University of Nevada, Reno

Rational Witt Classes of 4-Stranded Pretzel Knots

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by

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Abstract

The rational Witt classes of knots are invariants related to knot concordance. Although they are somewhat weaker than other concordance invariants, for example, the algebraic concordance classes, they are much easier to compute. The goal of this thesis is to obstruct sliceness of 4-stranded pretzel knots by obtaining specific numerical restrictions on the parameters of the knot. We begin by building the theory of bilinear forms over arbitrary fields and construct our main tool, the rational Witt ring. Some examples are presented and we address the specific case of 4-stranded pretzel knots, which exhibit interesting phenomena in the restrictions of the parameters to ensure triviality of their Witt class.

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Chapter 1

Bilinear Spaces

In this chapter we introduce the notion of bilinear forms and what we will call bilinear spaces in order to begin building the algebraic structures we will need to proceed. We will see that all bilinear forms can be described in terms of a matrix and will often refer to the form and its matrix interchangeably.

1.1 Preliminary Definitions

The goal here is to familiarize the reader with the terminology that will be used throughout this thesis. We will also state the Orthogonal Complement Theorem, which will be used when we begin building Witt rings.

Definition 1. Let $\mathbb{F}$ be a field and let V be a finite dimensional vector space over $\mathbb{F}$. A *bilinear form* on V is a function $B : V \times V \to \mathbb{F}$ that satisfies the following conditions for each $x, y, z \in V$ and $\alpha, \beta \in \mathbb{F}$:

(i) $B(\alpha x + \beta y, z) = \alpha B(x, z) + \beta B(y, z)$

(ii) $B(x, \alpha y + \beta z) = \alpha B(x, y) + \beta B(x, z)$.

The pair (V, B) is referred to as a *bilinear space*. If there is no danger of confusion of the vector space V, we will often refer to B as a bilinear space.

Definition 2. A bilinear space is said to be

(i) *symmetric* if $B(x, y) = B(y, x)$ for each $x, y \in V$,

(ii) *nondegenerate* if $B(x, y) = 0$ for every $y \in V$, then $x = 0$.

Remark 1. Unless specifically noted, all bilinear spaces are assumed to be symmetric and nondegenerate. In addition, we will restrict our attention to fields of characteristic not two.

The following definitions will be important to our constructions later.

Definition 3. Let (V, B) be a bilinear space. Then (V, B) is said to be *isotropic* if it contains an isotropic vector, i.e., a nonzero vector $v \in V$ such that $B(v, v) = 0$. If V does not contain an isotropic vector, B is said to be *anisotropic*. If V has a half dimensional subspace W on which $B(x, y) = 0$ for every $x, y \in W$, then B is said to be *totally isotropic*.

Definition 4. The *dimension* of a bilinear space (V, B) is the dimension of the vector space V, i.e., $\dim(V, B) = \dim(V)$.

We now give two definitions that will be used in the Orthogonal Complement Theorem.

Definition 5. Let (V, B) be a symmetric, nondegenerate bilinear space and suppose that S is a subspace of V. Then the *orthogonal complement* (with respect to B) of S, denoted $S^\perp$, is defined to be the set of all vectors in V that are orthogonal to every vector in S, i.e.,

$$S^\perp = \{v \in V : B(v, x) = 0 \text{ for every } x \in S\}.$$

Definition 6. A subspace S of the bilinear space (V, B) is said to be *nonsingular* if $S \cap S^{\perp} = 0$.

Remark 2. The one dimensional subspace $S = \mathbb{F}v$ of (V, B) is nonsingular if and only if v is an anisotropic vector. To see this, suppose $w \in S \cap S^{\perp}$ and $w \neq 0$. Then $w = \lambda v$ for some $\lambda \in \dot{\mathbb{F}}$. But then $B(v, w) = B(v, \lambda v) = \lambda B(v, v) \Rightarrow B(v, v) = 0$ and so v is isotropic.

The following theorem from [15] (Theorem 5.2.2) will be used throughout this thesis.

Theorem 1 (The orthogonal complement theorem)**.** If S is nonsingular subspace of the symmetric space (V, B), then

$$V = S \oplus S^{\perp}.$$

Definition 7. Let (V, B) be a one dimensional bilinear space. Then the form B will be denoted $\langle a \rangle$, where $\langle a \rangle : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is given by

$$\langle a \rangle (1, 1) = a$$

and extended linearly.

1.2 Matrices of Bilinear Forms

Here we show that given a basis for our vector space V, any bilinear form can be represented by a square matrix. We then show how to change bases to get different (though equivalent) matrix representatives for the form. The material here is largely from linear algebra.

Definition 8. Let (V, B) be a bilinear space of dimension n and choose a basis $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$ for V. Set $b_{ij} = B(e_i, e_j)$. Then

$$Q = [b_{ij}] = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

is a *matrix representative* of B with respect to the basis $\mathcal{B}$.

To see how this matrix Q determines the bilinear form B, let $x, y \in V$. Then with respect to the basis $\mathcal{B}$ we can write

$$x = x_1 e_1 + \ldots + x_n e_n \qquad \text{and} \qquad y = y_1 e_1 + \ldots + y_n e_n.$$

As column vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Then

$$B(x, y) = x^T Q y$$

where x^T denotes the *transpose* of x.

Remark 3. Since the matrix Q completely determines the bilinear form B on V, we will sometimes write $(V, B) \cong Q$ (or $B \cong Q$ when the vector space V is understood).

Suppose now that $\mathcal{B}_1 = \{e_1, \ldots, e_n\}$ and $\mathcal{B}_2 = \{f_1, \ldots, f_n\}$ are two bases for V.

Write the basis vectors of $\mathcal{B}_2$ as a linear combination of the basis vectors of $\mathcal{B}_1$, i.e.,

$$\begin{aligned} f_1 &= p_{11}e_1 + \ldots + p_{1n}e_n \\ f_2 &= p_{21}e_1 + \ldots + p_{2n}e_n \\ &\vdots \\ f_n &= p_{n1}e_1 + \ldots + p_{nn}e_n. \end{aligned}$$

Then $P = [p_{ij}]$ is the *transition matrix* from the basis $\mathcal{B}_1$ to the basis $\mathcal{B}_2$. We know that $\det P \neq 0$, and so P is a regular matrix. The inverse matrix of P, P^{-1}, is the transition matrix from $\mathcal{B}_2$ to $\mathcal{B}_1$.

Proposition 1. Let (V, B) be a bilinear space and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases for V. Suppose that P is the transition matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. If Q is the matrix representative of B with respect to $\mathcal{B}_1$, then PQP^T is the matrix representative for B with respect to $\mathcal{B}_2$.

Proof. Let $\mathcal{B}_1 = \{e_1, \ldots, e_n\}$ and $\mathcal{B}_2 = \{f_1, \ldots, f_n\}$ be two bases for V and suppose that $P = [p_{ij}]$ is the transition matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. Let $Q = [q_{ij}]$ and $S = [s_{ij}]$ be matrix representatives for B with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Then

$$\begin{aligned} s_{ij} = B(f_i, f_j) &= B(p_{i1}e_1 + \ldots + p_{in}e_n, p_{j1}e_1 + \ldots + p_{jn}e_n) \\ &= B\left(\sum_{k=1}^{n} p_{ik}e_k, \sum_{\ell=1}^{n} p_{j\ell}e_\ell\right) \\ &= \sum_{k=1}^{n}\sum_{\ell=1}^{n} B(p_{ik}e_k, p_{j\ell}e_\ell) \\ &= \sum_{k=1}^{n}\sum_{\ell=1}^{n} p_{ik} \cdot p_{j\ell} \cdot B(e_k, e_\ell) \\ &= \sum_{k=1}^{n}\sum_{\ell=1}^{n} p_{ik} \cdot p_{j\ell} \cdot q_{k\ell} \\ &= \sum_{k=1}^{n}\sum_{\ell=1}^{n} p_{ik} \cdot q_{k\ell} \cdot p_{\ell j}. \end{aligned}$$

This latter sum is easily seen to be the ij-entry of the matrix PQP^T. □

Remark 4. If S can be gotten from Q by means of a transition matrix, i.e., $S = PQP^T$, we will often write that $S \cong Q$.

We end this section with a definition and theorem that will be of later use.

Definition 9. Let (V, B) be a bilinear space over a field $\mathbb{F}$ and let Q be a matrix representative of B with respect to any basis of V. The *determinant* of the bilinear space (V, B) is defined as the determinant of Q modulo the square elements of $\mathbb{F}$. In particular, define $\det(B) = \det(Q) \in \dot{\mathbb{F}}/\dot{\mathbb{F}}^2$ where $\dot{\mathbb{F}}/\dot{\mathbb{F}}^2$ is the quotient group of the multiplicative group of nonzero elements of $\mathbb{F}$ by square elements. (Note that this definition is well posed by Proposition 1.)

It is useful to note that a matrix representative of a symmetric, nondegenerate bilinear form is a symmetric matrix with nonzero determinant. To see this, let (V, B) be a symmetric, nondegenerate bilinear space and suppose that $\{e_1, \ldots, e_n\}$ is a basis for V. Then

$$b_{ij} = B(e_i, e_j) = B(e_j, e_i) = b_{ji}.$$

So the matrix Q that represents B is equal to its transpose Q^T, i.e., the matrix is symmetric. The nondegeneracy of Q is implied by the next theorem.

Theorem 2. Let (V, B) be a symmetric, nondegenerate bilinear space over $\mathbb{F}$ and suppose that the characteristic of $\mathbb{F}$ is not two. Suppose that Q is a (symmetric) matrix representative of B with respect to some basis. Then Q is diagonalizable.

Proof. First note that if $\dim(V) = 1$, then Q is trivially diagonal since it is a 1×1 matrix. For induction suppose that Q is diagonalizable when $\dim(V) < n$ for some $n > 1$ and that now $\dim(V) = n$. It will suffice to find an orthogonal basis for V since then the matrix PQP^T will be a diagonal matrix, where P is the transition matrix from the basis of Q to this orthogonal basis.

First suppose that $v \in V$ is isotropic for every v. Then for each $v, w \in V$ we have

$$0 = B(v + w, v + w) = B(v, v) + 2B(v, w) + B(w, w) = 2B(v, w).$$

But since the characteristic of $\mathbb{F}$ is not two, we must have that $B(v, w) = 0$, rendering $B \equiv 0$. Since B is assumed nondegenerate, this can only occur if $\dim(V) = 0$, in which case B is diagonalizable.

Now suppose that V contains an anisotropic vector v. Let $S = \mathbb{F}v$ be the one dimensional subspace of V spanned by v. Then by Remark 2, S is nonsingular. By Theorem 1, $V = S \oplus S^\perp$. Since $\dim(S^\perp) < \dim(V)$, we can apply the induction hypothesis and so $S^\perp$ has an orthogonal basis $\{e_1, \ldots, e_{n-1}\}$. Clearly $B(v, e_i) = 0$ for each i by definition of $S^\perp$, and so $\{v, e_1, \ldots, e_{n-1}\}$ is an orthogonal basis for V. Thus Q is diagonalizable. $\square$

Since any symmetric matrix is diagonalizable (if the characteristic of the field is not 2), we may assume that Q is a diagonal matrix. Suppose that

$$Q = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{bmatrix}.$$

Then for $x, y \in V$ we have that

$$B(x, y) = x^T Q y = b_1 x_1 y_1 + \ldots + b_n x_n y_n.$$

We claim that $b_i \neq 0$ for each $i = 1, \ldots, n$. Suppose that $b_j = 0$ for some j. Define x to be the vector in which there is a one in the j^{th} coordinate and zeros everywhere else. In particular,

$$x = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T.$$

Then $B(x, y) = 0$ for every $y \in V$ and so B is degenerate. Thus $b_i \neq 0$ for each i and it is clear that the determinant of Q is nonzero.

Chapter 2

Witt Rings

In this chapter we consider the set of all bilinear forms over finite dimensional vector spaces and construct the Witt ring of a field. We will restrict our attention to fields of characteristic not two and refer the reader to [15] for a complete treatment of all fields, including characteristic two.

2.1 Hyperbolic Spaces

We begin with some definitions that will be needed in the construction of the Witt ring of a field. If we were concerned with *all* fields we would use the concept of metabolic spaces. As it is, we are only concerned with fields of characteristic not two, so we may restrict our study to that of hyperbolic spaces. The main result here describes alternate characterizations of a two-dimensional form being isotropic.

Definition 10. Let (V, B) be a bilinear space. Then two vectors $u, v \in V$ form a *hyperbolic pair* if u and v are both isotropic and $B(u, v) = 1$. The 2-dimensional space spanned by u and v is called a *hyperbolic plane* and is denoted $\mathbb{H}$. In particular, if $\{u, v\}$ is a hyperbolic pair in $\mathbb{H}$, then

$$\mathbb{H} \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{with respect to the basis } \{u, v\}.$$

The following proposition follows from Theorem 11.1.1 in [15].

Proposition 2. Let (V, B) be a bilinear space of dimension two over $\mathbb{F}$ (of characteristic not 2). Then the following are equivalent:

(i) (V, B) is isotropic.

(ii) $B \cong \begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix}$ for some $a \in \mathbb{F}$.

(iii) B is hyperbolic.

(iv) $B \cong \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(v) $B \cong \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$ for all $a \in \dot{\mathbb{F}}$.

(vi) $\det(B) = -1$.

Proof. (i)⇒(ii): Suppose (V, B) is isotropic. Then there is a nonzero vector $v \in V$ such that $B(v, v) = 0$. Since B is assumed to be nondegenerate, there exists a vector $w \in V$ such that $B(v, w) = x \neq 0$. But B is bilinear and so $B(v, \frac{1}{x}w) = \frac{1}{x}B(v, w) = \frac{1}{x}x = 1$. Set $u = \frac{1}{x}w$. Then u, v are linearly independent since otherwise $u = cv$ for some $c \in \mathbb{F}$ would imply that

$$B(v, u) = B(v, cv) = cB(v, v) = c \cdot 0 = 0.$$

But $B(v, u) = 1$ by construction of u, which is a contradiction.

Since v and u are linearly independent, we can construct the plane spanned by v and u. Then the matrix representing this plane with respect to the basis $\{v, u\}$ is

$$\begin{bmatrix} 0 & 1 \\ 1 & B(u,u) \end{bmatrix}.$$

Setting $a = B(u,u)$ gives (ii).

(ii)⇒(iii): Let $u \in V$. If $V \cong \begin{bmatrix} 0 & 1 \\ 1 & a \end{bmatrix}$ in the basis $\{v, u\}$ we will need to change basis to get a hyperbolic pair. Let $w \in V$ and write $w = xv + yu$. Then

$$\begin{aligned} B(v,w) &= B(v, xv + yu) \\ &= xB(v,v) + yB(v,u) \\ &= x \cdot 0 + y \cdot 1. \end{aligned}$$

For $\{v, w\}$ to be a hyperbolic pair, we need $B(v,w) = 1$ and so $y = 1$. Also

$$\begin{aligned} B(w,w) &= B(xv + yu, xv + yu) \\ &= B(xv + u, xv + u) \\ &= x^2 B(v,v) + 2xB(v,u) + B(u,u) \\ &= x^2 \cdot 0 + 2x \cdot 1 + a \\ &= 2x + a. \end{aligned}$$

Again, we need $B(w,w) = 0$ and so $x = -\frac{1}{2}a$. Thus $\{v, w\}$ with $w = -\frac{1}{2}av + u$ form a hyperbolic pair. (Note that $\frac{1}{2}a$ may not be defined if the characteristic of $\mathbb{F}$ is 2 so here we need the hypothesis that the characteristic of $\mathbb{F}$ is not equal to 2).

(iii)⇒(iv): As above, the pair $\{v, w\}$ forms a hyperbolic pair. Let Q be a matrix representative for B with respect to the basis $\{v, w\}$. Let

$$x = \frac{1}{2}v + w \qquad y = \frac{1}{2}v - w.$$

If

$$P = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{bmatrix}$$

is the transition matrix from $\{v, w\}$ to $\{x, y\}$ then we have

$$B \cong PQP^T = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(iv)⇒(v): Suppose that $Q \cong \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the matrix representative of B with respect to the basis $\{x, y\}$. Let $a \in \dot{\mathbb{F}}$. Define a new basis

$$f_1 = \frac{1}{2}(1 + a)x + \frac{1}{2}(1 - a)y$$
$$f_2 = \frac{1}{2}(1 - a)x + \frac{1}{2}(1 + a)y.$$

Then the transition matrix from $\{x, y\}$ to $\{f_1, f_2\}$ is

$$P = \begin{bmatrix} \frac{1}{2}(1+a) & \frac{1}{2}(1-a) \\ \\ \frac{1}{2}(1-a) & \frac{1}{2}(1+a) \end{bmatrix}.$$

Thus we have

$$B \cong PQP^T = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}.$$

(v)⇒(vi): $\det(B) = \det(PQP^T) = -a^2 = -1 \in \dot{\mathbb{F}}/\dot{\mathbb{F}}^2$.

(vi)⇒(i): Suppose that $B \cong \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (any diagonalization of (V, B)). Then $\det(B) = ab = -\lambda^2 = -1 \in \dot{\mathbb{F}}/\dot{\mathbb{F}}^2$. So $b = -a\lambda^2$ and $B \cong \begin{bmatrix} a & 0 \\ 0 & -a\lambda^2 \end{bmatrix}$. Then the vector $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ is an isotropic vector, i.e., it is nonzero and

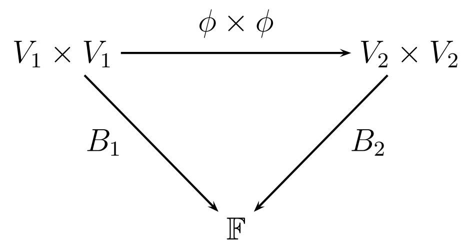
$$\begin{bmatrix} \lambda & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & -a\lambda^2 \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} = a\lambda^2 - a\lambda^2 = 0.$$

Thus (V, B) is isotropic. □

2.2 Equivalence of Bilinear Spaces

Here we introduce the notion of two bilinear spaces being isomorphic, which is the first step in what we will ultimately call Witt equivalence. The main result here is the equivalence relation of isomorphism.

Definition 11. Let V_i, $i = 1, 2$ be vector spaces and let $B_i : V_i \times V_i \to \mathbb{F}$ be symmetric bilinear forms. Then the pair (V_1, B_1) is *isomorphic* to the pair (V_2, B_2), denoted $(V_1, B_1) \cong (V_2, B_2)$, if there exists a vector space isomorphism $\phi : V_1 \to V_2$ such that the following diagram commutes:



i.e., $B_2(\phi(x), \phi(y)) = B_1(x, y)$ for all $(x, y) \in V_1 \times V_1$ We will sometimes write $(V_1, B_1) = (V_2, B_2)$ to mean $(V_1, B_1) \cong (V_2, B_2)$ when no confusion will arise.

The following proposition shows the relation between isomorphic bilinear spaces and their matrix representatives.

Proposition 3. Let (V_1, B_1) and (V_2, B_2) be bilinear spaces over $\mathbb{F}$ and suppose Q and S are matrix representatives of B_1 and B_2 (with respect to any bases), respectively. Then $(V_1, B_1) \cong (V_2, B_2)$ if and only if $Q \cong S$.

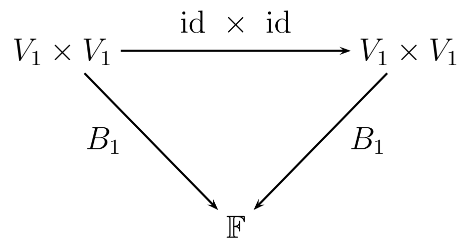
Proof. Theorem 4.1.2 in [15]. □

We now prove that the relation of isomorphism is an equivalence relation on the set of all symmetric, nondegenerate bilinear spaces.

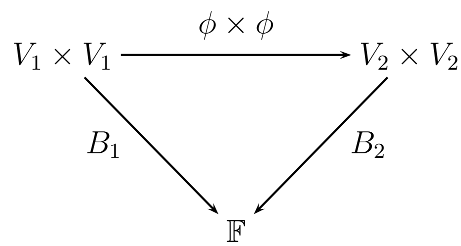
Proposition 4. Isomorphism of bilinear spaces is an equivalence relation on the set of all bilinear forms over $\mathbb{F}$. The set of isomorphism classes of bilinear forms over finite dimensional $\mathbb{F}$-vector spaces will be denoted $\mathfrak{B}_{\mathbb{F}}$.

Proof. Let $(V_1, B_1), (V_2, B_2), (V_3, B_3)$ be finite dimensional bilinear spaces over $\mathbb{F}$.

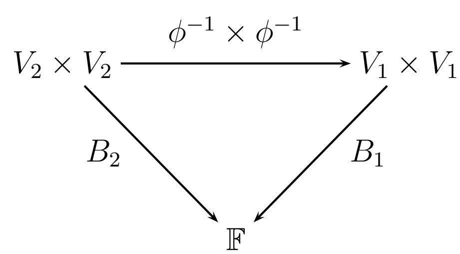
Reflexivity: Clearly $(V_1, B_1) \cong (V_1, B_1)$ since we can take $\phi : V_1 \to V_1$ to be the identity. Then ϕ is an isomorphism of vector spaces and the following diagram commutes:



Symmetry: Suppose $(V_1, B_1) \cong (V_2, B_2)$. Let $\phi : V_1 \to V_2$ be the vector space isomorphism such that the following diagram commutes:

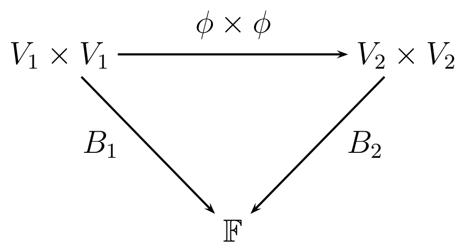


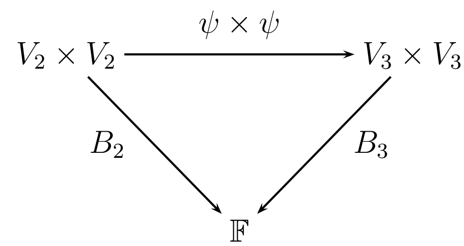
Since ϕ is an isomorphism, it has an inverse, ϕ^{-1}, which is also an isomorphism. Then the following diagram commutes:



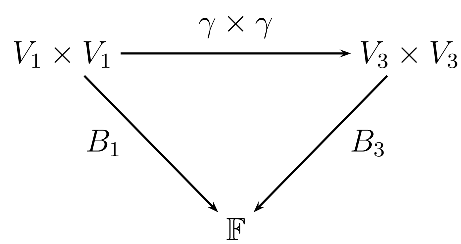
So $(V_2, B_2) \cong (V_1, B_1)$.

Transitivity: Suppose $(V_1, B_1) \cong (V_2, B_2)$ and $(V_2, B_2) \cong (V_3, B_3)$. Let $\phi : V_1 \to V_2$ and $\psi : V_2 \to V_3$ be the isomorphisms such that the following diagrams commute:





Define $\gamma : V_1 \to V_3$ as $\gamma = \psi \circ \phi$. Then γ is an isomorphism of vector spaces and the following diagram commutes:



So $(V_1, B_1) \cong (V_3, B_3)$ and we have an equivalence relation on $\mathfrak{B}_{\mathbb{F}}$. $\square$

2.3 The Witt Semigroup

We wish to make the set of isomorphism classes of bilinear spaces $\mathfrak{B}_{\mathbb{F}}$ into a group, but as we will see, initially we only get a semigroup. We will define a binary operation on $\mathfrak{B}_{\mathbb{F}}$ and show that under this operation, it is a semigroup.

Definition 12. Let $(V_1, B_1), (V_2, B_2) \in \mathfrak{B}_{\mathbb{F}}$. Define a binary operation on $\mathfrak{B}_{\mathbb{F}}$, called *direct orthogonal sum*, and denote it

$$(V_1, B_1) \oplus (V_2, B_2) = (V_1 \oplus V_2, B_1 \oplus B_2)$$

where $V_1 \oplus V_2$ is the ordinary direct orthogonal sum of vector spaces and $B_1 \oplus B_2$ is the direct orthogonal sum of forms, i.e., $B_1 \oplus B_2((x_1, y_1), (x_2, y_2)) = B_1(x_1, x_2) + B_2(y_1, y_2)$.

Note that vectors of V_1 pair trivially with vectors from V_2. More precisely, if $v \in V_1$ then $(v, 0) \in V_1 \oplus V_2$ and similarly, if $w \in V_2$ then $(0, w) \in V_1 \oplus V_2$. Then $B_1 \oplus B_2((v, 0), (0, w)) = B_1(v, 0) + B_2(0, w) = 0$.

If $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for V_1 and V_2, respectively, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $V_1 \oplus V_2$. So if Q and S are matrix representatives for B_1 and B_2 (with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$), respectively, then the block sum of matrices

$$\begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & S \end{bmatrix}$$

is a matrix representative for $B_1 \oplus B_2$ with respect to the basis $\mathcal{B}_1 \cup \mathcal{B}_2$. In this notation, if Q is an $n \times n$ matrix and S is an $m \times m$ matrix, then the $\mathbf{0}$'s represent an $n \times m$ and $m \times n$ block of zeros, respectively.

Proposition 5. The operation of direct sum on $\mathfrak{B}_{\mathbb{F}}$ is well defined. In particular, if (V_1, B_1) and (V_2, B_2) are two bilinear spaces over $\mathbb{F}$ (i.e., B_i is symmetric and nondegenerate for $i = 1, 2$), then $B_1 \oplus B_2$ is also a symmetric, nondegenerate bilinear form.

Proof. Since the matrix representatives of B_1 and B_2 are symmetric matrices with nonzero determinant, the block sum matrix that represents $B_1 \oplus B_2$ is also symmetric with nonzero determinant. In particular, $B_1 \oplus B_2$ is a symmetric, nondegenerate bilinear form. $\square$

Remark 5. Since a matrix representative of any symmetric bilinear form can be diagonalized, we will always assume that the matrix is a diagonal matrix and we will write the form as the direct sum of one dimensional forms. For example,

$$Q = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & 0\cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{nn} \end{bmatrix} = \langle b_{11} \rangle \oplus \langle b_{22} \rangle \oplus \ldots \oplus \langle b_{nn} \rangle.$$

Theorem 3. The pair $(\mathfrak{B}_{\mathbb{F}}, \oplus)$ forms a commutative semigroup.

Proof. Clearly, $\mathfrak{B}_{\mathbb{F}}$ is closed under the operation of direct sum by definition. It remains to see that the operation is associative, commutative, and has a unit.

Let $(V_1, B_1), (V_2, B_2), (V_3, B_3) \in \mathfrak{B}_{\mathbb{F}}$. Then

$$\begin{aligned} (V_1, B_1) \oplus ((V_2, B_2) \oplus (V_3, B_3)) &= (V_1, B_1) \oplus (V_2 \oplus V_3, B_2 \oplus B_3) \\ &= (V_1 \oplus (V_2 \oplus V_3), B_1 \oplus (B_2 \oplus B_3)) \\ &= ((V_1 \oplus V_2) \oplus V_3, (B_1 \oplus B_2) \oplus B_3) \\ &= (V_1 \oplus V_2, B_1 \oplus B_2) \oplus (V_3, B_3) \\ &= ((V_1, B_1) \oplus (V_2, B_2)) \oplus (V_3, B_3). \end{aligned}$$

It is easy to see that the unit for this operation is given by the 0-dimensional form. Thus, $\mathfrak{B}_{\mathbb{F}}$ forms a semigroup. It is commutative since $V_1 \oplus V_2 \cong V_2 \oplus V_1$ and if $\begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & S \end{bmatrix}$ is a matrix representative for $B_1 \oplus B_2$ (i.e., $B_1 \cong Q$ and $B_2 \cong S$) then the matrix

$$P = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

is a transition matrix from $\begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & S \end{bmatrix}$ to $\begin{bmatrix} S & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix}$ and so $B_1 \oplus B_2 \cong B_2 \oplus B_1$. □

The downfall of using just isomorphism classes of bilinear spaces, is of course, that we do not obtain inverses. This is easily seen in the fact that the identity in the semigroup is the zero form on the zero dimensional vector space. If (V_1, B_1) and (V_2, B_2) were inverses, we would have to have that $V_1 \oplus V_2 \cong 0$, but dimension of vector spaces is additive under direct sum. Hence, it is impossible for $V_1 \oplus V_2$ to be the 0-dimensional vector space if either (or both) are of positive dimension. We remedy this problem by considering yet another equivalence on $\mathfrak{B}_{\mathbb{F}}$.

2.4 Witt Equivalence and the Witt Group

Here we finally introduce the notion of Witt equivalence and make $\mathfrak{B}_{\mathbb{F}}$ into a group. An important result that is crucial to Witt equivalence is that any symmetric, nondegenerate bilinear space can be written as the direct sum of hyperbolic subspaces and an anisotropic subspace. In essence, this is what ensures the existence of inverses in the Witt group.

Theorem 4. Let (V, B) be a symmetric, nondegenerate bilinear space over a field $\mathbb{F}$. Then there are subspaces U and H of V such that U is anisotropic, H is a direct sum of hyperbolics, and $V = U \oplus H$ and $B = B|_U \oplus B|_H$. The isomorphism type of U and the number of hyperbolic summands in H are unique.

Proof. If V is anisotropic, then we can just take $H = 0$ and $U = V$. If V is isotropic, then there is a nonzero vector $v \in V$ such that $B(v, v) = 0$. Since B is nondegenerate, there must be a vector $w \in V$ with $B(v, w) \neq 0$. Let $\mathbb{H}_1$ be the plane spanned by $\{v, w\}$. Then

$$\mathbb{H}_1 = \begin{bmatrix} 0 & a \\ a & b \end{bmatrix}$$

for some $a, b \in \mathbb{F}$. By Proposition 2, $\mathbb{H}_1$ is hyperbolic. Now consider the orthogonal complement of $\mathbb{H}_1$, $\mathbb{H}_1^{\perp}$. Suppose there is a vector $u \in \mathbb{H}_1 \cap \mathbb{H}_1^{\perp}$ with $u \neq 0$. But then $B(u, x) = 0$ for every $x \in \mathbb{H}_1$, which is a contradiction to the fact that $B|_{\mathbb{H}_1 \times \mathbb{H}_1}$ is nondegenerate (since all hyperbolic spaces are nondegenerate). Thus $\mathbb{H}_1$ is nonsingular and by Theorem 1, $V = \mathbb{H}_1 \oplus \mathbb{H}_1^{\perp}$. Set $H = \mathbb{H}_1$ and $U = \mathbb{H}_1^{\perp}$. If U is anisotropic, we're done. Otherwise, U is isotropic and we can use the above process to split off another hyperbolic $\mathbb{H}_2$. Set now $H = \mathbb{H}_1 \oplus \mathbb{H}_2$ and $U = H^{\perp}$. We continue this process inductively until U is anisotropic. Note that this process will terminate after a finite number of steps since V is finite dimensional. We finally obtain a decomposition of $V = U \oplus H$ with U anisotropic and H a sum of hyperbolics. (This also works if V was hyperbolic to begin with taking $U = 0$ since technically speaking, the zero space is anisotropic). □

So we can decompose any space B into an anisotropic part and a hyperbolic part. We will call this decomposition the *Witt Decomposition*. We will construct an equivalence on $\mathfrak{B}_{\mathbb{F}}$ in which the hyperbolic part of B will be trivial.

Definition 13. Let (V_1, B_1) and (V_2, B_2) be two bilinear spaces. Define (V_1, B_1) to be *similar* to (V_2, B_2), denoted $(V_1, B_1) \sim (V_2, B_2)$, if

$$(V_1, B_1) \oplus \mathbb{H}_1 \oplus \ldots \oplus \mathbb{H}_k \cong (V_2, B_2) \oplus \mathbb{H}'_1 \oplus \ldots \oplus \mathbb{H}'_\ell$$

where $\mathbb{H}_i, \mathbb{H}'_i$ are hyperbolic planes for each i and $k, \ell \geq 0$.

Proposition 6. The relation of similarity of bilinear spaces is an equivalence relation.

Proof. Reflexivity: Let $(V, B) \in \mathfrak{B}_{\mathbb{F}}$. Then $(V, B) \sim (V, B)$ since $(V, B) \oplus 0 = (V, B) \oplus 0$.

Symmetry: Let $(V_1, B_1), (V_2, B_2) \in \mathfrak{B}_{\mathbb{F}}$ and suppose $(V_1, B_1) \sim (V_2, B_2)$. Then

$$(V_1, B_1) \oplus H \cong (V_2, B_2) \oplus H'$$

where H, H' are direct sums of hyperbolics. Reversing the equality gives that $(V_2, B_2) \sim (V_1, B_1)$.

Transitivity: Let $(V_1, B_1), (V_2, B_2), (V_3, B_3) \in \mathfrak{B}_{\mathbb{F}}$ and suppose that $(V_1, B_1) \sim (V_2, B_2)$ and $(V_2, B_2) \sim (V_3, B_3)$. Then

$$(V_1, B_1) \oplus \mathbb{H}_1 \oplus \ldots \oplus \mathbb{H}_k \cong (V_2, B_2) \oplus \mathbb{H}'_1 \oplus \ldots \oplus \mathbb{H}'_\ell$$
$$(V_2, B_2) \oplus \mathbb{H}''_1 \oplus \ldots \oplus \mathbb{H}''_m \cong (V_3, B_3) \oplus \mathbb{H}'''_1 \oplus \ldots \oplus \mathbb{H}'''_n$$

for some $k, \ell, m, n \geq 0$. So

$$\begin{aligned}(V_1, B_1) \oplus \mathbb{H}_1 \oplus \ldots \oplus \mathbb{H}_k \oplus \mathbb{H}''_1 \oplus \ldots \oplus \mathbb{H}''_m &=\cong (V_2, B_2) \oplus \mathbb{H}'_1 \oplus \ldots \oplus \mathbb{H}'_\ell \oplus \mathbb{H}''_1 \oplus \ldots \oplus \mathbb{H}''_m \\ &=\cong (V_3, B_3) \oplus \mathbb{H}'_1 \oplus \ldots \oplus \mathbb{H}'_\ell \oplus \mathbb{H}'''_1 \oplus \ldots \oplus \mathbb{H}'''_n.\end{aligned}$$

Thus $(V_1, B_2) \sim (V_3, B_3)$ and we have an equivalence relation on $\mathfrak{B}_{\mathbb{F}}$. □

Proposition 7. The operation of direct sum is well defined on $\mathfrak{B}_{\mathbb{F}}/\sim$.

Proof. Let $(V_1, B_1), (V_2, B_2), (V_3, B_3) \in \mathfrak{B}_{\mathbb{F}}$ and suppose that $(V_1, B_2) \sim (V_2, B_2)$. Then

$$(V_1, B_1) \oplus H \cong (V_2, B_2) \oplus H'$$

for H, H' direct sums of hyperbolics. Direct summing (V_3, B_3) to each side gives

$$(V_1, B_1) \oplus (V_3, B_3) \oplus H \cong (V_2, B_2) \oplus (V_3, B_3) \oplus H'$$
$$(V_1 \oplus V_3, B_1 \oplus B_3) \oplus H \cong (V_2 \oplus V_3, B_2 \oplus B_3) \oplus H'.$$

So $(V_1, B_1) \oplus (V_3, B_3) \sim (V_2, B_2) \oplus (V_3, B_3)$ and the operation is well defined. □

If H is a hyperbolic, then $(V, B) \oplus H \cong (V, B)$. This gives that any sum of hyperbolics acts as the identity element in $\mathfrak{B}_{\mathbb{F}}/\sim$. We can use this fact to finally build a group.

Theorem 5. The pair $(\mathfrak{B}_{\mathbb{F}}/\sim, \oplus)$ forms an Abelian group.

Proof. As before, we immediately obtain a commutative semigroup. It remains to see that we have inverses. Let (V, B) be a symmetric, nondegenerate bilinear space of dimension n. Define $-B : V \times V \to \mathbb{F}$ by $(-B)(x, y) = -1 \cdot B(x, y)$. Denote $(V, -B) = -(V, B)$. We claim that $(V, B) \oplus (V, -B) \cong \mathbb{H}_1 \oplus \ldots \oplus \mathbb{H}_n$, where $\mathbb{H}_i$ are hyperbolics. Since (V, B) is symmetric, it is diagonalizable and we can write $B = \langle b_1 \rangle \oplus \ldots \oplus \langle b_n \rangle$. Then $-B = \langle -b_1 \rangle \oplus \ldots \oplus \langle -b_n \rangle$. So

$$\begin{aligned} B \oplus (-B) &= (\langle b_1 \rangle \oplus \ldots \oplus \langle b_n \rangle) \oplus (\langle -b_1 \rangle \oplus \ldots \oplus \langle -b_n \rangle) \\ &= (\langle b_1 \rangle \oplus \langle -b_1 \rangle) \oplus \ldots \oplus (\langle b_n \rangle \oplus \langle -b_n \rangle) \\ &= \mathbb{H}_1 \oplus \ldots \oplus \mathbb{H}_n \end{aligned}$$

where $\langle b_i \rangle \oplus \langle -b_i \rangle$ is hyperbolic by Proposition 2. Thus $\mathfrak{B}_{\mathbb{F}}$ is an Abelian group. □

This group can be endowed with a multiplication operation (in particular, the tensor product) and as such, can be made into a ring. Only the group structure will be important to us, so we leave the description of the ring structure again to [2, 4, 5, 15]. Regardless, we will refer to this group as the *Witt ring over the field* $\mathbb{F}$ and denote it as $W(\mathbb{F})$.

The following is a characterization of being trivial in the Witt ring.

Theorem 6. Let (V, B) be a bilinear space. Then $(V, B) = 0 \in W(\mathbb{Q})$ if and only if (V, B) is totally isotropic.

Proof. Corollary 11.3.1 in [15]. □

2.5 Calculations of Witt Rings

In this section we calculate examples of the Witt ring over a few specific fields. The examples that will be most applicable to our purpose are those of finite fields ($\mathbb{F}_p = \mathbb{Z}_p$ for p a prime) and that of the rational field $\mathbb{Q}$. We will see that $W(\mathbb{Q})$ relies on the Witt rings of finite fields.

The following theorem provides a presentation of the Witt ring (as an Abelian group). We provide a partial proof of this theorem; the full details can be found in [6].

Theorem 7 (Presentation of the Witt Ring)**.** Let $\mathbb{F}$ be field of characteristic not equal to two. Then $W(\mathbb{F})$ is the Abelian group generated by the set $\{\langle a \rangle : a \in \dot{\mathbb{F}}\}$ modulo the subgroup generated by

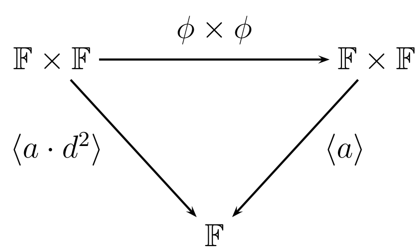
(i) $\langle a \cdot d^2 \rangle \ominus \langle a \rangle, \qquad a, d \in \dot{\mathbb{F}}$

(ii) $\langle a + b \rangle \oplus \langle ab(a+b) \rangle \ominus (\langle a \rangle \oplus \langle b \rangle), \qquad a, b \in \dot{\mathbb{F}},\ a + b \neq 0$

where $\ominus$ denotes the inverse operation of $\oplus$.

Proof. This partial proof will show that the above relations are satisfied in $W(\mathbb{F})$.

Starting with relation (i), note that the vector space associated with a one dimensional form is isomorphic to $\mathbb{F}$. Let $a, d \in \dot{\mathbb{F}}$ and consider the diagram



where $\phi(x) = d \cdot x$ for each $x \in \mathbb{F}$. Then $(\phi \times \phi)(1, 1) = (d, d)$ and we have

$$\langle a \rangle (d, d) = d^2 \langle a \rangle (1, 1)$$

and so the diagram commutes. Thus $\langle a \rangle = \langle a \cdot d^2 \rangle$, since ϕ is an isomorphism.

Now moving to relation (ii), we have the form $\langle a \rangle \oplus \langle b \rangle = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and suppose that we have this matrix with respect to the basis $\{e_1, e_2\}$. Then define a new basis as

$$f_1 = e_1 + e_2$$

$$f_2 = e_2.$$

The matrix representative with respect to this basis is

$$\begin{bmatrix} a+b & b \\ b & b \end{bmatrix}.$$

Diagonalizing this matrix gives

$$\begin{bmatrix} a+b & 0 \\ 0 & \frac{ab}{a+b} \end{bmatrix} = \langle a+b \rangle \oplus \left\langle \frac{ab}{a+b} \right\rangle = \langle a+b \rangle \oplus \left\langle \frac{ab}{a+b} \cdot (a+b)^2 \right\rangle = \langle a+b \rangle \oplus \langle ab(a+b) \rangle .$$

□

Example 1. Let $\mathbb{F} = \mathbb{C}$ and let B be a symmetric, nondegenerate bilinear form over $\mathbb{C}$. Then

$$B = \langle a_1 \rangle \oplus \ldots \oplus \langle a_n \rangle, \qquad a_i \in \dot{\mathbb{C}}.$$

Let $a \in \dot{\mathbb{C}}$. Then $\langle a \rangle = \langle 1 \cdot (\sqrt{a})^2 \rangle = \langle 1 \rangle \in W(\mathbb{C})$. Similarly, if $b \in \dot{\mathbb{C}}$ then $\langle b \rangle = \langle -1(i\sqrt{b})^2 \rangle = \langle -1 \rangle \in W(\mathbb{C})$. So

$$\langle a \rangle \oplus \langle b \rangle = \langle 1 \rangle \oplus \langle -1 \rangle = 0 \in W(\mathbb{C}).$$

Then we have

$$\langle a_1\rangle \oplus \langle a_2\rangle \oplus \ldots \oplus \langle a_n\rangle = \begin{cases} 0 & ; \quad \text{if } n \text{ is even} \\ \langle 1\rangle & ; \quad \text{if } n \text{ is odd.} \end{cases}$$

Thus $W(\mathbb{C}) \cong \mathbb{Z}_2$.

Example 2. Let $\mathbb{F} = \mathbb{R}$ and $a \in \mathbb{R}$. If $a > 0$ then $\langle a\rangle = \langle 1 \cdot (\sqrt{a})^2\rangle = \langle 1\rangle$. If $a < 0$ then $\langle a\rangle = \left\langle -1 \cdot (\sqrt{-a})^2\right\rangle = \langle -1\rangle$. So we have

$$\langle a_1\rangle \oplus \langle a_2\rangle \oplus \ldots \oplus \langle a_n\rangle = \begin{cases} \underbrace{\langle 1\rangle \oplus \ldots \oplus \langle 1\rangle}_{k} = k & ; \quad \exists k+m \text{ positive } a_i, m \text{ negative } a_i \\ \underbrace{\langle -1\rangle \oplus \ldots \oplus \langle -1\rangle}_{\ell} = \ell & ; \quad \exists \ell + r \text{ negative } a_i, r \text{ positive } a_i. \end{cases}$$

Thus $W(\mathbb{R}) \cong \mathbb{Z}$ with the isomorphism being $\langle a\rangle \mapsto \text{sign}(a)$.

For the case of finite fields, $\mathbb{Z}_p$, we will need to know more information about the prime p. As before, we will consider only fields of characteristic not two, which in this case is equivalent to saying that p is an odd prime. We prove several propositions before calculating the Witt rings of finite fields.

Proposition 8. Let p be an odd prime. Then -1 is a square in $\mathbb{Z}_p$ if and only if $p \equiv 1 \pmod 4$.

Proof. First suppose that -1 is a square in $\mathbb{Z}_p$. Then $-1 \equiv \lambda^2 \pmod p$ for some $\lambda \in \mathbb{Z}_p$. Then $(\lambda^2)^2 \equiv (-1)^2 \pmod p \Rightarrow \lambda^4 \equiv 1 \pmod p$. In particular, the order of λ in the multiplicative group $\dot{\mathbb{Z}}_p$ is 4. Since $\dot{\mathbb{Z}}_p$ is cyclic (see Corollary 19 on page 314 in [3]), it has an element of order 4, and by Lagrange's Theorem, the order of $\dot{\mathbb{Z}}_p$ equals $p-1$ and is divisible by 4. So $p-1 = 4k$ for some $k \in \mathbb{Z}$. Thus $p \equiv 1 \pmod 4$.

Now suppose that $p \equiv 1 \pmod 4$. Then $p-1 = 4k$ for some $k \in \mathbb{Z}$. In particular, 4 divides $p-1$, the order of $\dot{\mathbb{Z}}_p$. Since $\dot{\mathbb{Z}}_p$ is cyclic, it has an element of order 4 by Lagrange's Theorem, say λ. Then

$$\begin{aligned} \lambda^4 &\equiv 1 \pmod p \equiv (-1)^2 \pmod p \\ (\lambda^2)^2 &\equiv (-1)^2 \pmod p \\ \lambda^2 &\equiv \pm 1 \pmod p. \end{aligned}$$

But then $\lambda^2 \equiv -1 \pmod p$ since otherwise λ has order 2 (it was chosen to have order 4). Thus -1 is a square in $\mathbb{Z}_p$. □

Proposition 9. Let p be an odd prime and consider the multiplicative group $\dot{\mathbb{Z}}_p$. Then there are $p-1$ elements in $\dot{\mathbb{Z}}_p$ and exactly half of them are squares in $\dot{\mathbb{Z}}_p$. More precisely, $\dot{\mathbb{Z}}_p / \dot{\mathbb{Z}}_p^2 \cong \mathbb{Z}_2$.

Proof. This follows from the observation that $\lambda^2 \equiv (p-\lambda)^2 \pmod p$, so that the map $\dot{\mathbb{Z}}_p \to \dot{\mathbb{Z}}_p^2 : \lambda \mapsto \lambda^2$ is $2:1$. □

It is now clear that if $\alpha, \beta \in \dot{\mathbb{Z}}_p$ are nonsquare elements, then $\langle \alpha \rangle = \langle \beta \rangle$.

Proposition 10. Let p be a prime. Then for each $\lambda \in \dot{\mathbb{Z}}_p$,

$$\langle \lambda \rangle \oplus \langle \lambda \rangle \cong \langle 1 \rangle \oplus \langle 1 \rangle .$$

Proof. If λ is a square in $\mathbb{Z}_p$, say $\lambda \equiv d^2 \pmod p$ for some d, we have $\langle \lambda \rangle \oplus \langle \lambda \rangle = \langle 1 \cdot d^2 \rangle \oplus \langle 1 \cdot d^2 \rangle = \langle 1 \rangle \oplus \langle 1 \rangle$. If -1 is a square in $\mathbb{Z}_p$, then we have $\langle \lambda \rangle \oplus \langle \lambda \rangle = \langle \lambda \rangle \oplus \langle -1\lambda \rangle = \mathbb{H}$, where $\mathbb{H}$ is hyperbolic, and hence the statement is trivially true.

Now suppose that neither of $-1, \lambda$ is a square in $\mathbb{Z}_p$. Observe that $\dot{\mathbb{Z}}_p^2$ and $1 + \dot{\mathbb{Z}}_p^2$ have the same number of elements, namely $\frac{p-1}{2}$. Note that $1 = 1^2 \in \dot{\mathbb{Z}}_p^2 \setminus (1 + \dot{\mathbb{Z}}_p^2)$.

Thus there exists an element α of the form $\alpha = 1 + a^2$ with α not a square. Since α and λ are not squares, there is an element $b \in \dot{\mathbb{Z}}_p$ such that $\lambda = \alpha \cdot b^2$ by Proposition 9. So we have $\lambda = (1 + a^2)b^2 = b^2 + (ab)^2$.

Suppose that $\langle 1 \rangle \oplus \langle 1 \rangle \cong \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with respect to the basis $\{e_1, e_2\}$. Define a new basis

$$f_1 = \frac{1}{\alpha} e_1 - \frac{a}{\alpha} e_2 = \frac{1}{1 + a^2} e_1 - \frac{a}{1 + a^2} e_2$$
$$f_2 = abe_1 + be_2.$$

Consider the transition matrix from $\{e_1, e_2\}$ to $\{f_1, f_2\}$

$$P = \begin{bmatrix} \frac{1}{1+a^2} & -\frac{a}{1+a^2} \\ ab & b \end{bmatrix}.$$

and note that $\det(P) = \frac{b}{1+a^2} + \frac{a^2 b}{1+a^2} = \frac{b+a^2 b}{1+a^2} = \frac{b(1+a^2)}{1+a^2} = b \in \dot{\mathbb{Z}}_p$. Then we have

$$P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P^T = \begin{bmatrix} \frac{1}{1+a^2} & 0 \\ 0 & b^2 + (ab)^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+a^2} & 0 \\ 0 & \lambda \end{bmatrix} = \left\langle \frac{1}{1 + a^2} \right\rangle \oplus \langle \lambda \rangle .$$

So we have $\langle 1 \rangle \oplus \langle 1 \rangle \cong \left\langle \frac{1}{1+a^2} \right\rangle \oplus \langle \lambda \rangle$. Since $1 + a^2$ is not a square, $\frac{1}{1+a^2}$ is also a nonsquare. Since λ is not a square, $\frac{1}{1+a^2} = \lambda \cdot d^2$ for some $d \in \dot{\mathbb{Z}}_p$ (again, by Proposition 9). Thus

$$\langle 1 \rangle \oplus \langle 1 \rangle \cong \left\langle \frac{1}{1 + a^2} \right\rangle \oplus \langle \lambda \rangle \cong \left\langle \lambda d^2 \right\rangle \oplus \langle \lambda \rangle \cong \langle \lambda \rangle \oplus \langle \lambda \rangle .$$

□

Proposition 11. Let (V, B) be an n-dimensional bilinear space over $\mathbb{Z}_p$. Then (V, B) is isotropic if $n \geq 3$.

Proof. If (V, B) has dimension 3 or greater, then either

$$B = \langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle \oplus B' \quad \text{or}$$
$$B = \langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle \beta \rangle \oplus B' \quad \text{or}$$
$$B = \langle 1 \rangle \oplus \langle \beta \rangle \oplus \langle \beta \rangle \oplus B' \quad \text{or}$$
$$B = \langle \beta \rangle \oplus \langle \beta \rangle \oplus \langle \beta \rangle \oplus B'$$

where β is any nonsquare element. By Proposition 10, each of the above cases can be reduced:

$$\langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle \oplus B' = \langle 1 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle \oplus B' = \langle -1 \rangle \oplus B'$$
$$\langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle \beta \rangle \oplus B' = \langle -\beta \rangle \oplus \langle -\beta \rangle \oplus \langle \beta \rangle \oplus B' = \langle -\beta \rangle \oplus B'$$
$$\langle 1 \rangle \oplus \langle \beta \rangle \oplus \langle \beta \rangle \oplus B' = \langle 1 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle \oplus B' = \langle -1 \rangle \oplus B'$$
$$\langle \beta \rangle \oplus \langle \beta \rangle \oplus \langle \beta \rangle \oplus B' = \langle \beta \rangle \oplus \langle -\beta \rangle \oplus \langle -\beta \rangle \oplus B' = \langle -\beta \rangle \oplus B'.$$

In any case, we have split off a hyperbolic from B and obtained a form of dimension 2 less. □

With these observations in hand, we compute the Witt rings of finite fields.

Example 3. Let $p \equiv 3 \pmod 4$ and consider the possible forms over $\mathbb{Z}_p$. The only 0-dimensional form is the zero form. In $\mathbb{Z}_p$, an element is either a square or not a square by Proposition 9. So the possible 1-dimensional forms are

$$\langle a \rangle = \begin{cases} \langle 1 \rangle & ; \quad a \text{ is a square} \\ \langle -1 \rangle & ; \quad a \text{ is not a square.} \end{cases}$$

The possible 2-dimensional forms are then

$$\begin{aligned} &\langle 1 \rangle \oplus \langle 1 \rangle \\ &\langle 1 \rangle \oplus \langle -1 \rangle \\ &\langle -1 \rangle \oplus \langle 1 \rangle \\ &\langle -1 \rangle \oplus \langle -1 \rangle . \end{aligned}$$

Note that $\langle 1 \rangle \oplus \langle -1 \rangle = \langle -1 \rangle \oplus \langle 1 \rangle = 0$. Also,

$$\begin{aligned} (\langle 1 \rangle \oplus \langle 1 \rangle) \oplus (\langle -1 \rangle \oplus \langle -1 \rangle) &= \langle 1 \rangle \oplus (\langle 1 \rangle \oplus \langle -1 \rangle) \oplus \langle -1 \rangle \\ &= \langle 1 \rangle \oplus 0 \oplus \langle -1 \rangle \\ &= \langle 1 \rangle \oplus \langle -1 \rangle \\ &= 0. \end{aligned}$$

So $\langle 1 \rangle \oplus \langle 1 \rangle = -(\langle -1 \rangle \oplus \langle -1 \rangle)$. We obtain

$$W(\mathbb{Z}_p) = \{0, \langle 1 \rangle, \langle -1 \rangle, \langle 1 \rangle \oplus \langle 1 \rangle\}$$

where $\langle -1 \rangle = \langle -1 \rangle \oplus (\langle -1 \rangle \oplus \langle 1 \rangle) = (\langle -1 \rangle \oplus \langle -1 \rangle) \oplus \langle 1 \rangle = \langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle$. It remains to show that $\langle 1 \rangle \oplus \langle 1 \rangle$ is not hyperbolic (otherwise it would be 0 in $W(\mathbb{Z}_p)$). Note that

$1 \in \mathbb{Z}_p$ is a square $(1 = 1^2)$ and that $\det(\langle 1\rangle \oplus \langle 1\rangle) = \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \in \dot{\mathbb{Z}}_p/\dot{\mathbb{Z}}_p^2$. By Proposition 2, $\langle 1\rangle \oplus \langle 1\rangle$ is not hyperbolic since other wise it would have determinant $-1 \in \dot{\mathbb{Z}}_p/\dot{\mathbb{Z}}_p^2$. Thus we obtain the isomorphism, $W(\mathbb{Z}_p) \cong \mathbb{Z}_4$.

Example 4. Let $p \equiv 1 \pmod 4$ and consider the possible forms over $\mathbb{Z}_p$. Again the only 0-dimensional form is the zero form. Since -1 is a square in this case, $\langle 1\rangle = \langle -1\rangle$. So the possible 1-dimensional forms are

$$\langle a\rangle = \begin{cases} \langle 1\rangle & ; \quad a \text{ is a square} \\ \langle \beta\rangle & ; \quad \text{for any } \beta \in \dot{\mathbb{Z}}_p \setminus \dot{\mathbb{Z}}_p^2 \text{ and } a \text{ is not a square.} \end{cases}$$

The possible 2-dimensional forms are

$$\langle 1\rangle \oplus \langle 1\rangle$$
$$\langle 1\rangle \oplus \langle \beta\rangle$$
$$\langle \beta\rangle \oplus \langle 1\rangle$$
$$\langle \beta\rangle \oplus \langle \beta\rangle .$$

Note that $\langle 1\rangle \oplus \langle \beta\rangle = \langle \beta\rangle \oplus \langle 1\rangle$. Also $\langle 1\rangle \oplus \langle 1\rangle = \langle 1\rangle \oplus \langle -1\rangle = 0$ and $\langle \beta\rangle \oplus \langle \beta\rangle = \langle \beta\rangle \oplus \langle -\beta\rangle = 0$. So we obtain

$$W(\mathbb{Z}_p) = \{0, \langle 1\rangle, \langle \beta\rangle, \langle 1\rangle \oplus \langle \beta\rangle\}.$$

It remains to show that $\langle 1\rangle \oplus \langle \beta\rangle$ is not hyperbolic (otherwise it would be 0 in $W(\mathbb{Z}_p)$). Suppose that $\langle 1\rangle \oplus \langle \beta\rangle$ were hyperbolic and let $v = (a, b)$, $a, b \in \mathbb{Z}_p$ be an isotropic vector. Then

$$
\begin{aligned}
0 &= (\langle 1\rangle \oplus \langle \beta\rangle)(v,v) \\
&= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\
&= a^2 + \beta b^2
\end{aligned}
$$

But then $\beta = -\frac{a^2}{b^2} \Rightarrow \beta$ is a square, which is a contradiction. So $\langle 1\rangle \oplus \langle \beta\rangle$ is not hyperbolic. Thus we obtain the isomorphism $W(\mathbb{Z}_p) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Remark 6. We will not directly compute the Witt ring of $\mathbb{Z}_2$ since this field has characteristic two. We will use the fact that $W(\mathbb{Z}_2) \cong \mathbb{Z}_2 = \{0, \langle 1\rangle\}$ in the calculation of the Witt ring of $\mathbb{Q}$. The calculation is worked out in [15].

2.6 The Witt Ring of $\mathbb{Q}$

We now calculate the Witt ring of $\mathbb{Q}$, $W(\mathbb{Q})$, to be used later in our application to knots. Recall that $W(\mathbb{Q})$ is generated by one dimensional forms $\langle a\rangle$ where $a \in \dot{\mathbb{Q}}$. We first define a homomorphism $\partial_p : W(\mathbb{Q}) \to W(\mathbb{Z}_p)$ for each prime p.

Let p be an odd prime and let $a \in \mathbb{Q}$. Write $a = p^n \cdot \frac{c}{d}$ where c, d are relatively prime to p and $n \in \mathbb{Z}$. Define $\partial_p : W(\mathbb{Q}) \to W(\mathbb{Z}_p)$ on generators (and extend linearly) by

$$
\partial_p \langle a\rangle = \partial_p \left\langle p^n \cdot \frac{c}{d} \right\rangle = \begin{cases} 0 & ; \quad n \text{ is even} \\ \langle c \cdot d\rangle & ; \quad n \text{ is odd.} \end{cases}
$$

We claim that ∂_p is well defined. To see this we will check our relations from the above presentation of the Witt ring (Theorem 7):

$$\partial_p \left\langle \lambda^2 \cdot a \right\rangle = \partial_p \left\langle a \right\rangle$$

$$\partial_p(\langle a\rangle \oplus \langle b\rangle) = \partial_p(\langle a+b\rangle \oplus \langle ab(a+b)\rangle).$$

First,

$$\begin{aligned}
\partial_p \left\langle \lambda^2 a \right\rangle &= \partial_p \left\langle p^{2m}\frac{e^2}{f^2} \cdot p^n \frac{c}{d} \right\rangle \\
&= \begin{cases} 0 & ; \quad n \text{ is even} \\ \langle e^2 f^2 cd\rangle = \langle cd\rangle & ; \quad n \text{ is odd} \end{cases} \\
&= \partial_p \left\langle a \right\rangle .
\end{aligned}$$

Now suppose that $a = p^n \cdot \frac{c}{d}$ and $b = p^m \cdot \frac{e}{f}$. Then

$$\partial_p(\langle a\rangle \oplus \langle b\rangle) = \begin{cases} 0 & ; \quad m, n \text{ are even} \\ \langle cd\rangle & ; \quad m \text{ is even, } n \text{ is odd} \\ \langle ef\rangle & ; \quad m \text{ is odd, } n \text{ is even} \\ \langle cd\rangle \oplus \langle ef\rangle & ; \quad m, n \text{ are odd.} \end{cases}$$

There are several cases for $\partial_p(\langle a+b\rangle) \oplus \partial_p(\langle ab(a+b)\rangle)$. Consider the case that $n > m > 0$. Then

$$a + b = p^m \left(\frac{p^{n-m}cf + ed}{df} \right).$$

So we have

$$\partial_p(\langle a+b\rangle) = \begin{cases} 0 & ; \quad m \text{ is even} \\ \langle df(p^{n-m}cf+ed)\rangle = \langle efd^2\rangle = \langle ef\rangle & ; \quad m \text{ is odd.} \end{cases}$$

Similarly,

$$ab(a+b) = p^n p^{2m} \frac{ce}{df}\left(\frac{p^{n-m}cf+ed}{df}\right)$$

and so

$$\partial_p(\langle ab(a+b)\rangle) = \begin{cases} 0 & ; \quad n \text{ is even} \\ \langle d^2f^2(p^{n-m}c^2ef+ce^2d)\rangle = \langle cdd^2e^2f^2\rangle = \langle cd\rangle & ; \quad n \text{ is odd.} \end{cases}$$

Direct summing these gives

$$\begin{aligned} \partial_p(\langle a\rangle \oplus \langle b\rangle) \oplus \partial_p(\langle ab(a+b)\rangle) &= \begin{cases} 0 & ; \quad m, n \text{ are even} \\ \langle cd\rangle & ; \quad m \text{ is even, } n \text{ is odd} \\ \langle ef\rangle & ; \quad m \text{ is odd, } n \text{ is even} \\ \langle cd\rangle \oplus \langle ef\rangle & ; \quad m, n \text{ are odd.} \end{cases} \\ &= \partial_p(\langle a\rangle \oplus \langle b\rangle). \end{aligned}$$

The cases of the other parities of n and m are similar.

Also, ∂_p is surjective. To see this, first let $p \equiv 3 \pmod 4$. We note that

$$\partial_p \langle p \rangle = \partial_p \left\langle p^1 \cdot 1 \right\rangle = \langle 1 \rangle .$$

Since we have an element in $W(\mathbb{Q})$ that maps to a generator $W(\mathbb{Z}_p)$, ∂_p is surjective when $p \equiv 3 \pmod 4$. Now let $p \equiv 1 \pmod 4$. Then

$$\partial_p \langle p \rangle = \langle 1 \rangle \qquad \text{and}$$
$$\partial_p \langle \beta \cdot p \rangle = \langle \beta \rangle .$$

Again, we have mapped to generators, and so ∂_p is surjective for all odd primes p.

For $p = 2$ (where the characteristic is two), we define $\partial_2 : W(\mathbb{Q}) \to W(\mathbb{Z}_2)$ as

$$\partial_2 \langle a \rangle = \partial_2 \left\langle 2^n \cdot \frac{c}{d} \right\rangle = \begin{cases} 0 & ; \qquad n \text{ is even} \\ \langle 1 \rangle & ; \qquad n \text{ is odd.} \end{cases}$$

Now define $\sigma : W(\mathbb{Q}) \to \mathbb{Z}$ on generators (and extend linearly) as

$$\sigma \langle a \rangle = \text{sign}(a).$$

The function σ will be called the *signature*. We claim that σ is also well defined. To see this we observe that $\sigma \langle \lambda^2 a \rangle = \sigma \langle a \rangle$ since $\lambda^2 > 0$. For the second relation, we have several cases:

First suppose that $a, b > 0$. Then $a + b > 0$ and $ab(a + b) > 0$ and we have

$$\sigma(\langle a \rangle \oplus \langle b \rangle) = \sigma(\langle a \rangle) + \sigma(\langle b \rangle) = 2 \qquad \text{and}$$
$$\sigma(\langle a + b \rangle \oplus \langle ab(a + b) \rangle) = \sigma(\langle a + b \rangle) + \sigma(\langle ab(a + b) \rangle) = 2.$$

Now suppose that $a < 0$ and $b > 0$. If $|a| < |b|$, then $a + b > 0$ and $ab(a + b) < 0$. Similarly, if $|b| < |a|$, then $a + b < 0$ and $ab(a + b) > 0$. In either case we have

$$\sigma(\langle a\rangle \oplus \langle b\rangle) = \sigma(\langle a\rangle) + \sigma(\langle b\rangle) = 0 \qquad \text{and}$$
$$\sigma(\langle a+b\rangle \oplus \langle ab(a+b)\rangle) = \sigma(\langle a+b\rangle) + \sigma(\langle ab(a+b)\rangle) = 0.$$

The cases where $a > 0$ and $b < 0$ are completely symmetric to the one above. Lastly, suppose that $a, b < 0$. Then $a + b < 0$ and $ab(a + b) < 0$ so we have

$$\sigma(\langle a\rangle \oplus \langle b\rangle) = \sigma(\langle a\rangle) + \sigma(\langle b\rangle) = -2 \qquad \text{and}$$
$$\sigma(\langle a+b\rangle \oplus \langle ab(a+b)\rangle) = \sigma(\langle a+b\rangle) + \sigma(\langle ab(a+b)\rangle) = -2.$$

Thus the signature σ is well defined.

Also, define $\iota : \mathbb{Z} \to W(\mathbb{Q})$ to be the inclusion map where

$$n \mapsto \underbrace{\langle 1\rangle \oplus \ldots \oplus \langle 1\rangle}_{n}.$$

Theorem 8. The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} W(\mathbb{Q}) \xrightarrow{\partial} \bigoplus_{p=\text{ prime}} W(\mathbb{Z}_p) \longrightarrow 0$$

is split exact with $\sigma : W(\mathbb{Q}) \to \mathbb{Z}$ being a splitting homomorphism. Here $\partial = \bigoplus_{p=\text{ prime}} \partial_p$. In particular,

$$W(\mathbb{Q}) \cong \mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}_4.$$

Proof. Theorem 2.1 in [6]. □

Chapter 3

Knots

We are now finally ready to apply our construction of $W(\mathbb{Q})$ to knots. We will begin with a description of the linking form of a knot and use the rational Witt class of this form as a knot invariant related to knot concordance. Either the topological, smooth, or algebraic concordance groups can be used. We will see that having a trivial Witt class will be a prerequisite for a knot being slice. The main results will come in the form of specific restrictions on 4-stranded pretzel knots to have trivial Witt class. More background on knots can be found in [1, 9, 11, 12, 14].

3.1 Linking Forms

Basic definitions about knots embedded in the 3-sphere will be assumed. Familiarity with Seifert surfaces will also be assumed. We will take as convention the positive and negative crossings in a regular projection of oriented curves in S^3 as in Figure 3.1.

Using this convention, we can assign a linking number to a pair of oriented curves in a Seifert surface of a link. Suppose we have two curves α and β, each with orien-

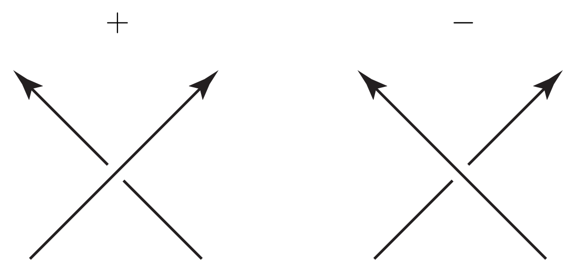


Figure 3.1: Sign Convention for Crossings in a Regular Projection.

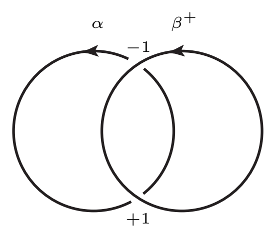


Figure 3.2: Examples of Linking Numbers.

tation, and we wish to see how they link together. Choose β and push it off in the positively oriented direction of the Seifert surface so as to make it disjoint from α. Denote this push off of β as β^+. Each crossing in a regular projection of $\alpha \cup \beta^+$ is given a value of ± 1, respecting the convention of positive and negative crossings as in Figure 3.1. For example, see Figure 3.2.

To get the *linking number* of the curve α with the curve β, we add up all the crossings and then divide by two. In Figure 3.2, the left diagram has linking number 0, which we'll denote $\text{lk}(\alpha, \beta) = 0$. This should be clear since it is obvious that α and β do not link; we can separate them just by pulling them apart. In the right diagram, we have $\text{lk}(\alpha, \beta) = \frac{1+1}{2} = 1$. Again, we can see that α and β do in fact link, but the linking number will depend on the value of the crossings.

Definition 14. Let K be a knot and Σ an oriented Seifert surface for K. Then the *linking* or *Seifert form* on Σ, $\mathrm{lk} : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$, is given by

$$\mathrm{lk}(\alpha, \beta) = \text{ linking number of } \alpha \text{ and } \beta^+$$

where α, β are oriented curves in Σ and β^+ is the push-off of β in the positively oriented direction of Σ. We will arrange these values into a matrix, called a *Seifert matrix*, as follows:

For $\alpha_1, \ldots, \alpha_{2g} \in H_1(\Sigma)$ a basis for $H_1(\Sigma)$, the *Seifert matrix* consists of all pairs of linking numbers

$$\mathrm{lk} = \begin{bmatrix} \mathrm{lk}(\alpha_1, \alpha_1^+) & \mathrm{lk}(\alpha_1, \alpha_2^+) & \cdots & \mathrm{lk}(\alpha_1, \alpha_{2g}^+) \\ \mathrm{lk}(\alpha_2, \alpha_1^+) & \mathrm{lk}(\alpha_2, \alpha_2^+) & \cdots & \mathrm{lk}(\alpha_2, \alpha_{2g}^+) \\ \vdots & \ddots & \ddots & \vdots \\ \mathrm{lk}(\alpha_{2g}, \alpha_1^+) & \mathrm{lk}(\alpha_{2g}, \alpha_2^+) & \cdots & \mathrm{lk}(\alpha_{2g}, \alpha_{2g}^+) \end{bmatrix}.$$

Example 5. A Seifert surface Σ for the right-handed trefoil ($T_{2,3}$) with generators for $H_1(\Sigma)$ is shown in Figure 3.3. The Seifert matrix is then

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Note that in general the Seifert matrix is not symmetric as typically, $\mathrm{lk}(\alpha, \beta) \neq \mathrm{lk}(\beta, \alpha)$. It is true, however, that the Seifert matrix plus its transpose ($\mathrm{lk} + \mathrm{lk}^T$) is a symmetric and nondegenerate matrix ([14]). We can apply our previous constructions to this symmetric matrix (diagonalizing, etc.) to obtain the Witt class of a knot.

We now view this symmetrized linking form by extending linearly as $(\mathrm{lk} + \mathrm{lk}^T)$:

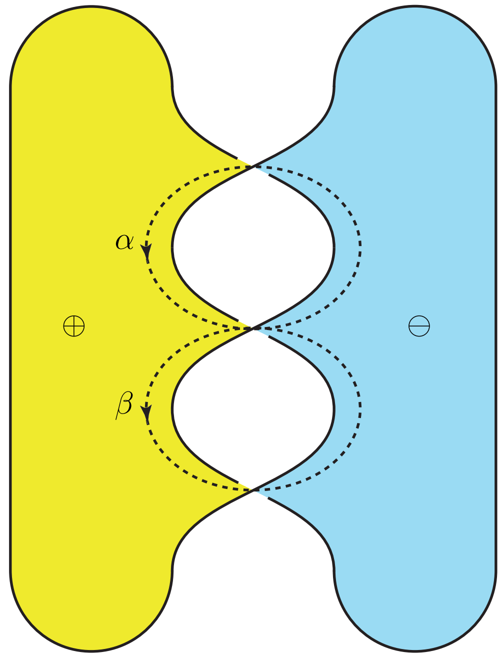


Figure 3.3: Seifert Surface Σ for $T_{2,3}$ with Generators for $H_1(\Sigma)$.

$H_1(\Sigma;\mathbb{Q}) \times H_1(\Sigma;\mathbb{Q}) \to \mathbb{Q}$. Recall that since a Seifert surface Σ is orientable, it is homeomorphic to a genus g surface for some g. Then $H_1(\Sigma) \cong \mathbb{Q}^{2g}$.

Definition 15. The *determinant* of a knot K is defined to be the determinant of the symmetrized linking matrix, i.e.,

$$\det(K) = \det(\mathrm{lk} + \mathrm{lk}^T).$$

3.2 The Rational Witt Class of a Knot

In this section we define the rational Witt class of a knot K and discuss its independence on the Seifert surface chosen for K. Proofs of the assertions in this section can be found in [9, 14].

Definition 16. The *rational Witt class*, $\phi(K)$, of the knot K is

$$\phi(K) = (H_1(\Sigma;\mathbb{Q}), \mathrm{lk} + \mathrm{lk}^T) \in W(\mathbb{Q}).$$

Clearly, this definition could easily depend on our choice of Seifert surface for the knot K. It is proven in [9] that if Σ_1 and Σ_2 are two oriented Seifert surfaces for a knot K, then there are a finite number of stabilizations/destabilizations[1] connecting Σ_1 and Σ_2.

It is not hard to see that the linking form lk' of a stabilized Seifert surface Σ is related to the linking form lk of Σ as

$$\mathrm{lk}' = \left[\begin{array}{ccc|cc} & & & 0 & 0 \\ & \mathrm{lk} & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \ldots & 0 & 0 & 1 \\ 0 & \ldots & 0 & 0 & 0 \end{array}\right]$$

with respect to a particular choice of basis. Accordingly, the rational Witt classes of $\mathrm{lk} + \mathrm{lk}^T$ and $\mathrm{lk}' + \mathrm{lk}'^T$ are equal since the forms differ by a hyperbolic.

Theorem 9. Let K_1 and K_2 be knots. Then $\phi(K_1\#K_2) = \phi(K_1) \oplus \phi(K_2)$, where $\#$ denotes the connect sum of knots.

Proof. This essentially comes down to a Mayer-Vietoris sequence. Let Σ_1 and Σ_2 be Seifert surfaces for K_1 and K_2, respectively. Then a Seifert surface for $K_1\#K_2$, which we will denote $\Sigma_1\#\Sigma_2$, is given by gluing a band $[0,1] \times [0,1]$ between them, respecting orientation.[2] For a pictorial description, see Figure 3.4.[3] Then the exact

[1]Stabilizing a Seifert surface Σ is the process of connect summing it with a torus T^2, so as to form $\Sigma\#T^2$. Destabilization is the inverse of this process.

[2]Here $\Sigma_1\#\Sigma_2$ denotes the boundary connected sum as opposed to the ordinary connected sum of surfaces.

[3]Here we used genus one surfaces for the illustrations, but this construction works with surfaces of any genus.

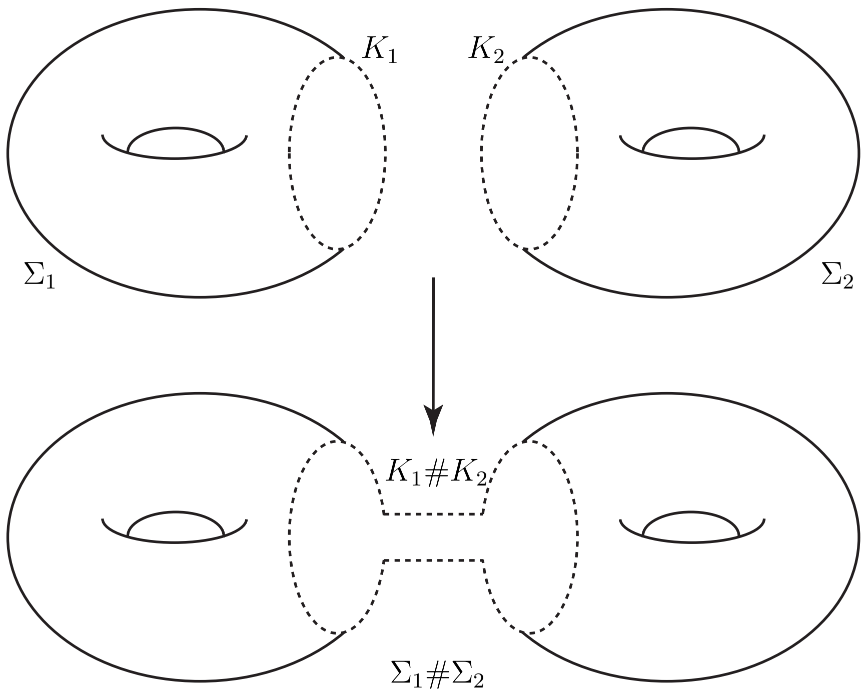


Figure 3.4: Process of Obtaining $\Sigma_1 \# \Sigma_2$ from Σ_1 and Σ_2.

Mayer-Vietoris sequence (in reduced homology) associated to Σ_1 and Σ_2 is

$$\ldots \longrightarrow \widetilde{H}_1(\Sigma_1 \cap \Sigma_2) \longrightarrow \widetilde{H}_1(\Sigma_1) \oplus \widetilde{H}_1(\Sigma_2) \longrightarrow \widetilde{H}_1(\Sigma_1 \# \Sigma_2) \longrightarrow \widetilde{H}_0(\Sigma_1 \cap \Sigma_2) \longrightarrow \ldots$$

Note that $\Sigma_1 \cap \Sigma_2 = [0,1]$ and so $\widetilde{H}_1(\Sigma_1 \cap \Sigma_2) = \widetilde{H}_1([0,1]) = 0$. Also, $\widetilde{H}_0(\Sigma_1 \cap \Sigma_2) = 0$. Then we can write the corresponding portion of this exact sequence as

$$0 \longrightarrow \widetilde{H}_1(\Sigma_1) \oplus \widetilde{H}_1(\Sigma_2) \longrightarrow \widetilde{H}_1(\Sigma_1 \# \Sigma_2) \longrightarrow 0.$$

Thus, we must have that $\widetilde{H}_1(\Sigma_1) \oplus \widetilde{H}_1(\Sigma_2) \cong \widetilde{H}_1(\Sigma_1 \# \Sigma_2)$. In particular,

$$H_1(\Sigma_1 \# \Sigma_2) \cong H_1(\Sigma_1) \oplus H_1(\Sigma_2).$$

If $\mathrm{lk}_i + \mathrm{lk}_i^T$ is the symmetrized linking form for Σ_i, then the linking form for $\Sigma_1 \# \Sigma_2$ is the block sum of matrices

$$\begin{bmatrix} \mathrm{lk}_1 + \mathrm{lk}_1^T & \mathbf{0} \\ \mathbf{0} & \mathrm{lk}_2 + \mathrm{lk}_2^T \end{bmatrix}$$

since the generators for $H_1(\Sigma_1)$ will link trivially with the generators for $H_1(\Sigma_2)$. This is because the generators for $H_1(\Sigma_1)$ can easily be made disjoint from the generators of $H_1(\Sigma_2)$ (for example, by stretching the band connecting Σ_1 and Σ_2 as necessary). □

This discussion, along with the observation that the unknot has trivial rational Witt class, has shown the following proposition:

Proposition 12. Let $\mathcal{K}$ be the semigroup of knots with the operation of connect sum. Then the map $\phi : \mathcal{K} \to W(\mathbb{Q})$ is a semigroup homomorphism.

We now introduce the relationship between the rational Witt ring and knot concordance.

Definition 17. An oriented knot $K \subseteq S^3$ is called *slice* if there exists a smoothly embedded 2-disk D^2 in the 4-ball D^4 such that $\partial(D^4, D^2) = (S^3, K)$. Two oriented knots K_0 and K_1 are called *concordant* if $K_0 \#(-K_1)$ is slice (where $-K$ is the reverse mirror image of K).

The notion of concordance between knots is an equivalence relation and its equivalence classes with respect to the operation of connect sum form an Abelian group, $\mathcal{C}$, referred to as the *Concordance group*. It can be shown that the inverse of $K \in \mathcal{C}$ is given by $-K$.

Theorem 10. Let K be a knot. If K is slice, then $(H_1(\Sigma; \mathbb{Q}), \mathrm{lk} + \mathrm{lk}^T)$ is totally isotropic. In particular (by Theorem 6), $\phi(K) = 0 \in W(\mathbb{Q})$.

Proof. Proposition 8.E.19 in [14]. □

3.3 Calculations of Rational Witt Classes of Some Knots

We finally begin to compute some examples of the rational Witt class of specific knots.

Example 6. Right-Handed Trefoil ($T_{2,3}$ torus knot): In Example 5 we computed the Seifert matrix with respect to the chosen Seifert surface and generators to be

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Then the symmetrized linking form is

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

Diagonalizing this matrix gives

$$\begin{bmatrix} -2 & 0 \\ 0 & -\frac{3}{2} \end{bmatrix} = \langle -2 \rangle \oplus \left\langle -\frac{3}{2} \right\rangle \in W(\mathbb{Q}).$$

It is usually not a trivial task to compute the order of a knot from this calculation. For instance, in general, by just looking at this Witt class we do not know if it is trivial, of order 2, 4 or infinite order. To check this, we need to express this form in terms of generators, and we do this by considering the maps ∂_p for *every* prime p and the signature map σ. Obviously, this could be a very time consuming task, but it is possible to do so for a concrete choice of knot. We will continue this example with a computation of the Witt class of $T_{2,3}$ in terms of generators.

First note that the signature $\sigma(T_{2,3}) = -2$. This immediately tells us that the right-handed trefoil is of infinite order in $W(\mathbb{Q})$. For completeness, we now consider the maps ∂_p for each prime p. First suppose $p \equiv 3 \pmod 4$. For $p = 3$ we have

$$\partial_3(\langle -2\rangle) = \partial_3\left(\left\langle 3^0 \cdot \frac{-2}{1}\right\rangle\right) = 0 \in W(\mathbb{Z}_3) \cong \mathbb{Z}_4$$

and

$$\partial_3\left(\left\langle -\frac{3}{2}\right\rangle\right) = \partial_3\left(\left\langle 3^1 \cdot \frac{-1}{2}\right\rangle\right) = \langle -2\rangle = \langle 1\rangle \in W(\mathbb{Z}_3).$$

Now we have $\partial_3(\langle -2\rangle \oplus \langle -\frac{3}{2}\rangle) = 0 \oplus \langle 1\rangle = \langle 1\rangle \in W(\mathbb{Z}_3) \cong \mathbb{Z}_4$. Thus for $p = 3$ we associate $1 \in \mathbb{Z}_4$ to $T_{2,3}$. For all other primes $p \equiv 3 \pmod 4$ we will have $-2 = p^0 \cdot \frac{-2}{1}$ and $-\frac{3}{2} = p^0 \cdot \frac{-3}{2}$ and so we will associate $0 \in \mathbb{Z}_4$. Similarly, we will associate $0 = (0, 0) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for all primes $p \equiv 1 \pmod 4$. Thus we obtain the rational Witt class of $T_{2,3}$

$$\phi(T_{2,3}) = (\underbrace{-2}_{\mathbb{Z}}, \underbrace{0, \ldots}_{\mathbb{Z}_2^\infty}, \underbrace{1, 0, \ldots}_{\mathbb{Z}_4^\infty}) \in W(\mathbb{Q}).$$

Here we have used the notation $\mathbb{Z}_2^\infty = \bigoplus_{i=1}^\infty \mathbb{Z}_2$ and $\mathbb{Z}_4^\infty = \bigoplus_{i=1}^\infty \mathbb{Z}_4$. The reader may recall that when $p \equiv 1 \pmod 4$, $W(\mathbb{Z}_p) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In the above notation we really consider the $\mathbb{Z}_2$ entries in pairs so that every pair of entries lies in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, but for simplification of notation, we ignore the extraneous parentheses.

Example 7. Torus knots $T_{2,n}$, $n > 0$ and odd: In general, the symmetrized linking form for $T_{2,n}$ after diagonalization is

$$\langle -2\rangle \oplus \left\langle -\frac{3}{2}\right\rangle \oplus \left\langle -\frac{4}{3}\right\rangle \oplus \ldots \oplus \left\langle -\frac{n}{n-1}\right\rangle.$$

First we claim that

$$\langle -2\rangle \oplus \left\langle -\frac{3}{2}\right\rangle \oplus \left\langle -\frac{4}{3}\right\rangle \oplus \ldots \oplus \left\langle -\frac{n}{n-1}\right\rangle = \langle n\rangle \oplus \underbrace{\langle -1\rangle \oplus \ldots \oplus \langle -1\rangle}_{n}.$$

We get this by applying our relations from the construction of $W(\mathbb{Q})$. We proceed by induction on n. First, for $n = 3$ we have

$$\begin{aligned}
\langle 3\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle &= \langle 3 + (-1)\rangle \oplus \langle 3(-1)(3 + (-1))\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle \\
&= \langle 2\rangle \oplus \langle -6\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle \\
&= \langle 2\rangle \oplus \langle -6\rangle \oplus \langle -1 + (-1)\rangle \oplus \langle (-1)(-1)(-1 + (-1))\rangle \\
&= \langle 2\rangle \oplus \langle -6\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle \\
&= \langle -2\rangle \oplus \langle -6\rangle \\
&= \langle -2\rangle \oplus \left\langle -6\left(\frac{1}{2}\right)^2\right\rangle \\
&= \langle -2\rangle \oplus \left\langle -\frac{3}{2}\right\rangle.
\end{aligned}$$

This is what we computed as the form for $T_{2,3}$ in the previous example. For induction, suppose that the result holds for each parameter less than n. Then

$$\begin{aligned}
\langle n\rangle \oplus \underbrace{\langle -1\rangle \oplus \ldots \oplus \langle -1\rangle}_{n} &= \langle n+(-1)\rangle \oplus \langle n(-1)(n+(-1))\rangle \oplus \underbrace{\langle -1\rangle \oplus \ldots \oplus \langle -1\rangle}_{n-1} \\
&= \langle n-1\rangle \oplus \langle -n(n-1)\rangle \oplus \underbrace{\langle -1\rangle \oplus \ldots \oplus \langle -1\rangle}_{n-1} \\
&= \langle n-1\rangle \oplus \underbrace{\langle -1\rangle \oplus \ldots \oplus \langle -1\rangle}_{n-1} \oplus \langle -n(n-1)\rangle \\
&= \langle -2\rangle \oplus \left\langle -\frac{3}{2}\right\rangle \oplus \ldots \left\langle -\frac{n-1}{n-2}\right\rangle \oplus \left\langle -n(n-1)\left(\frac{1}{n-1}\right)^2\right\rangle \\
&= \langle -2\rangle \oplus \left\langle -\frac{3}{2}\right\rangle \oplus \ldots \left\langle -\frac{n-1}{n-2}\right\rangle \oplus \left\langle -\frac{n}{n-1}\right\rangle .
\end{aligned}$$

When $n < 0$, a similar result holds for $T_{2,n}$:

$$\langle n\rangle \oplus \underbrace{\langle 1\rangle \oplus \ldots \oplus \langle 1\rangle}_{-n} = \langle 2\rangle \oplus \left\langle \frac{3}{2}\right\rangle \oplus \ldots \oplus \left\langle \frac{n}{n-1}\right\rangle .$$

Example 8. Figure-8 knot: A knot diagram for the figure-8 knot, K is given in Figure 3.5. Using this, we can obtain a Seifert surface Σ for K and choose generators for $H_1(\Sigma, \mathbb{Z})$ as in Figure 3.6. The Seifert matrix with respect to this basis is

$$\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

which gives a symmetrized linking form of

$$\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}.$$

Diagonalizing this matrix gives

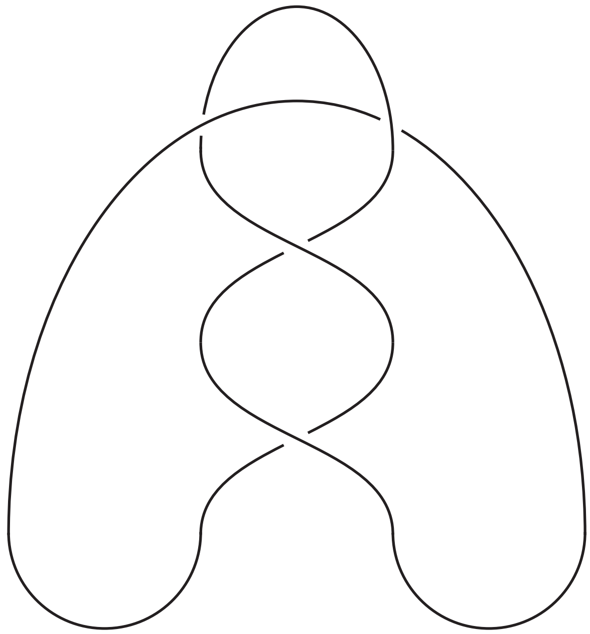


Figure 3.5: Figure-8 Knot.

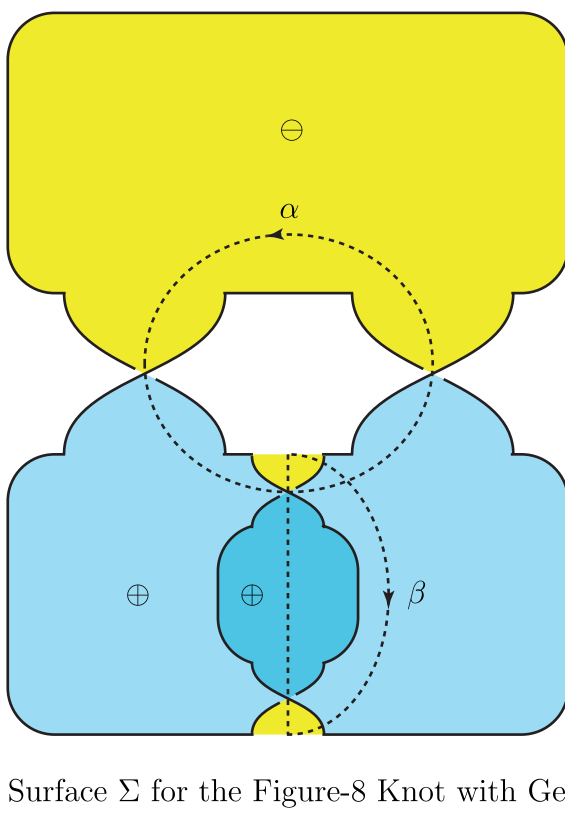


Figure 3.6: Seifert Surface Σ for the Figure-8 Knot with Generators for $H_1(\Sigma)$.

$$\begin{bmatrix} 2 & 0 \\ 0 & -\frac{5}{2} \end{bmatrix} = \langle 2 \rangle \oplus \left\langle -\frac{5}{2} \right\rangle \in W(\mathbb{Q}).$$

So $\sigma(K) = 0$. Again we consider the maps ∂_p for every prime p. The only prime which does not associate a trivial form to $\langle 2 \rangle \oplus \left\langle -\frac{5}{2} \right\rangle$ is $p = 5 \equiv 1 \pmod 4$. We then have

$$\partial_5 \left(\left\langle -\frac{5}{2} \right\rangle \right) = \partial_5 \left(\left\langle 5^1 \cdot \frac{-1}{2} \right\rangle \right) = \langle -2 \rangle .$$

The question now comes down whether -2 is a square in $\mathbb{Z}_5$, which it is not. Thus we associate $(0, 1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ to the form. We finally obtain the Witt class of K

$$\phi(K) = (\underbrace{0}_{\mathbb{Z}}, \underbrace{0, 1, 0, \ldots}_{\mathbb{Z}_2^\infty}, \underbrace{0, \ldots}_{\mathbb{Z}_4^\infty}).$$

We can now see that the figure-8 knot is of order 2.

3.4 Pretzel Knots

In this section we state our results for 4-stranded pretzel knots. Theorem 12 and Corollary 1 are new results which obstruct sliceness by restricting the parameters of the knot. These 4-stranded pretzel knots exhibit interesting restrictions that are unusual amongst pretzel knots.

Remark 7. A prerequisite for a knot K to have trivial rational Witt class is that $\det(K) = \pm m^2$ for some $m \in \mathbb{Z}$. This is clear since the determinant is defined modulo square elements and hyperbolics have determinant -1.

Example 9. A 4-stranded pretzel knot can be compactly written as $P(p, q, r, s)$ where

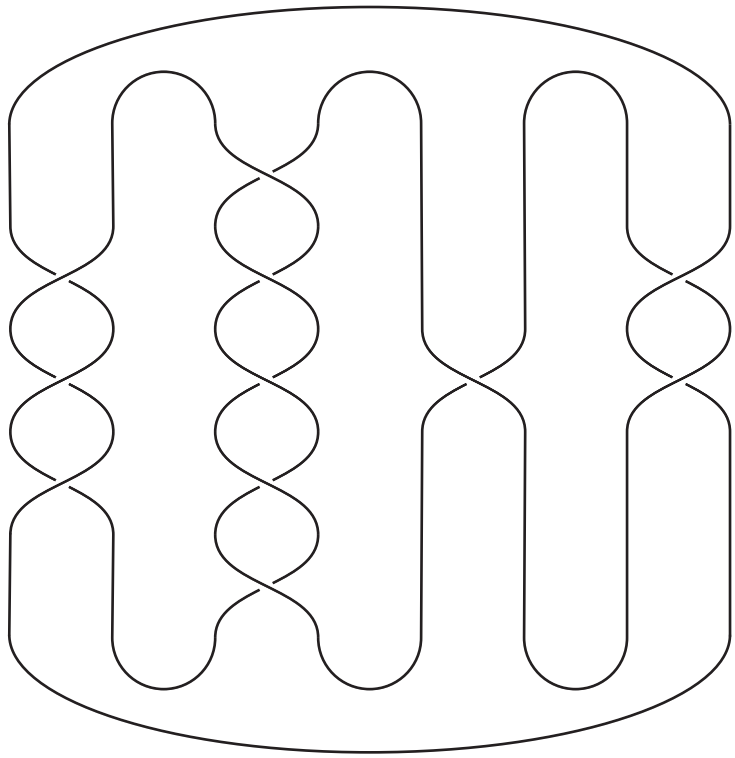


Figure 3.7: Pretzel Knot $P(3,-5,-1,2)$.

$p,q,r,s \in \mathbb{Z}$. The parameters p,q,r,s are the number of twists in each band. For example, see Figure 3.7.

We will be concerned with pretzel knots $P(p,q,r,s)$ where p,q,r are odd and $s \neq 0$ is even. We are not restricting generality here since if at least two of the parameters are even, we get a link instead of a knot. Further, we can arrange the even parameter to be the last of the four listed.

Theorem 11. Let $K = P(p,q,r,s)$ be a 4-stranded pretzel knot with p,q,r odd and $s \neq 0$ even. Then the Witt class of K is

$$\phi(K) = \langle p\rangle \oplus \langle q\rangle \oplus \langle r\rangle \oplus \langle s\rangle \oplus \langle -pqrs \cdot \det(K)\rangle \oplus \begin{cases} \bigoplus_{i=1}^{|p+q+r+s|} \langle 1\rangle & ; \text{ if } p+q+r+s<0 \\ \\ \bigoplus_{i=1}^{|p+q+r+s|} \langle -1\rangle & ; \text{ if } p+q+r+s>0 \end{cases}$$

where $\det(K) = pqr + pqs + prs + qrs$.

Proof. Theorem 1.3 in [7] and applying Lemma 6.1 in [8]. □

With the Witt class of $P(p, q, r, s)$ in hand, we can now proceed to finding specific restrictions on the parameters to obstruct sliceness. To begin, let a be an odd prime. We will further assume that $\det(P(p, q, r, s)) = \pm m^2$ for some $m \in \mathbb{Z}$ (see Remark 7). Consider the maps ∂_a. There are several cases to address:

- a is relatively prime to each of p, q, r, s
- a is relatively prime to exactly three of p, q, r, s
- a is relatively prime to exactly two of p, q, r, s
- a is relatively prime to exactly one of p, q, r, s
- a is not relatively prime to any of p, q, r, s

Here we will address the first three cases. First suppose that a is relatively prime to each of p, q, r, s. In particular, $a \nmid p, q, r, s$. Then

$$\partial_a \langle p \rangle = \partial_a \left\langle a^0 \cdot p \right\rangle = 0 \in W(\mathbb{Z}_a)$$
$$\partial_a \langle q \rangle = \partial_a \left\langle a^0 \cdot q \right\rangle = 0 \in W(\mathbb{Z}_a)$$
$$\partial_a \langle r \rangle = \partial_a \left\langle a^0 \cdot r \right\rangle = 0 \in W(\mathbb{Z}_a)$$
$$\partial_a \langle s \rangle = \partial_a \left\langle a^0 \cdot s \right\rangle = 0 \in W(\mathbb{Z}_a)$$
$$\partial_a \langle -pqrs \cdot \det(K) \rangle = \partial_a \left\langle a^0(-pqrs \cdot \det(K)) \right\rangle = 0 \in W(\mathbb{Z}_a).$$

Clearly, the final summands of $\langle \pm 1 \rangle$ only contribute to the signature of the knot and lie in the kernel of ∂_a. It then follows that

$$\partial_a(\phi(K)) = 0 \in W(\mathbb{Z}_a).$$

Now suppose that a is relatively prime to exactly three of p, q, r, s. Without loss of generality, suppose that $a \big| p$, say $p = a^\lambda b$, and $a \not\big| \; q, r, s$. As above, we obtain $0 \in W(\mathbb{Z}_a)$ for $\partial_a \langle q \rangle, \partial_a \langle r \rangle, \partial_a \langle s \rangle$. If λ is even, we also obtain 0 for $\partial_a \langle p \rangle$ and $\partial_a \langle -pqrs \cdot \det(K) \rangle$. Otherwise, if λ is odd we obtain

$$\partial_a \langle p \rangle = \partial_a \left\langle a^\lambda \cdot b \right\rangle = \langle b \rangle$$

For $\langle -pqrs \cdot \det(K) \rangle$, first note that we can write $\det(K) = \pm m^2 = pqr + pqs + prs + qrs = p(qr + qs + rs) + qrs$. Reducing this modulo a gives

$$qrs \equiv \pm m^2 \pmod{a}.$$

So

$$\partial_a \langle -pqrs \cdot \det(K) \rangle = \partial_a \left\langle a^\lambda(-bqrs \cdot \det(K)) \right\rangle = \langle -b \rangle .$$

Finally,

$$\partial_a(\phi(K)) = \langle b\rangle \oplus \langle -b\rangle = 0 \in W(\mathbb{Z}_a).$$

Now suppose that a is relatively prime to exactly two of p, q, r, s. Without loss of generality, suppose $a \big| p, q$, say $p = a^{\lambda} \cdot b$ and $q = a^{\gamma} \cdot c$, and $a \not| \, r, s$. Then

$$\phi(K) = \left\langle a^{\lambda} b\right\rangle \oplus \langle a^{\gamma} c\rangle \oplus \langle r\rangle \oplus \langle s\rangle \oplus \left\langle -a^{\lambda+\gamma} bcrs \cdot \det(K)\right\rangle \oplus \langle \pm 1\rangle \oplus \ldots \oplus \langle \pm 1\rangle .$$

Clearly $\langle r\rangle$, $\langle s\rangle$, and all the $\langle \pm 1\rangle$ lie in the kernel of ∂_a. We must now consider the parity of λ and γ. If λ and γ are both even, so is $\lambda + \gamma$, and then $\partial_a(\phi(K)) = 0 \in W(\mathbb{Z}_a)$.

Suppose that λ and γ are both odd. Without loss of generality, we can assume that $\lambda \geq \gamma$. Then $\lambda + \gamma$ is even and so $\partial_a(\phi(K)) = \langle b\rangle \oplus \langle c\rangle$. Then

$$\begin{aligned}\det(K) = \pm m^2 = pqr + pqs + prs + qrs &= a^{\lambda+\gamma} bcr + a^{\lambda+\gamma} bcs + a^{\lambda} brs + a^{\lambda} crs \\ &= a^{\gamma}(a^{\lambda} bcr + a^{\lambda} bcs + a^{\lambda-\gamma} brs + crs).\end{aligned}$$

In particular, a^{γ} divides m^2. But an even power of a must divide m^2 and γ is odd, so a divides $\pm\frac{m^2}{a^{\gamma}}$ and hence, divides $(a^{\lambda} bcr + a^{\lambda} bcs + a^{\lambda-\gamma} brs + crs)$. Then $a^{\lambda-\gamma} brs + crs$ must be divisible by a. Since crs is not divisible by a, $a^{\lambda-\gamma} brs$ must not be divisible by a. Thus, $\lambda = \gamma$. So we have $brs + crs = (b + c)rs \equiv 0 \pmod{a}$. Since r, s are relatively prime to a, we have $b + c \equiv 0 \pmod{a}$ and so $c \equiv -b \pmod{a}$. We finally obtain

$$\partial_a(\phi(K)) = \langle b \rangle \oplus \langle c \rangle = \langle b \rangle \oplus \langle -b \rangle = 0 \in W(\mathbb{Z}_a).$$

Now consider the case that λ is odd and γ is even with $\lambda > \gamma > 0$. Then

$$\det(K) = \pm m^2 = a^{\lambda+\gamma}bcr + a^{\lambda+\gamma}bcs + a^{\lambda}brs + a^{\gamma}crs = a^{\gamma}(a^{\lambda}bc(r+s) + rs(a^{\lambda-\gamma}b+c)).$$

In particular, a^{γ} divides m^2. Since γ is even, say $\gamma = 2\mu$, $m = a^{\mu}d$ for some d. So $m^2 = (a^{\mu}d)^2 = a^{2\mu}d^2 = a^{\gamma}d^2$. The above equation then becomes

$$\pm a^{\gamma}d^2 = a^{\gamma}(a^{\lambda}bc(r+s) + rs(a^{\lambda-\gamma}b + c)).$$

Dividing by a^{γ} gives

$$\pm d^2 = a^{\lambda}bc(r+s) + rs(a^{\lambda-\gamma}b + c)$$

and reducing modulo a we finally obtain

$$\pm d^2 \equiv crs \pmod{a}.$$

Now

$$\partial_a \left\langle -a^{\lambda+\gamma}bcrs \cdot \det(K) \right\rangle = \langle -b \rangle .$$

Thus,

$$\partial_a(\phi(K)) = \langle b \rangle \oplus \langle -b \rangle = 0 \in W(\mathbb{Z}_a).$$

Now suppose that $\gamma > \lambda > 0$. As above, we obtain $\pm m^2 = a^{\lambda+\gamma}bc(r+s) + a^{\lambda}brs +$

$a^\gamma crs$. Dividing by a^λ gives

$$\pm\frac{m^2}{a^\lambda} = a^\gamma bc(r+s) + brs + a^{\gamma-\lambda}crs.$$

Since the right-hand side of this equation is an integer, so is the left-hand side. Since $a \big| m^2$, an even power of a must divide m^2. But λ is odd, so a must divide $\frac{m^2}{a^\lambda}$, which is a contradiction since the right-hand side is not divisible by a.

This argument is symmetric in λ and γ, so the other cases follow. We have shown the following theorem:

Theorem 12. Let $K = P(p,q,r,s)$ be a 4-stranded pretzel knot with p, q, r odd and $s \neq 0$ even. Suppose that $\det(K) = \pm m^2$ for some integer m and that the signature of K, $\sigma(K) = 0$. If the rational Witt class of K, $\phi(K)$, is not trivial in $W(\mathbb{Q})$, then there exists an odd prime a which divides at least three of p, q, r, s.

Corollary 1. Let $K = P(p,q,r,s)$ be a 4-stranded pretzel knot with p, q, r, s mutually relatively prime. Then $\phi(K) = 0 \in W(\mathbb{Q})$ if and only if $\det(K) = \pm m^2$ for some integer m and $\sigma(K) = 0$.

As mentioned at the beginning of this section, this phenomenon is not universal among all pretzel knots as there exist examples of knots which have determinant plus or minus a square and zero signature which have nontrivial rational Witt class. For example, see Examples 1.7-10 in [7].

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