y_{max}]^3\right] - \left[\theta\bar{y}[\hat{y}_t]^2 - \frac{\theta}{3}[\hat{y}_t]^3\right]$$

$$Z_{t+1}^2 = \frac{\theta}{3}[[\hat{y}_t]^3 - [y_{max}]^3] + \theta\bar{y}[[y_{max}]^2 - [\hat{y}_t]^2]$$

$$Z_{t+1}^2 = \frac{\theta}{3}\left[\left[\frac{\gamma Z_t^2}{\phi}\right]^3 - [y_{max}]^3\right] + \theta\bar{y}\left[[y_{max}]^2 - \left[\frac{\gamma Z_t^2}{\phi}\right]^2\right]$$

Let $Z_t^2 = Z_t$

$$Z_{t+1} = \frac{\theta}{3}\left[\left[\frac{\gamma Z_t}{\phi}\right]^3 - [y_{max}]^3\right] + \theta\bar{y}\left[[y_{max}]^2 - \left[\frac{\gamma Z_t}{\phi}\right]^2\right]$$

$$Z_{t+1} = \frac{\theta}{3}\left[\frac{\gamma Z_t}{\phi}\right]^3 - \theta\bar{y}\left[\frac{\gamma Z_t}{\phi}\right]^2 - \frac{\theta}{3}[y_{max}]^3 + \theta\bar{y}[y_{max}]^2$$

$$Z_{t+1} = \frac{\theta}{3}\left[\frac{\gamma Z_t}{\phi}\right]^3 - \theta\bar{y}\left[\frac{\gamma Z_t}{\phi}\right]^2 + \left[\bar{y} - \frac{y_{max}}{3}\right]\theta[y_{max}]^2$$