Evaluation of the Fractional Theis Solution

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Hydrogeology

By
Andrea N. Gehlhausen

Dr. Rina Schumer/ Thesis Advisor

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We recommend that the thesis prepared under our supervision by

ANDREA N. GEHLHAUSEN

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Rina Schumer, Ph.D, Advisor

Greg Pohll, Ph.D., Committee Member

David Prudic, Ph.D., Graduate School Representative

Marsha H. Read, Ph. D., Associate Dean, Graduate School

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ABSTRACT

The Theis solution is the most widely used solution for determination of confined aquifer parameters. Because each aquifer is a unique system, sometimes an aquifer cannot accurately be described with the assumptions of the Theis solution. To offer a more general solution for aquifers where piezometric head at a given position is governed in a non-local manner, a generalized groundwater flow equation was developed by Cloot and Botha (2006). This thesis analyzes the flow equation, based on non-integer order derivatives and its numerical implementation developed by Cloot and Botha. However, their equation was found faulty and a new robust derivation of a fractional Theis solution developed by Baeumer et al. (2009) is presented. The new fractional Theis equation solution type curves (drawdown and radius of influence) are evaluated for relationships between aquifer properties and a fractional Theis solution. The fractional Theis solution is then used to reanalyze the aquifer test data used by Cloot and Botha (2006) in their analysis. Conclusions are made on the necessity of using a fractional Theis equation for the given data and suggestions are made for future work with a fractional Theis solution.
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INTRODUCTION

A confined aquifer is a saturated permeable geologic unit capable of transmitting significant quantities of water under a hydraulic gradient. The unit, found at depth, is bounded on top and bottom by less permeable units. As a result of its confined setting, the water level in a penetrating well, reflecting hydraulic head, rises above the top of the aquifer (Freeze & Cherry, 1979). Beyond its elementary definition, each aquifer is a unique system, with variations in hydraulic properties and geometry. Two parameters describing the properties of a confined aquifer are transmissivity and storativity. Transmissivity is the ability or capacity of an aquifer to transmit water through its entire thickness (Bear, 1979). Transmissivity $T = Kb$, has units $[L^2/T]$, where $K$ is the hydraulic conductivity and $b$ is the thickness of the aquifer. Storativity is the volume of water an aquifer releases from storage, measured in unit surface area per unit decline in head and is a dimensionless value (Fetter, 2001). Aquifer parameters are a function of the fluid and porous media properties and are calculated with semi-analytical solutions suited to the unique physical system they describe.

An aquifer test is typically used to estimate hydraulic parameters at the field scale. An aquifer test is a controlled field experiment where a well is pumped and the piezometric head, or water level, is recorded as a function of time. The difference between the static water level and the pumping water at a given time, called drawdown, can be recorded and analyzed at either the pumping well or at observation wells. Analysis begins with selection of appropriate boundary and initial conditions to best fit the specific characteristics of the aquifer being studied (Anderson & Woessner, 2002).
These conditions are then applied to the partial differential equation governing groundwater flow. Assumptions about aquifer characteristics allow the governing equation to be simplified and solved mathematically; resulting in an analytical mathematical drawdown vs. time solution of a known conceptual model. The solution is compared, through curve matching, with the observed aquifer test drawdown data. The result of the comparison is a set of values that can be plugged into the original mathematical solutions for a specific determination of aquifer parameters (Fetter, 2001).

The most commonly used mathematical solution to the radial groundwater flow equation is the Theis solution. The Theis solution is specific to transient flow in aquifers under the assumptions of a single, homogeneous, isotropic, non-leaky, confined aquifer of uniform thickness (Figure 1). The Theis solution treats the well as a line source and therefore does not take into consideration the water derived from storage within the well (Batu, 1998). The well is assumed vertical with full penetration and screening.

![Figure 1: Confined aquifer under pumping conditions with a fully penetrating well (Fetter 2001).](image)
The governing equation (PDE) describing flow in a horizontal confined aquifer in the two-dimensional polar coordinate system is

\[
\frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} = \frac{S}{T} \frac{dh}{dt},
\]

(1.1)

where
- \( h \) is the hydraulic head (L; m or ft),
- \( S \) is the storativity (dimensionless),
- \( T \) is transmissivity (L²/T; m²/d, ft²/d),
- \( t \) is time, and
- \( r \) is the radial distance from the pumping well.

The solution (Theis, 1935) for equation (1.1) is based on the following conditions:

Initial condition:

\[
h(r, 0) = h_0 \text{ for all } r,
\]

(1.2)

Boundary condition:

1. \( h(\infty, t) = h_0 \text{ for all } t \),

(1.3)

2. \( \lim_{r \to 0} \left( r \frac{\partial h}{\partial r} \right) = \frac{Q}{2\pi T} \).

(1.4)

The initial condition (1.2) states drawdown is initially zero everywhere in the aquifer, thus pre-pumping hydraulic head is constant throughout the aquifer. Boundary condition one (1.3) states drawdown is always zero at an infinite distance from the well. Boundary condition two (1.4) states the discharge, \( Q \left( \text{L}^3/\text{T} \right) \), is \( Q=0 \) for \( t<0 \), and \( Q = \text{constant} \) for \( t \geq 0 \) signifying constant discharge throughout the pumping period. The constant pumping rate \( (Q) \) at the well is:

\[
-Q = \lim_{r \to 0} 2\pi Tr \frac{dr}{dh}
\]

(1.5)
implying flow toward the well is equal to the discharge just outside of it (Fetter 2001).

The Theis solution in terms of drawdown is

\[ h_0 - h = \frac{Q}{4\pi T} \int_{u}^{\infty} \frac{e^{-u}}{a} da, \]  

(1.6)

where

\[ u = \frac{r^2 S}{4\pi T}. \]  

(1.7)

The integral \( \int_{u}^{\infty} \frac{e^{-u}}{a} da \), is a well known exponential integral and can be approximated by an infinite series, \(-0.5772 - \ln u + u - \frac{u^2}{2 \cdot 2!} + \frac{u^3}{3 \cdot 3!} - \frac{u^4}{4 \cdot 4!} + ...\), yielding

\[ h_0 - h = \frac{Q}{4\pi T} \left[ -0.5772 - \ln u + u - \frac{u^2}{2 \cdot 2!} + \frac{u^3}{3 \cdot 3!} - \frac{u^4}{4 \cdot 4!} + ... \right]. \]

(1.8)

This equation is frequently rewritten

\[ h_0 - h = \frac{Q}{4\pi T} W(u), \]

(1.9)

where \( W(u) \) is the dimensionless well function (Fetter, 2001).

Estimation of aquifer parameters \( S \) and \( T \) using type curve matching is performed by graphing field data and \( W(u) \) vs. \( 1/u \) on log-log graphs of equal scale. The two curves are superimposed and adjusted until the field data best fits the type curve while maintaining the axes of the graphs parallel to each other. Match points are chosen where the field data and solution curve match exactly. The corresponding values of \( W(u) \), \( 1/u \), \( s(t) \) and \( t \) for the match point are then substituted into the following calculation for \( T \) and \( S \).

Transmissivity is calculated as
\[ T = \frac{Q}{4\pi s} W(u), \quad (1.10) \]

and storativity is

\[ S = \frac{4Ttu}{r^2}, \quad (1.11) \]

where \( s = h_0 - h(t) \) = drawdown.

(Schwartz & Zhang, 2003).

In many aquifers, drawdown is adequately reproduced by the Theis solution. When a deviation occurs between observations and the theoretically expected values it indicates the assumptions used in the solution do not adequately describe the observed aquifer. Alternate solutions to the groundwater flow equation, used when Theis assumptions are not applicable, are obtained with different initial and boundary conditions. These solutions are used to treat unconfined aquifers, leaky aquifers, large diameter wells, well bore storage, and anisotropic or heterogeneous aquifers. It has been suggested that the groundwater flow equation itself is not always applicable in cases where fracture flow is present (Freeze & Cherry, 1979).

**OBJECTIVE**

The goal of this thesis is to test the utility of a multi-dimensional non-integer (fractional) order Theis equation and its solution. When this research began, the goal was to evaluate the application of the fractional Theis equation and numerical solution developed by Cloot and Botha (2006). As the research progressed, however, it became clear that their implementation was not correct. As a result, the first section of this thesis is an analysis of the previous work. Next, a robust derivation of a fractional Theis
solution that is a true generalization of the classical Theis equation, as developed by Baeumer et al. (2009), will be described. The results will be compared to those of Cloot and Botha (2006) and a re-evaluation of the aquifer test data they analyzed will be performed. Finally, we provide an interpretation of the relationship between fractional derivatives in the fractional Theis equation and aquifer properties and evaluation of potential applications.

PREVIOUS APPLICATION OF FRACTIONAL PDEs

Fractional differential equations have been applied to hydrologic phenomena. One application was a fractional Darcy’s law from a fractional Newton’s law of viscosity (Ochoa-Tapia, et al., 2007). Fractional advection-dispersion equations are known to be the governing equations for statistical models of particle movement with heavy-tailed jump distributions or heavy-tailed waiting times between jumps and have been used to reproduce aquifer solute transport (e.g. Schumer, et al., 2003; Benson, 2000). The hydrologic application most relevant to this thesis is the development of a fractional Theis equation (Cloot & Botha, 2006). The equation was developed to treat anomalous drawdown observed during fractured-rock aquifer tests where the best-fit conventional radial flow model to their data overestimated drawdown at late time.

BACKGROUND

Fractional Derivatives

Fractional calculus is concerned with rational-order, rather than strictly integer order, derivatives and integrals. The term “fractional” calculus is a misnomer, a better name might be “differentiation and integration to an arbitrary order”; thus it could be
rational, irrational, positive or negative, real or complex (Miller & Ross, 1993). Whereas integer order derivatives are dependent only on the local behavior of a function (e.g. the slope at a specific point), the fractional derivative is a non-local function and depends on the character of the entire function (Schumer, et al., 2001) The fractional order derivatives form a continuum between their integer order counterparts (Figure 2).

(a).

(b).

Figure 2a and 2b: (a): Plot of the function \( f(x) = 2x \) and its 0.2, 0.4, 0.6, and 1\textsuperscript{st} derivatives and (b): Plot of the function \( f(x) = e^{2x} \) and its 0.2, 0.4, 0.6, 0.8, and 1\textsuperscript{st} derivatives.
The Grunwald representation of a fractional derivative clarifies the nature of a fractional derivative. To obtain it, begin with the notion that in Fourier space, a fractional derivative is the fractional power of an integer order derivative (following Schumer et al., 2009)

\[
\frac{d^\alpha}{dx^\alpha} f(k) = (i k)^\alpha f(k)
\]  

Applying this idea to the finite-difference approximation of a derivative

\[
\frac{d}{dx} f \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]  

with the Fourier transform

\[
\frac{df(k)}{dx} \approx \frac{e^{i k \Delta x} \hat{f}(k) - \hat{f}(k)}{\Delta x} = \frac{e^{i k \Delta x} - 1}{\Delta x} \hat{f}(k), \text{ and}
\]

\[
\frac{d^\alpha}{dx^\alpha} f(k) \approx \left(\frac{e^{i k \Delta x} - 1}{\Delta x}\right)^\alpha \hat{f}(k) = \frac{1}{(\Delta x)^\alpha} \sum_{n=0}^{\infty} \left(\begin{array}{c} \alpha \\ n \end{array}\right) e^{i k n \Delta x} (-1)^n.
\]

The finite-difference operator in real space is

\[
D^\alpha_{\Delta x} f(x) = \frac{1}{(\Delta x)^\alpha} \sum_{n=0}^{\infty} (-1)^n \left(\begin{array}{c} \alpha \\ n \end{array}\right) f(x + \Delta x - n \Delta x)
\]

and can be written as the weighted average of all points along the function

\[
D^\alpha_{\Delta x} f(x) = \frac{1}{(\Delta x)^\alpha} \sum_{n=0}^{\infty} w_n f(x + [\alpha] \Delta x - n \Delta x)
\]

where \([\alpha]\) is the largest natural number less than \(\alpha\) and the weights are \(w_0 = 1\) and \(w_{n+1} = w_n \frac{n - \alpha}{n + 1}\).
This definition demonstrates that the fractional derivative at a point is a non-local function whose value depends not only on points immediately surrounding it, but also in all points on a function with decreasing weight with distance proportional to the power law \( n^{-1+\alpha} \).

**Numerical Treatment of fractional derivatives**

Numerical methods are used to approximate complex equations, such as the partial differential equations that govern the flow of water in aquifers of various types, for computer-based analysis. Numerical methods typically generate a set of algebraic equations by approximating the partial-differential equations (governing equation, boundary conditions, and initial conditions) that form the mathematical model (Bear, 1979).

In finite-difference approximations of the integer order derivative (a local function), the value in a given cell at the next time step is dependent only upon values at adjacent cells (figure 3a). However, the change in the value of a variable being operated on by a fractional derivative is dependent on the variable value in all cells (Figure 3b). The dependence on other cells decreases as a power law with distance. Theoretically, this dependence extends infinite distances, requiring additional accounting at model boundaries.
Figure 3: (a) In finite-difference implementations of a derivative, the time step change in a variable depends on the values of the function at adjacent cells. (b) In finite-difference implementations of a fractional derivative, the time-step change in variable depends on values of the function at all cells.

Previous application of a fractional Theis equation

Cloot and Botha (2006) used a generalized groundwater flow equation with non-integer order derivatives to fit drawdown data with a poor match to classical solutions. A fractional equation allows for situations where head change at a given point in an aquifer is non-local. Non-local behavior is observed when properties of the piezometric head field are dependent on their specific position as well as the global spatial distribution of that field in the aquifer. The non-local weighting in fractional derivatives,

\[ w_0 = 1 \text{ and } w_{n+1} = w_n \frac{n-\alpha}{n+1}, \]

allows the properties in the direct vicinity of the specific point to have a greater influence over the fluid flow than the head at a distance.
Cloot and Botha (2006) began with the classical groundwater flow equation with
the Theis assumptions:

\[
S_t \frac{dh}{dt} = K \frac{d}{dr} \left[ r^{n-1} \frac{dh}{dr} \right], \tag{1.16}
\]

where

- \( K \) is the hydraulic conductivity,
- \( n = 1, 2 \) or \( 3 \) is the dimension of flow, and
- with the boundary conditions

\[
h(r,0) = h_0, \\
h(r,t)_{r \to \infty} = h_0, \text{ and} \tag{1.17}
\]

\[
Q = \frac{2\pi^2}{\Gamma\left(\frac{n}{2}\right)} r_w^{n-1} (Kd)^{3-n} \frac{dh}{dr} [r_w, t = 0]
\]

where

- \( Q \) is the discharge rate of a borehole with a radius \( r_w \),
- \( d \) is the thickness of flow, in this case the width of the fracture, and
- \( h_0 \) is the initial piezometric head in the aquifer.

To explicitly include the possible effect of flow geometry into the mathematical model
the radial component of piezometric head, \( \frac{dh}{dr} \), is replaced by the complementary
fractional derivative of order \( \alpha = m - \rho \):

\[
\frac{\partial^{\alpha} h(r,t)}{\partial r^{\alpha}} = \frac{(-1)^{n+1}}{\Gamma(\rho)} d^{m-\rho-1} \frac{\partial^m}{\partial r^m} \int_0^\infty (s-r)^{\rho-1} h(s,t) ds \tag{1.18}
\]

where \( s \) is drawdown, providing a generalized form of the classical equation governing
the flow.
The additional parameter $\mu = m - \rho$, $0 < \rho \leq 1$, is viewed as a new variable to characterize flow through the geological formations.

The same transformation also generates a more general form of the boundary condition at the borehole

$$Q = \left( -1 \right)^{m+1} \frac{2\pi n}{\Gamma \left( \frac{n}{2} \right) \Gamma \left( \rho \right)} \left( r^n \right)^{n-1} \left( K d \right)^{2-n-m-\rho} \frac{\partial}{\partial r^n} \left[ \int_{r}^{\infty} (s-r)^{\rho-1} \Phi(s,t) ds \right]$$  \hspace{1cm} (1.20)

Cloot and Botha (2006) were unable to obtain an analytical solution for the complete set of equations (1.19) and (1.20) and initial condition (1.17) and used numerical methods to solve them. They reached solution using a two-step numerical approximation. First, the range of the interval was discretized into a set of line elements, $\Delta r_j$, $j=1,2,...$ with equal sizes, and $h(s,t)$ was approximated over the element $(r_j, r_{j+1})$ with its average $\bar{h}(s,t) = \frac{h_j(t) + h_{j+1}(t)}{2}$; $s \in [r_j, r_{j+1}]$. Second, this approximation and the implicit backward finite difference approximation for the time derivative was used to approximate the generalized equation (1.19) as

$$r_i^{n-1} \left( h_i^{k+1} - h_i^k \right) = \frac{Kd^{m-p-1}}{V \left[ r_i^{n-1}, F(r_i) \right]}$$ \hspace{1cm} (1.21)

Where

$\Delta t$ is the timestep,

$V \left[ r_i^{n-1}, F(r_i) \right]$ a suitable finite difference approximation for the factor.
From the generalized form of Darcy’s law (Cloot and Botha, 2006) the change in pressure at the borehole as pumping commences causes the drawdown to be a decreasing function of the order of the derivative. Because the impact of the head gradient in the direct vicinity of the borehole will have a greater influence on the function, the model includes the weighting factor, \( W = \frac{1}{(s - r)^{1-p}} \), implemented during the integration process.

The numerical solution for the fractional Theis equation developed by Cloot and Botha (2006) is restricted because the solution applies a one-sided fractional derivative (akin to backwards Euler) that can only approximate on one side of the pumping field and only data from the pumping well can be used in the analysis. This is limiting for aquifer test applications because an analysis cannot be completed at observation wells.

One objective of our study is to expand the generalized groundwater flow equation created by Cloot and Botha to radial flow, so that evaluation is not limited to the pumping well only but expanded to include monitoring well observations as well. Initially when evaluating the concept of a fractional groundwater flow equation, the equation developed by Cloot and Botha (2006) was coded in Matlab using a Grunwald representation of a fractional derivative (Baeumer et al., 2009). In 1-D, the generalized

\[
(-1)^{m+1} \frac{\partial}{\partial r} \left( r^{n-1} F(r,t) \right) = 0. 
\] (1.22)
groundwater flow equation produced the set of type curves as shown in Figure 4.

Next, a two-dimensional generalized groundwater flow equation was developed using the Grunwald representation of a fractional derivative. This was done by creating a left and right flux matrix of Grunwald weights to drive the system. However, the 2-D solution led to negative drawdown (Figure 5).
Figure 5: Type curves for a two-dimensional groundwater flow equation using the Grunwald representation of a fractional derivative. In two dimensions, negative drawdown occurred indicating an error in the mathematical equation.

An evaluation of the numerical mass balance revealed the source of the error in the generalized Theis equation (1.19). The 2-D conservation of mass equation in radial coordinates is:

\[
\frac{1}{r} \frac{\partial (r \rho V_r)}{\partial r} + \frac{1}{r} \frac{\partial (\rho V_\theta)}{\partial \theta} + \frac{\partial \rho}{\partial t} = 0
\]  

(1.23)

The \( \frac{1}{r} \) term corrects for the fact that moving outward from the pumping well, the area, \( d\theta dr \), increases (Figure 6).
Figure 6: (a) 2-D radial flow in polar coordinates (b) Non-local behavior. The size of the arrows reflects the weight.

As a result of the non-local nature of fractional derivatives, the head near and far from a source influences the drawdown. In 1-D (Figure 2a and b), $\Delta x$ remains the same throughout. However, when the generalized groundwater flow equation was placed in radial coordinates for two dimensions as seen in equation (1.19) rewritten here:


\[
S_0 \frac{\partial h}{\partial t} = \frac{(-1)^{m+1}}{\Gamma(\rho)} \frac{1}{r^{n-1}} (Kd)^{m-\rho-1} \frac{\partial}{\partial r} \left[ r^{n-1} \frac{\partial^m}{\partial r^m} \int_s^{\infty} (s-r)^{\rho-1} h(s,t) ds \right],
\]

the changing area and the weights need to be corrected for the increasing radius. Cloot and Botha (2006) only account for the change in area. The weighting factor further away from the well needs to be lowered to account for the increase in radius. Because the weights were not lowered, too much water was brought toward the pumping well from distant locations resulting in draw up (seen in the negative drawdown values of Figure 5) rather than drawdown.

**Derivation of a Fractional Theis Equation**

In previous work, the fractional Theis equation was obtained by replacing integer-order terms in the classical Theis equation with non-integer order terms. No derivation was provided, nor was a solution to the equation obtained. When this implementation failed, the derivation of the fractional Theis equation was revisited for the development of a correct generalization of the fractional Theis equation and its solution.

For clarity, the classical Theis equation is derived and then generalized. The solution to any PDE with initial condition \(=\delta(x,t=0)\), an impulse in space at time=zero, is called a Green’s function solution. The solution to the same PDE with any initial condition can be obtained by convolving the Green’s function solution with the new initial condition, where convolution is defined \((f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau\).

The Gaussian Greens function solution to the 2-D, isotropic, homogeneous, confined groundwater flow equation(1.1) has form (Carlslaw & Jaeger, 1959)
\[ h(t) = \int_0^t \frac{1}{2\pi} \left( \frac{T}{S} \right)_s \exp \left( -\frac{r^2 S}{4T S} \right) ds = \frac{1}{4\pi} T \int_0^t \frac{1}{S} \exp \left( -\frac{r^2 S}{4T S} \right) ds. \]  

Let \( u = \frac{r^2 S}{4T S} \) such that \( du = -\frac{r^2 S}{4T S^2} ds \) and \( s = -\frac{r^2 S}{4T} \frac{ds}{du} \). Then let \( u_{1=0} = \infty, u_{1=t} = u \) and

\[
h_{1,t}(t) = \frac{Q}{4\pi} \int_0^t \frac{1}{-u} ds \exp(-u) ds
\]
\[= \frac{1}{4\pi} \frac{T}{S} \int_u^\infty \frac{1}{u} du
\]
\[= \frac{1}{4\pi} \frac{T}{S} W(u) \tag{1.25}\]

where \( W(u) = \int_u^\infty \frac{e^{-u}}{u} du \) is the exponential integral commonly known as the well function. Using convolution with constant \( Q \) at an infinitesimally small point as the new initial condition

\[
\frac{\partial h}{\partial t} = \frac{T}{S} \nabla^2 h + Q(t-s) \delta
\]

results in the Theis equation

\[
h(r,t) = \int \frac{1}{4\pi} \frac{T}{S} W(u) Q(t-s) ds = \frac{Q}{4\pi} \frac{T}{S} W(u) \tag{1.27}\]

The Green’s function solution to the fractional Theis equation is an alpha stable distribution and variance \( 2\frac{T}{S} \) (Baeumer et al., 2009).

\[
h(r,t) = \int_0^\infty \int_0^\infty \frac{r^2 S}{2\pi T_s} \frac{1}{\tau^2} \frac{1}{\alpha^2} \left( \frac{\tau s^{2/\alpha}}{2^\alpha \pi^\alpha T_s} \right) \exp \left( -\frac{1}{\tau} \right) ds d\tau \tag{1.28}\]
The Gaussian density is alpha-stable with tail parameter alpha=2. This demonstrates that
the fractional Theis equation is a generalization of the classical case. Let \( u = \frac{r^\alpha S}{2^\alpha T^\alpha s} \), toind
\[
h(r,t) = \frac{1}{2^\alpha \pi \frac{T^\alpha}{S} r^{2-\alpha}} W_\alpha(u), \tag{1.29}
\]
where the fractional well function is
\[
W_\alpha(x) = \int_x^\infty \int_0^\infty \frac{1}{\tau s^{2-2/\alpha}} g_{\alpha/2}(\tau s^{2/\alpha}) \exp\left(-\frac{1}{\tau}\right) d\tau ds.
\tag{1.30}
\]
Convolve the Greens solution with a constant flux, \( Q \), at an infinitesimal well
\[
h(r,t) \int_0^\infty \int_0^\infty \frac{1}{s^{2/\alpha}} s^{\alpha/2} \left(\frac{\tau}{s^{2/\alpha}}\right) \frac{S^{2/\alpha}}{4\pi T^{2/\alpha}} \exp\left(-\frac{r^2 S^{2/\alpha}}{4\tau T^{2/\alpha}}\right) Q(t-s) d\tau ds
\[
= \frac{Q}{S} \int_0^\infty \int_0^\infty \frac{1}{s^{2/\alpha}} s^{\alpha/2} \left(\frac{r^2 S^{2/\alpha}}{4(T_a s)^{2/\alpha}}\right) ds
\tag{1.31}
\]
to find the fractional Theis solution
\[
h(r,t) = \frac{Q}{2^\alpha \pi Tr^{2-\alpha}} W_\alpha(u), \tag{1.32}
\]
where, again, \( u = \frac{r^\alpha S}{2^\alpha T^\alpha s} \). The fractional Theis equation is a generalization of the classical
case such that when \( \alpha=2 \) we obtain the classical Theis equation (Baeumer et al., 2009).
ANALYSIS

Type Curve Behavior

As pumping commences in a confined aquifer, assuming only vertical compaction, pressure head is decreased resulting in a decrease of fluid pressure and an increase in effective stress. The decreased fluid pressure causes the fluid to expand, the new increase in volume replacing the water removed by pumping. The increased effective stress on the solid material of the aquifer causes the solid to compact leaving less pore space for water storage. The expansion of water combined with the degree of compaction results in a specific volume of water being released from storage (Freeze & Cherry, 1979).

In a confined aquifer, a large pressure head drop over significant areas is needed to create a substantial volume of water released from storage. The aquifer releases only a small volume of water from storage per unit volume of earth material resulting in a deep and aerially extensive drawdown curve. Nearby observation wells also experience significant drops in water level as the pressure head drop spreads over large areas. Typical storativity values of a confined aquifer range from $5 \times 10^{-3}$ to $5 \times 10^{-5}$ (Kruseman & de Ridder, 2000).

Type curves, known solutions to an equation, for the fractional Theis solution were generated using a numerical solution implemented in Matlab developed by Baeumer et al (2009) The type curves are shown in Figure 7. An analysis of the type curves offers insight into the physical system they represent.
Figure 7: Fractional Theis equation type curves for orders of the fractional derivative $\alpha$, between 1.2 and 2 in (a) 1-D, (b) 2-D, (c) 3-D. As $\alpha$ decreases the non-local behavior of the function is evident.
The dimension of the groundwater flow equation is equal to the number of directions in which the groundwater moves within an aquifer. 1-D flow can be viewed as pipe flow, 2-D as pie shaped and 3-D as wedge shaped. In 1-, 2- and 3- dimensional flow, the fractional Theis type curve changes shape as α decreases from a value of two. The height of the type curve is dependent on the order of the fractional derivative, α. As α decreases the non-local behavior of the function increases and curves become flatter and at a wider distance from the pumping well, as the effects of pumping propagate more and more quickly. The fractional Darcy’s law allows for increased flow for a given K value than the traditional Darcy’s law. When α =2, the fractional Theis equation is equal to the classical Theis equation.

Drawdown in a 1-dimensional flow system is more restricted because flow to the well is only from 1 direction whereas in 2- and 3- dimensional systems, radial and spherical, respectively. The restricted 1-D flow requires a larger pressure head drop to produce the same volume of water resulting in an increased drawdown curve (Figure 8).
Figure 8: Drawdown vs. Time observations for a fractional value $\alpha$ in 1, 2, and 3 dimensions. An increase in dimension results in a smaller drawdown curve.

The fractional Darcy’s law/fractional groundwater flow equation also affects the radius of influence of a pumping well. The radius of influence is the boundary around a pumping well where drawdown = 0. As the order of the fractional derivative, $\alpha$, decreases from 2, the radius of influence becomes wider about the well. Decreased drawdown and a gentler slope are found at the pumping well for $\alpha=2$, the Theis equation. This can be attributed to the non-local effect of the fractional Theis equation as the effects of pumping propagate more and more quickly for lower values of $\alpha$. This behavior is consistent in 1-, 2- and 3- dimensions (Figure 9).
Figure 9: Fractional Theis equation type curves for orders of $\alpha$ between 2 and 1.2 in (a) 1-D, (b) 2-D, (c) 3-D. As $\alpha$ decreases from 2 the radius of influence becomes greater about the well in 1-D and less for 2 and 3-D.
Figure 10: The behavior of a fractional Theis equation on the radius of influence over time in 1-, 2-, and 3- dimensions. The greatest radius of influence is observed under 1- dimensional flow.

A 3-dimensional flow field yields more water in the direct vicinity of the pumping well than 1 or 2 dimensional flow fields for a given hydraulic conductivity. This results in a greater radius of influence in one dimension where the effects of pumping are observed at greater distances. This behavior is observed in the fractional solution in 1, 2, and 3 dimensions (Figure 10).

For a constant value of $\alpha$, storativity and transmissivity produces curves consistent with their classical interpretation (Fig 9a and 9b). Transmissivity is the rate which water can be transmitted horizontally. As pumping commences, water flows toward the pumping well at the rate determined by the transmissivity. A higher value of transmissivity results in a wider flatter cone of depression because a higher rate of water is available to move to the pumping well in the horizontal direction. An aquifer with a lower transmissivity value results in a greater cone of depression around the pumping
well with a steeper slope as not enough water can be supplied from the horizontal direction and increased vertical drawdown occurs (Figure 11).

The effect of changing transmissivity on radius of influence was also evaluated for consistency with the standard conceptual model. Higher transmissivity values allowed for more horizontal movement of water, resulting in small drawdown curves with a greater radius of influence. Lower transmissivity values that were not able to move water horizontally at an adequate rate resulted in higher drawdown curves with a smaller radius of influence (Figure 12).

![Figure 11: The behavior of a fractional Theis equation on drawdown with time under changing transmissivity values. Higher transmissivity values result in flatter more extensive drawdown curves.](image-url)
A lower storativity value means less water is released from storage per unit change in head thus requiring a greater drop in head to supply water from the aquifer to the pumping well. Lower storativity values lead to a larger radius of influence and more drawdown (Figure 13 and 14). Larger values of storativity mean more water is released from storage per unit change in head thus requiring less of a drop in head to supply water to the pumping well. Higher storativity values lead to a smaller radius of influence and less drawdown.

Figure 12: The behavior of a fractional Theis equation on the radius of influence under changing transmissivity values. Higher transmissivity values result in a shallower greater radius of influence.
Figure 13: The behavior of a fractional Theis equation on drawdown with time under changing storativity values. Lower storativity values result in greater drawdown with time.

Figure 14: The behavior of a fractional Theis equation on the radius of influence under changing storativity values. Higher storativity values result in a smaller radius of influence and less drawdown.
Previous application of the generalized groundwater equation

Cloot and Botha (2006) applied the generalized groundwater flow equation to aquifer test data from borehole UO5 at the Campus Test site at the University of Free State, South Africa. The Campus Test Site is geologically composed of layered sedimentary rocks (mudstones and sandstones) with bedding plane fractures present. The site, part of the Karoo aquifer system, is underlain by three aquifers. The top aquifer, located in the upper 13m of layered mudstone, is unconfined. A layer of confining carbonaceous shale, 0.5 to 4m in thickness, separates the first and second aquifers. The second aquifer is composed of sandstone, approximately 10m thick, and serves as the primary aquifer. The third aquifer is composed of mudstone layers greater than 100m thick (Riemann et al., 2002). The formation displays the geometry of a porous medium, but has large variations in porosity (van Tonder et al., 2001)

The second (primary) aquifer is characterized by a horizontal bedding plane fracture at its center plane. The fracture aperture is approximately 10mm, and the surrounding 200mm of sandstone has a higher permeability than the rest of the aquifer (Figure 15) (Cloot & Botha, 2006). Significant water is not stored within the bedding plane fracture itself but rather the fracture serves as the primary conduit for flow, allowing water to be transmitted rapidly.
Figure 15: The three aquifer system underlying the Campus Test Site (R) and a close up of the primary sandstone aquifer, Borehole UO5, and the horizontal bedding plane fracture (L).

Borehole UO5, located on The Campus Test Site, is intersected by the bedding plane fracture in the second aquifer. A constant rate pump test was performed on the borehole at a rate of 1.25 l/s. Water levels were measured at intervals in borehole UO5 as well as observation boreholes UO6 (distance 6m), UP15 (distance 22m), and UP16 (distance 32m). All boreholes were intersected by the same bedding plane fracture. The drawdown measurements for each borehole were similar in value (Figure 16) (Riemann et al., 2002).
Figure 16: Drawdown versus time data for the aquifer test on borehole UO5 at the Campus Test Site. Drawdown data is shown for the pumping well (UO5) and 3 observation wells (UO6, r=6m), (UP15, r=22m) and (UP16, r=32m).

Aquifer test drawdown data from borehole UO5, the pumping well, was compared to the traditional Theis solution and found to match reasonably well at early time, while underestimating expected drawdown at late time (Cloot and Botha, 2006). Cloot and Botha determined the “Theis model does not accommodate a time dependent solution that can account properly for the dynamical evolution of the piezometric value.” The traditional Theis analysis completed by Cloot and Botha (2006) yielded a transmissivity of 19m²/d and an unrealistic value of 8.6 for storativity (Figure 17).

To address the inability of the Theis analysis to properly account for the dynamics of the system, Cloot and Botha (2006) reanalyzed the UO5 pump test data with the
generalized groundwater flow model. An analysis was not performed for the observation wells because their numerical implementation only permitted evaluation of pumping well data. For the pumping well analysis, both the order of the derivative and the aquifer fracture thickness were treated as free parameters, and the storativity and transmissivity values obtained with the previous Theis analysis were retained. The generalized groundwater flow model reached a best-fit curve match, with an order of derivative of $\alpha = 1.15$ and a flow thickness $d=0.04m$ (Figure 17).

![Figure 17: Comparison of the drawdown observed during a constant rate test on borehole UO5 at the Campus Test Site of the University of Free State in South Africa with the fitted Theis curve and the solution to the generalized groundwater flow model with $\alpha=1.15$ and $d=40mm$ (Cloot & Botha, 2006).](image)

Cloot and Botha (2006) concluded that the generalized fractional groundwater flow equation was superior to the traditional Theis solution for determination of aquifer properties from UO5 aquifer test data. They maintained that the non-local nature of the fractional groundwater flow equation was required because it allowed for more general
situations where groundwater flow at a given point is determined by the system as a whole, where the entities closest to the point in question are given a larger weight of influence than those further away. They argued that integration to one side of the pumping well only is justified through the concept that the model accounts for the flow geometry as experienced by the fluid particles to date and that the geometry of the aquifer lying ahead (the future) of a given particle has no influence on the behavior. Under Cloot and Botha (2006), each particle within the aquifer, located at greater distances from the pumping well, is viewed as an image of the behavior of the flow along the half infinite line with the sum of the particle images creating a history of movement. They found the non-local nature of the generalized model caused the flow equation to lose its parabolic character (Cloot & Botha, 2006).

Cloot and Botha (2006), also introduce an additional coefficient, \( d^{\mu-1} \), as a new geometrical description independent of the physical dimension assumed for the flow description. They offer this coefficient combined with the hydraulic conductivity \( K \) offers a new hydraulic parameter, \( K_f = K d^{\mu-1} \).

**Application of the updated generalization of the fractional Theis equation**

When the mathematics of the generalized flow equation were found flawed the corrected generalization of the Theis equation was developed by (Baumer et al., 2009). A reanalysis of the aquifer test data analyzed by Cloot and Botha (2006) was performed with the corrected equation.

Re-analysis began with an evaluation of aquifer test data with the traditional Theis curve solution for both the pumping and the observation well data. The analysis was
implemented in Aqtesolve where a best-fit curve match was found for each well and a storativity and transmissivity value were calculated.

Table 1: Estimated aquifer parameter values of storativity and transmissivity with the Theis solution (1935) at the Campus Test Site at the University of Free State, South Africa.

<table>
<thead>
<tr>
<th>Borehole</th>
<th>Storativity</th>
<th>Transmissivity (m²/d)</th>
<th>Distance from PW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pumping Well</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UO5</td>
<td></td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Observation Well</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>UO6</td>
<td>4x10⁻³</td>
<td>15</td>
<td>6m</td>
</tr>
<tr>
<td>UP15</td>
<td>3x10⁻⁴</td>
<td>14</td>
<td>22m</td>
</tr>
<tr>
<td>UP16</td>
<td>1x10⁻⁴</td>
<td>14</td>
<td>32m</td>
</tr>
</tbody>
</table>
Next, the observed data was analyzed with the one dimensional fractional Theis equation implemented in Matlab. Observed data were matched to fractional Theis curve solutions to find a best fit.

The observed data was then analyzed with the two dimensional fractional Theis equation, where again observed data was matched to fractional Theis curve solutions to find a best fit. In two dimensions a best fit was achieved with an $\alpha=2$, the traditional Theis solution. This 2-D analysis confirmed that in two dimensions the traditional Theis solution is not only a good fit but also the best fit to the Campus Test Site aquifer test site data.

Figure 18: Traditional Theis solution analysis of aquifer test data for observation wells (a) UO6, (b) UP15, and (c) UP16, at the Campus Test Site at the University of Free State, South Africa.
Table 2: Best-fit $\alpha$ value and corresponding transmissivity for a 1-dimensional fractional Theis solution with aquifer test data from the Campus Test Site at the University of Free State, South Africa.

<table>
<thead>
<tr>
<th>Borehole</th>
<th>1-D Best- fit</th>
<th>Transmissivity (m$^2$/d)</th>
<th>Distance from PW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pumping Well</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U05</td>
<td>1.19</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Observation Well</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U06</td>
<td>1.12</td>
<td>74</td>
<td>6m</td>
</tr>
<tr>
<td>UP15</td>
<td>1.12</td>
<td>76</td>
<td>22m</td>
</tr>
<tr>
<td>UP16</td>
<td>1.12</td>
<td>9</td>
<td>32m</td>
</tr>
</tbody>
</table>

Table 3: Best-fit $\alpha$ value and corresponding storativity and transmissivity for a 2-dimensional fractional Theis solution with aquifer test data from the Campus Test Site at the University of Free State, South Africa.

<table>
<thead>
<tr>
<th>Borehole</th>
<th>2-D Best- fit</th>
<th>Transmissivity (m$^2$/d)</th>
<th>Distance from PW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pumping Well</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U05</td>
<td>2.0</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>Observation Well</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U06</td>
<td>2.0</td>
<td>11</td>
<td>6m</td>
</tr>
<tr>
<td>UP15</td>
<td>2.0</td>
<td>10</td>
<td>22m</td>
</tr>
<tr>
<td>UP16</td>
<td>2.0</td>
<td>10</td>
<td>32m</td>
</tr>
</tbody>
</table>
Figure 19: 1-D Fractional Theis solution analysis of aquifer test data for observation wells (a) UO6, (b) UP15, and (c) UP16, at the Campus Test Site at the University of Free State, South Africa.
Figure 20: 2-D Fractional Theis solution analysis of aquifer test data for observation wells (a) UO6, (b) UP15, and (c) UP16, at the Campus Test Site at the University of Free State, South Africa.
DISCUSSION
Analysis of the UO5 aquifer test data brings into question the choice of aquifer data used by Cloot and Botha (2006) to test a fractional Theis solution. With only the pumping well, the value of transmissivity but not storativity can be determined, necessitating the usage of the observation well aquifer test data (Fetter, 2001). When observation well data was analyzed with the traditional Theis solution a good fit with consistent values for both transmissivity and storativity at each well was obtained. In order to improve the fit the aquifer test data from the Campus Test Site drawdown must flatten more quickly than the traditional Theis solution curve predicts. This fact supports the use of a fractional Theis equation.

The analysis of Cloot and Botha (2006) has three shortcomings. First, as mentioned previously, the equation does not account for the changing weights associated with the changing area in multiple dimensions. Second, the analysis of borehole UO5 is performed by using values for storativity and transmissivity determined by the traditional Theis solution. Cloot and Botha (2006) found and used a storativity of 8.6, a value not physically possible given the definition of storativity. This method of using values for parameters reached with the traditional Theis solution is also questionable because the fractional transmissivity does not have the same units as classical transmissivity. A dimensional analysis shows hydraulic conductivity has the units $\left[\frac{L^a}{T}\right]$, resulting in transmissivity units of $\left[\frac{L^a \cdot L}{T}\right]$. Using the generalized groundwater flow equation developed by Cloot and Botha (2006), new values of storativity and transmissivity with
the fractional derivative cannot be determined. Third, the aquifer test drawdown data from observation wells was not (and can not) be evaluated.

We intended to validate our work (the fractional Theis equation) with a reanalysis of the Campus Test Site Aquifer Test Data that reportedly required the flexibility of a generalized flow equation. An initial analysis of the data in Table 1 with inclusion of observation wells suggests that the Test Site data is well modeled by the radial Theis solution (see discussion above). Because the Test Site wells intersect a fracture, we also analyzed the data using the fractional Theis solution in one and multiple dimensions (Table 2). The 1-dimensional fractional type curves fit pumping and observation well data closely for all time. In two dimensions a fractional Theis curve reached a best fit when $\alpha=2$, or a traditional Theis. Values decreasing from 2 were unable to create a good match. More information about the nature of the fracture underlying the Campus Test Site is necessary to substantiate the claim that a fractional Theis Equation is required here. If the fracture intersects the pumping well and monitoring wells such that pumping results in 1-D pipe flow, then the excellent fit of the $\alpha=1.12$ solution would be acceptable. However, it is more likely that radial flow is occurring at this site leading us to conclude that the Theis equation is an acceptable model for flow at this site.

**CONCLUSION**

Analysis of pumping test data for aquifer parameter determination is often highly simplified mathematically. In some scenarios late time data is unable to match the expected type curve solution. One possible answer is an analysis with a multi-dimensional non-integer (fractional) order Theis equation and its solution, which is characterized by non local behavior.
This thesis took a first step in evaluating the utility of a multi-dimensional non-integer (fractional) order Theis solution. An initial evaluation of the fractional Theis equation and its numerical implementation developed by Cloot and Botha (2006) was performed and found faulty when applied to more than 1-dimensional. In response to this failure, a robust derivation of a fractional Theis solution that is a true generalization of the classical Theis equation, (Baeumer, et al., 2009), was described. The resulting fractional Theis solution type curves were evaluated for the impact of variation in aquifer properties on drawdown in time and space. The fractional Theis solution was then used to analyze the Campus data set used previously to show the efficacy of the solution. A reanalysis showed that a traditional Theis solution fit both pumping and observation well data well, only requiring the more general fractional Theis solution if fracture flow at the site could be shown one dimensional. with the given site information and the data set was not in need of a fractional Theis solution.

**Future work**

1) Dimensional analysis of the fractional Darcy’s law shows that the units of transmissivity are \( \left[ \frac{L^\alpha \cdot L}{T} \right] \). If this model is to find useful application in water supply studies an interpretation of this parameter is needed, where \( T \) can offer a meaningful value. One area that this model may find immediate application is as an aquifer test to estimating hydraulic conductivity for use in the previously developed fractional advection dispersion equation in contaminant transport.

2) Analysis of the fractional Theis equation with an aquifer test data set that requires flexible derivatives.
3) A thorough evaluation of the potential applications of the fractional Theis equation needs to be performed.

WORKS CITED


