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Asymptotic Distributions of the Estimated Survival Functions of the Exponential, Geometric and BEG Distributions

A thesis submitted in partial fulfillment of the requirements of the degree of Master of Science in Mathematics

by

Shakil K. Shrestha

Dr. Anna Panorska / Thesis Advisor

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We recommend that the thesis
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SHAKIL KUMAR SHRESTHA

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Anna Panorska, Ph.D., Advisor

Tomasz Kozubowski, Ph.D., Committee Member

Frederick Harris, Ph.D., Graduate School Representative

Marsha H. Read, Ph. D., Dean, Graduate School

August, 2011

Abstract

In this work, we derive the limiting normal distributions of the maximum likelihood estimators of the survival functions for the exponential, geometric and (bivariate) BEG distributions. We discuss the monotonicity of the variance of the limiting distribution for exponential and geometric cases. Finally, we illustrate the convergence of the empirical distribution of the survival function to the limiting normal law using Monte Carlo simulations.

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1. **Introduction**

We begin with definitions of exponential, geometric, and BEG distributions, along with some applications of these models. Let $X_1, X_2, \ldots$ be independent, identically distributed (i.i.d.) exponential variables with parameter $\beta > 0$ and probability density function (p.d.f.)

$$f(x) = \beta e^{-\beta x}, \quad x > 0, \tag{1}$$

and let N be a geometric random variable with the p.d.f.

$$h(n) = \mathbb{P}(N = n) = p(1-p)^{n-1}, \quad n = 1, 2, \ldots, \tag{2}$$

independent of the X_i's. We denote these distributions by $\mathcal{EXP}(\beta)$ and $\mathcal{GEO}(p)$, respectively. Since the exponential distribution is closed under geometric compounding, the random sum

$$X = \sum_{i=1}^{N} X_i \tag{3}$$

has an exponential distribution with parameter $p\beta$ (see, e.g., Arnold, 1973). Further, the joint distribution of N and $\sum_{i=1}^{N} X_i$ is BEG distribution with parameters β and p introduced and described in Kozubowski and Panorska (2005). We say that the random vector $(X, N) \sim \mathcal{BEG}(\beta, p)$.

In this thesis, we are primarily interested in the estimation of the error of the probabilities computed using $\mathcal{EXP}(\beta)$, $\mathcal{GEO}(p)$, and $\mathcal{BEG}(\beta, p)$ models. More precisely, suppose we have a sample $X_1, \ldots, X_n$ coming from an exponential distribution with parameter β (and mean $\frac{1}{\beta}$). Further, suppose we want to estimate P(X>x), where

$X \sim \mathcal{EXP}(\beta)$, based on that sample. Our estimate will be the common maximum likelihood estimate (MLE) $\hat{S}(x)$ of the survival function $S(x)$ as follows

$$P(X > x) \approx \hat{S}(x) = e^{-\hat{\beta}x}.$$

The question here is what is the error of this estimate of $P(X>x)$? That is, what is the error of $\hat{S}(x)$ as an estimator of $S(x)$. Estimation of this error for the exponential, goemetric, and BEG model's survival functions is the main goal of this work. We concentrate on the survival function because it provides probabilities of all events. We start with a review of the literature concerning these distributions and their applications.

Geometric distribution is often used to model discrete *time till the first event*, such as the number of days till a rainy day, the number of years till the next earthquake, the number of time periods till the stock index goes up, etc. Exponential distribution is often thought of as a continuous equivalent of the geometric distribution. In fact, exponential distribution is frequently used to model the waiting time till an event. It has applications as a waiting time in queing theory (engineering, quality control, sciences), and it serves as the waiting time between events in the Poisson process. Poisson process is probably the most common statistical model for counting events used in engineering and science. It describes the number of earthquakes, arrivals of customers for service, arrivals of calls at a switchboard, occurences of floods, radioactive decay and other natural and man-made disasters (see DeGroot and Schervish, 2002). Addtionally, in hydrology, the exponential distribution is used to analyze such variables as monthly and annual maximum values of daily rainfall and river discharge volumes. Further, geometric distribution is a special case of the negative binomial distribution with parameter $r = 1$. More precisely, if $X_1, \ldots, X_r$ are independent geometrically distributed variables with parameter p, the sum

$$\mathbb{Z} = \sum_{m=1}^{r} X_m$$

follows a negative binomial distribution with parameter r and $p\epsilon(0,1)$. Clearly, the case $r = 1$ produces the geometric distribution itself.

Among other properties, the geometric distribution is memoryless. That means that if we repeat an experiment until we obtain the first success, then, given that the first success has not occured, the conditional probability of the number of additional trails does not depend on how many failures have been observed. The geometric distribution is in fact the only memoryless discrete distribution. Exponential distribution is also memoryless. That means that if $X \sim \mathcal{EXP}(\beta)$, then for every s, t >0

$$P(X \geq s + t | X \geq s) = P(X \geq t).$$

Geometric sums such as (3) arise in many fields, including hydrology, climate research, and finance (see, e.g., Kalashnikov, 1997; Kozubowski et al. 2011, Biondi et al., 2008). In hydroclimatic problems we often consider processes indicating water availability, such as precipitation, stream flow, or the Palmer Drought Severity Index over North America (see Cook and Krusic, 2003). These processes are often described and studied using *episodes*. An episode is represented as a random vector of magnitude and duration, where duration is the number of time intervals (e.g. years) the process remains continuously above (or below) a reference level, while magnitude is the sum (3) of all process values for a given duration. In drought/flood analysis, the joint distribution of magnitude X and duration N of an episode is of primary practical importance for water resource managers, risk assessment, civil engineering projects, and the insurance industry (see, e.g., Shiau and Shen, 2001; Biondi et al., 2002; Kim et al., 2003, González and Valdés, 2003). The BEG model proved very useful in hydroclimatic research (see Biondi et al., 2005, 2008).

Further, the BEG model showed very good fit to financial data, see Kozubowski and Panorska (2005). In finance, the X_i's are the log-returns X_i=$\log P_i - \log P_{i-1}$, $i = 1, 2, \ldots,$ corresponding to the values $P_0, P_1, \ldots$ of a process such as the daily price of a stock, or an index, or currency exchange rate. In this context, N represents a period of growth (or decline) in value, where the consecutive X_i's are positive (negative).

This thesis is organized as follows. In Section 2, we recall definations and basic properties of the BEG model. In Section 3, we derive the error of the estimated survival function from exponential, geometric, and BEG distributions. In Section 4, we discuss the asymptotic distribution of the estimated survival function of the BEG distribution. In Section 5, we present a Monto Carlo illustration of our theoretical results. Section 6 is an appendix with numerical code used for Monto Carlo simulations.

2. **Definition and Methods**

2.1. **Exponential, Geometric and BEG distributions.** We begin with the maximum likelihood estimators (MLE) for the parameters of exponential and geometric distributions.

Let X$\sim \mathcal{EXP}(\beta)$. Then the mean and the variance of X are (see DeGroot and Schervish, 2002)

$$E[X] = \frac{1}{\beta} \quad \text{and} \quad Var[X] = \frac{1}{\beta^2}.$$

The moment generating function (m.g.f) of $\mathcal{EXP}(\beta)$ is

$$\psi(t) = \frac{\beta}{\beta - t} \qquad \text{for} \quad t < \beta.$$

Let $X_1, X_2 \ldots, X_n$ be a random sample from an $\mathcal{EXP}(\beta)$ with p.d.f defined in (1). Then, the MLE of the parameter β can be obtained by maximizing the log-likelihood function

$$\log \mathbb{L}(\beta) = n \log \beta - \beta \sum_{i=1}^{n} x_i.$$

By differentating the above equation with with respect to β and equating it to 0, we get a unique MLE as,

$$\hat{\beta} = \frac{1}{\bar{X}}. \tag{4}$$

Next, we discuss some of the basic properties of the geometric distribution $\mathcal{GEO}(p)$. Consider a sequence of independent Bernoulli trails in which the outcome of any trail is either 1 (success) or 0 (failure) and the probability of success on any trail is p (see

DeGroot and Schervish, 2002). Let X denote the number of trails needed for the first success to occur. Then, X is said to have a geometric distribution with parameter p and probability distribution function defined in (2).

If X$\sim \mathcal{GEO}(p)$, then the expected value and variance of X are

$$E[X] = \frac{1}{p} \quad \text{and} \quad Var[X] = \frac{1-p}{p^2}.$$

The moment generating function (m.g.f) of X is given by

$$M(t) = \frac{pe^t}{1-(1-p)e^t}, \qquad t < \log(1/(1-p)).$$

The MLE of p for the geometric distribution can be obtained my maximizing the log-likelihood function

$$\log \mathbb{L}(p) = n \log p + \sum_{i=1}^{n} x_i - n \log(1-p), \quad 0 < p < 1.$$

By differentating the above equation with respect to p and equating it to 0, we get a unique MLE of p as

$$\hat{p} = \frac{1}{\bar{X}}. \tag{5}$$

Now we present the formal definition and recall some properties the BEG distribution. The BEG vector, which stands for **b**ivariate distribution with **e**xponential and **g**eometric marginals, was developed by Kozubowski and Panorska, 2005. That paper included the definition of BEG distribution along with the joint p.d.f, c.d.f, survival function of BEG vector, its marginal and several conditional distributions, alternative stochastic representation, discussion of stability, and estimation of parameters. We start with recalling the formal definition of the BEG vector.

Definition 2.1. *A random vector (X, N) with the stochastic representation*

$$(X, N) \stackrel{d}{=} \left(\sum_{i=1}^{N} X_i, N \right), \tag{6}$$

where the $X_i's$ are i.i.d exponential variables with p.d.f (1) and N is a geometric variable with p.d.f (2), independent of the $X_i's$, is said to have a BEG distribution with parameters $\beta > 0$ and $p \in (0,1)$. This distribution is denoted by $\mathcal{BEG}(\beta, p)$. The joint p.d.f. of (X, N) is

$$g(x, n) = \frac{p\beta^n}{(n-1)!}[x(1-p)]^{n-1}e^{-\beta x}, \quad x > 0, \quad n = 1, 2, \ldots \tag{7}$$

We will be interested in the survival function of the BEG distribution, which is given by

$$S(x, n) = e^{-p\beta x}\left(1 - \sum_{k=0}^{n-1} e^{-(1-p)\beta x}\frac{[(1-p)\beta x]^k}{k!}\right) + (1-p)^n \sum_{k=0}^{n-1} e^{-\beta x}\frac{[\beta x]^k}{k!}. \tag{8}$$

We will often use the notion of convergence in distribution. We define this notion here.

Definition 2.2. *Let $X_1, X_2, \ldots$ be a sequence of random variables, and for each n=1,2,......, let S_n and F_n denote the survival function and the c.d.f of X_n, respectively. Also, let X^* denote another random variable for which the c.d.f. is F^*. We shall assume that F^* is a continuous function over the entire line. Then, the sequence $X_1, X_2, \ldots$ is said to converge in distribution to the random variable X^*, denoted by*

$X_n \xrightarrow{d} X^*$, *if*

$$\lim_{x\to\infty} F_n(x) = F^*(x) \quad \text{for} \quad -\infty{<}x{<}\infty.$$

In literature, convergence in distribution to a normal distribution is called asymptotic normality.

Definition 2.3. *(see Serfling, 1980) A sequence of random variables X_n is asymptotically normal with mean μ_n and variance σ_n^2, denoted by $AN(\mu_n, \sigma_n^2)$, if $\sigma_n > 0$ for all n sufficiently large and*

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1). \tag{9}$$

For a multivariate case, a sequence of random vectors X_n is asymptotically normal with mean vector μ_n and covariance matrix Σ_n if Σ_n has nonzero diagonal elements for all n sufficiently large, and, for every vector λ such that $\lambda\Sigma_n\lambda^{'} > 0$, for all n sufficiently large, the sequence $\lambda X_n^{'}$ is AN$(\lambda X_n^{'}, \lambda\Sigma_n\lambda^{'})$ (see Serfling, 1980). Here, μ_n is a sequence of vector constants and Σ_n a sequence of covariance matrix constants. Below are two theorems that provide asymptotic distribution of a function of sequence of random variables or vectors. Both results follow Serfling (1980).

Theorem 2.1. *Suppose that X_n is $AN(\mu, \sigma_n^2)$, with $\sigma_n \to 0$. Let g be a real-valued function, diffrentiable at x=μ, and with $g^{'}(\mu) \neq 0$. Then*

$$g(X_n) \quad is \quad AN(g(\mu), [g^{'}(\mu)]^2\sigma_n^2).$$

Now, the following theorem extends Theorem 2.1 to a multivariate case.

Theorem 2.2. *Suppose that* $\mathbf{X_n} = (X_{n1}, \ldots, X_{nk})$ *is* $AN(\mu, b_n^2\mathbf{\Sigma})$*, with* $\mathbf{\Sigma}$ *a covariance matrix and* $b_n \to 0$*. Let* $g(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))$*,* $\mathbf{x} = (x_1, \ldots, x_k)$*, be a vector-valued function for which each component function* $g_i(\mathbf{x})$ *is real-valued and has a nonzero differential* $g_i(\mu; \mathbf{t})$*,* $\mathbf{t} = (t_1, \ldots, t_k)$*, at* $\mathbf{x} = \mu$*. Put*

$$\mathbf{D} = \left[\left. \frac{\partial g_i}{\partial x_j} \right|_{\mathbf{x}=\mu} \right]_{m \times k}.$$

Then

$$g(\mathbf{X_n}) \qquad is \qquad AN(g(\mu), b_n^2 \mathbf{D}\mathbf{\Sigma}\mathbf{D}')$$

In our case, we have $\mathbf{x} = (x_1, x_2)$ and j=1,2.

In our thesis we use Anderson-Darling (A-D) test for normality using significance level 0.05. If the test rejects normality then we conclude with 95% confidence that the data does not come from a normal distribution. However, if the test accepts the hypothesis of normality then it states that no significant departure from the normality was detected.

Remark 2.1. Anderson-Darling test has known problems when ties are present in the data set. A tie is when identical values occur more then once in a data set. If the significant number of ties is present in the data set, then Anderson-Darling test will frequently reject the hypothesis of normality, regardless of how well that data fits the normal distribution. But in our work, there are no ties and hence A-D test works well.

3. **Estimation of Error of the Survival Function for Exponential and Geometric Distributions**

In this section, we develop error estimates of the survival functions for the exponential and geometric distributions. For each model, we present the theoretical derivation of the limiting distribution of the estimated survival functions $\hat{S}(x)$. We also discuss monotonicity of the asymptotic variance of $\hat{S}(x)$ and present large sample confidence intervals for $\hat{S}(x)$. We begin with exponential distribution.

3.1. **Limiting distribution of the survival function of an exponential distribution.** We begin our presentation with the limiting distribution of $\hat{S}(x)$. Our main interest is in the asymptotic (large sample) variance η^2 of $\hat{S}(x)$, which is a measure of error when computing probabilities such as $P(X > x) = S(x) \approx \hat{S}(x)$. In the derivation of the asymptotic variance of $\hat{S}(x)$, we use Theorem 2.1 from Section 2. The survival function $S(x)$ of an exponential random variable with the p.d.f defined in (1) is given by

$$S(x) = e^{-\beta x}, \qquad x > 0. \tag{10}$$

The MLE of $S(x)$ is

$$\hat{S}(x) = e^{-\hat{\beta} x},$$

where $\hat{\beta}$ is the MLE of β given by (4).

We start with a result describing the asymptotic distribution of $\hat{S}(x)$.

Proposition 3.1. *Let* $X_1, X_2, \ldots, X_n$ *be a random sample from* $\mathcal{EXP}(\beta)$ *and let* $X \sim \mathcal{EXP}(\beta)$, $\hat{S}(x){=}S_n(x){=}e^{-\hat{\beta}x}$, *where* $\hat{\beta}{=}\frac{1}{\bar{X}}$. *Then,*

$$\sqrt{n}(S_n(x) - e^{-\beta x}) \xrightarrow{d} N(0, \eta^2), \tag{11}$$

where

$$\eta^2 = \tfrac{x^2\beta^2}{e^{2\beta x}}.$$

Proof : Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the sample mean from the exponential sample. Since $VarX_i = \frac{1}{\beta^2}{<}\infty$, then by the Central Limit Theorem, $\bar{X}_n$ is asymptotically normal; that is for sufficiently large n (see Navidi, 2006),

$$\bar{X}_n \sim AN\left(EX_i, \frac{VarX_i}{n}\right). \tag{12}$$

The above relation can be written as,

$$\bar{X}_n \sim AN\left(\frac{1}{\beta}, \frac{1}{n\beta^2}\right).$$

where $EX_i = \mu{=}\frac{1}{\beta}$ and $VarX_i = \sigma^2{=}\frac{1}{\beta^2}$.
Since the survival function (10) is a function of β, its MLE is a function of the MLE of β, i.e $\hat{\beta} = \frac{1}{\bar{X}}$, and we get MLE of $\hat{S}(x)$ as follows:

$$\hat{S}(x) = S_n(x) = e^{-\hat{\beta}x} = e^{-\frac{x}{\bar{X}}} = g(\bar{X}). \tag{13}$$

Since $\mu = \frac{1}{\beta}$, we have

$$g(\mu){=}\ S(x){=}e^{-\beta x}\ {=}e^{-\frac{x}{\mu}}.$$

Since the function g is differentiable, so by Theorem 2.1 we obtain

$$\sqrt{n}\left(g(\bar{X}) - g(\mu)\right) \xrightarrow{d} N(0, \eta^2), \tag{14}$$

where

$$\eta^2 = \sigma^2 \cdot \left[g^{'}(x)\Big|_{x=\mu} \right]^2 . \tag{15}$$

Now, let

$$g(s) = \exp{(-\frac{x}{s})},$$

so that

$$g^{'}(s) = \frac{x}{s^2} e^{-\frac{x}{s}} . \tag{16}$$

Thus, using (15) and (16), we have

$$\begin{aligned} \eta^2 &= \frac{1}{\beta^2} \left[\frac{x}{s^2} e^{-\frac{x}{s}} \Big|_{s=\frac{1}{\beta}} \right]^2 \\ &= \frac{1}{\beta^2} \left[x\beta^2 \cdot e^{-\beta x} \right]^2 \end{aligned}$$

Hence

$$\eta^2 = x^2 \beta^2 e^{-2\beta x} \tag{17}$$

which completes the proof of Proposition 3.1.

Now we will turn to the monotonicity of the asymptotic variance of $\hat{S}(x)$.

3.2. **Monotonicity of the asymptotic variance of $\hat{S}(x)$ of an exponential distribution.** We are interested in the behaviour of Var $\hat{S}(x)$ for changing x and sample size n. From (17), the variance is

$$Var[\hat{S}(x)] = \frac{\beta^2 x^2}{n e^{2\beta x}}.$$

Let

$$f(x) = \frac{\beta^2 x^2}{n e^{2\beta x}}, \qquad x>0. \tag{18}$$

Differentiating $f(x)$ with respect to x we get

$$\begin{aligned} f^{'}(x) &= \frac{\partial}{\partial x}\left(\frac{\beta^2 x^2}{n e^{2\beta x}}\right) \\ &= \frac{\beta^2}{n}\left(\frac{2xe^{2\beta x} - 2x^2\beta e^{2\beta x}}{(e^{2\beta x})^2}\right). \end{aligned}$$

Hence

$$f^{'}(x) = \frac{\beta^2}{n}\left(\frac{2x - 2\beta x^2}{e^{2\beta x}}\right). \tag{19}$$

Letting $f^{'}(x) = 0$, we get

$$x = x^2\beta.$$

Solving for x yields

$$x = \frac{1}{\beta} \quad \text{or} \quad x = 0. \tag{20}$$

Now we differentiate (19) with respect to x and obtain

$$\begin{aligned} f''(x) &= \frac{2\beta^2}{n}\left[\frac{e^{2\beta x}(1-2\beta x) - 2\beta e^{2\beta x}(x - x^2\beta)}{(e^{2\beta x})^2}\right] \\ &= \frac{2\beta^2}{n}\left[\frac{(1-2\beta x) - 2\beta(x - x^2\beta)}{e^{2\beta x}}\right]. \end{aligned}$$

Hence, we obtain

$$f''(x) = \frac{2\beta^2}{n}\left[\frac{(1 - 4\beta x + 2\beta^2 x^2)}{e^{2\beta x}}\right]. \tag{21}$$

Substituting $x = \frac{1}{\beta}$ and x= 0 yields

$$f''(\frac{1}{\beta}) = \frac{-2\beta^2}{ne^2} < 0, \quad \text{and} \quad f''(0) = \frac{2\beta^2}{n} > 0,$$

which shows that $f(x)$ has maximum at $x = \frac{1}{\beta}$ and a minimum at $x = 0$.
Subsituting $x = \frac{1}{\beta}$ in $f(x)$, we see that the maximum value of $f(x)$ is

$$f(\frac{1}{\beta}) = \frac{1}{ne^2}. \tag{22}$$

This shows that as $n \to \infty$, $f(x) \to 0$. It means that for every x, as the sample size n increases, variance of the estimated survival function η^2 decreases.

Now we discuss the behaviour of $\hat{S}(x)$ when $x \to 0$ and when $x \to \infty$. We take equation (18) and consider β and n to be constant. First, we note that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{\beta^2 x^2}{ne^{2\beta x}}\right) = 0.$$

We again consider (18) and see the behaviour of $f(x)$ for $x \to \infty$. Using L' Hospital rule, we obtain

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{\beta^2 x^2}{ne^{2\beta x}}\right) = 0.$$

The graph in the Figure 1 presents the asymptotic variance of $\hat{S}(x)$ as the function of x, where $\beta = 2$ and n=1.

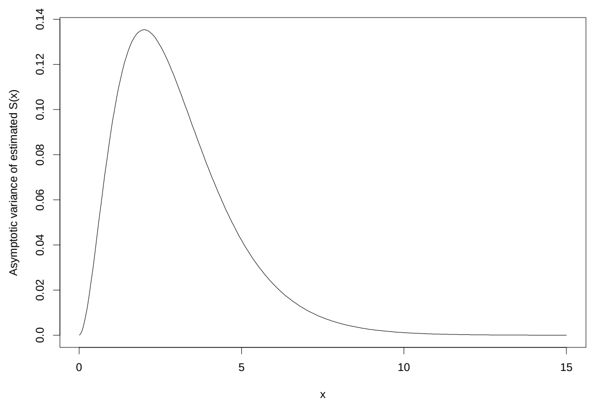

FIGURE 1. Graph of the asymptotic variance of the estimate of the survival function for the exponential distribution with parameter $\beta = 0.25$ and $n = 1$.

3.3. **Asymptotic confidence interval for $\hat{S}(x)$ for an exponential distribution.** Here we shall derive an asymptotic confidence interval for the survival function of an exponential distribution. By Proposition 3.1, we have

$$\sqrt{n}\left(\hat{S}(x) - S(x)\right) \xrightarrow[n\to\infty]{d} N(0, \eta^2). \tag{23}$$

Hence, we have

$$\hat{S}(x) \sim AN\left(S(x), \frac{\eta^2}{n}\right).$$

Thus we can construct a $(1-\alpha)100\%$ Confidence Interval for $S(x)$, using

$$Pr\left(-Z_{\frac{\alpha}{2}} \le \frac{\hat{S}(x) - S(x)}{\eta/\sqrt{n}} \le Z_{\frac{\alpha}{2}}\right) = 1-\alpha, \tag{24}$$

where $Z_{\frac{\alpha}{2}}$ is the $100(1-\frac{\alpha}{2})^{th}$ percentile of the standard normal distribution. Upon reformulating equation (24), we have

$$Pr\left(\hat{S}(x) - Z_{\frac{\alpha}{2}}\frac{\eta}{\sqrt{n}} \le S(x) \le \hat{S}(x) + Z_{\frac{\alpha}{2}}\frac{\eta}{\sqrt{n}}\right) = 1-\alpha.$$

Thus, a $(1-\alpha)100\%$ Confidence Interval for $S(x)$ is

$$\hat{S}(x) \pm Z_{\frac{\alpha}{2}} \cdot \frac{\eta}{\sqrt{n}}.$$

We summarize the above derivation in the following proposition.

Proposition 3.2. *Let $X_1, X_2 \ldots, X_n$ be a random sample from $\mathcal{EXP}(\beta)$. Then for any $x > 0$, an approximate $(1-\alpha)100\%$ Confidence Interval for S(x) is*

$$\hat{S}(x) \pm Z_{\frac{\alpha}{2}} \cdot \frac{\eta}{\sqrt{n}}, \tag{25}$$

where η^2 is the asymptotic variance of $\hat{S}(x)$ given by $\eta^2 = \frac{x^2\beta^2}{e^{2\beta x}}$.
Since in practice we don't know β, we may use $\hat{\beta}$ in the above formula to get approximate confidence interval for $S(x)$.

3.4. **Limiting distribution of the survival function of a geometric distribution.** Here, we shall follow the same steps as we did in the exponential case. We begin with the derivation of the asymptotic variance of the estimated survival function $\hat{S}(x)$ using Theorem 2.1 from Section 2. The survival function $S(x)$ of the geometric distribution with the p.d.f in (2) is given by

$$S(x) = (1-p)^x, \quad x > 0. \tag{26}$$

We begin with the proposition providing the asymptotic variance of the MLE of $S(x)$ given by $\hat{S}(x) = (1-\hat{p})^x$, where $\hat{p}=\frac{1}{\bar{X}}$.

Proposition 3.3. *Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathcal{GEO}(p)$, and $\hat{S}(x) = S_n(x) = (1-\hat{p})^x$, then,*

$$\sqrt{n}(S_n(x) - (1-p)^x) \xrightarrow{d} N(0, \eta^2), \tag{27}$$

where $\quad \eta^2 = x^2 p^2 (1-p)^{(2x-1)}$.

Proof: Since $VarX_i=\frac{1-p}{p^2} \;<\infty$, then by the Central Limit Theorem (See Navidi, 2006), for sufficiently large n, we have

$$\bar{X}_n \sim AN\left(\frac{1}{p}, \frac{1-p}{np^2}\right).$$

Next, the survival function can be written as a function of the MLE $\hat{p}$, and thus a function of $\bar{X}_n$ as follows:

$$\hat{S}(x) = S_n(x) = (1-\hat{p})^x = (1-\frac{1}{\bar{X}})^x = g(\bar{X}) \tag{28}$$

Thus by the Central Limit Theorem, we have

$$\sqrt{n}\left(g(\bar{X}) - g(\mu)\right) \xrightarrow[n\to\infty]{d} N(0, \eta^2). \tag{29}$$

where η^2 is defined in (15).

Now, let

$$g(s) = \left(1 - \frac{1}{s}\right)^x.$$

so that

$$g^{'}(s) = \frac{x}{s^2}\left(1 - \frac{1}{s}\right)^{x-1}. \tag{30}$$

Thus, using (15) and (30), we have,

$$\begin{aligned}\eta^2 &= \left(\frac{1-p}{p^2}\right)\left[\frac{x}{s^2}\left(1 - \frac{1}{s}\right)^{x-1}\Bigg|_{s=\frac{1}{p}}\right]^2 \\ &= \left(\frac{1-p}{p^2}\right)\left[xp^2(1-p)^{x-1}\right]^2\end{aligned}$$

Hence,

$$\eta^2 = x^2p^2(1-p)^{(2x-1)}, \tag{31}$$

which concludes the proof of Proposition 3.3.

3.5. **Monotonicity of the asymptotic variance of $\hat{S}(x)$ for geometric distribution.** In this section, we investigate the behaviour of the asymptotic variance η^2 of the estimated survival function of the geometric distribution as we change x and the sample size n. The variance of $\hat{S}(x)$ is given in (31) by

$$Var[\hat{S}(x)] = \frac{x^2 p^2 (1-p)^{(2x-1)}}{n}.$$

Let

$$f(x) = \frac{x^2 p^2 (1-p)^{(2x-1)}}{n}. \tag{32}$$

Now differentiating $f(x)$ with respect to x we have,

$$f^{'}(x) = \frac{p^2}{n}\left[2x(1-p)^{(2x-1)} + 2x^2(1-p)^{(2x-1)}\log(1-p)\right].$$

Hence,

$$f^{'}(x) = \frac{2xp^2(1-p)^{(2x-1)}}{n}\left[1 + x\log(1-p)\right]. \tag{33}$$

To check the maximum or minimum, we set $f^{'}(x)$ to 0, resulting in

$$(1-p)^{(2x-1)} + x(1-p)^{(2x-1)}\log(1-p) = 0.$$

The solutions of the above equation are

$$x = 0 \quad \text{or} \quad x_0 = \frac{-1}{\log(1-p)}; \quad p < 1. \tag{34}$$

Now, differentating equation (33) with respect to x, we have

$$\begin{aligned}
f^{''}(x) &= \frac{2p^2(1-p)^{2x}}{n(1-p)}\left[1 + 2xlog(1-p) + 2xlog(1-p)\left(1 + xlog(1-p)\right)\right] \\
&= \frac{2p^2(1-p)^{2x}}{n(1-p)}\left[1 + 2xlog(1-p) + 2xlog(1-p) + 2x^2log^2(1-p)\right] \\
&= \frac{2p^2(1-p)^{2x}}{n(1-p)}\left[1 + 2x^2log^2(1-p) + 4xlog(1-p)\right].
\end{aligned}$$

Substituting x from (34) in the above equation, we obtain

$$f''(x_0) = \frac{-2p^2(1-p)^{\frac{-2}{\log(1-p)}}}{n(1-p)} < 0, \quad p < 1,$$

and

$$f''(0) = \frac{2p^2}{n(1-p)} > 0,$$

which shows that $f(x)$ is maximized at $x_0 = \frac{-1}{\log(1-p)}$ and is minimized at $x = 0$. Subsituting x_0 in $f(x)$, we see that the maximum value of $f(x_0)$ is

(35) $$f(x_0) = \frac{p^2(1-p)^{-\left[\frac{2}{\log(1-p)}+1\right]}}{n\log^2(1-p)}.$$

Taking limit of $f(x_0)$ as $n \to \infty$, we see that $f(x) \to 0$. It means that as the sample size n increases, η^2 decreases.

Now we will discuss the behaviour of the function $f(x)$ as x approches zero and infinity. In this case we consider p and n as constants. We begin with the limit of f at zero.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2p^2(1-p)^{(2x-1)}}{n} = 0.$$

We again consider $f(x)$ to see its behaviour for $x \to \infty$. We have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2p^2(1-p)^{(2x-1)}}{n}.$$

Applying twice L' Hospital rule to the right hand side of the above equation we have

$$\lim_{x\to\infty} f(x) = \frac{p^2}{-n(1-p)}\left[\lim_{x\to\infty}\frac{x}{(1-p)^{-2x}\log(1-p)}\right]$$
$$= \frac{p^2}{2n\log^3(1-p)}\left[\lim_{x\to\infty}\frac{1}{(1-p)^{-2x}}\right]$$
$$= 0$$

The graph in the Figure(2) presents asymptotic variance of $\hat{S}(x)$ as a function of x at p=0.25 and n=1.

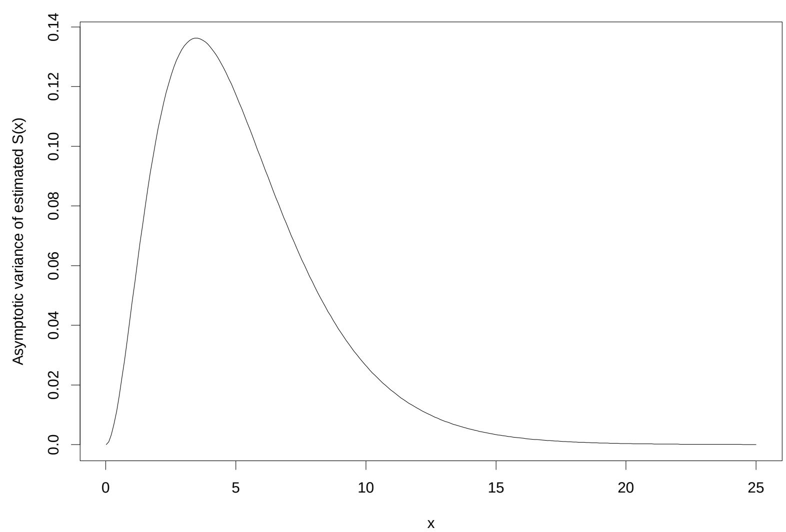

FIGURE 2. Graph of the asymptotic variance of the estimate of the survival function for the geometric distribution with parameter p=0.25

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3.6. **Asymptotic confidence interval for geometric distribution.** Here we present an asymptotic confidence interval for the survival function of the geometric distribution. The technique is similar to that used for the confidence interval in the exponential case.

Proposition 3.4. *Let $X_1, X_2 \dots, X_n$ be a random sample from $\mathcal{GEO}(p)$. Then for any $x > 0$, an approximate $(1-\alpha)100\%$ Confidence Interval for S(x) is*

$$\hat{S}(x) \pm Z_{\frac{\alpha}{2}} \cdot \frac{\eta}{\sqrt{n}}, \tag{36}$$

where η^2 is the asymptotic variance of $\hat{S}(x)$, equal to $\eta^2 = x^2 p^2 (1-p)^{(2x-1)}$.

Proof: By proposition 3.3,

$$\sqrt{n}\left(\hat{S}(x) - S(x)\right) \xrightarrow[n\to\infty]{d} N(0, \eta^2). \tag{37}$$

Hence, we have

$$\hat{S}(x) \sim AN\left(S(x), \frac{\eta^2}{n}\right).$$

Thus we can construct an asymptotic (for large n) $(1-\alpha)100\%$ Confidence Interval for $S(x)$ using

$$Pr\left(-Z_{\frac{\alpha}{2}} \le \frac{\hat{S}(x) - S(x)}{\eta/\sqrt{n}} \le Z_{\frac{\alpha}{2}}\right) = 1-\alpha, \tag{38}$$

where $Z_{\frac{\alpha}{2}}$ is the $100(1-\frac{\alpha}{2})^{th}$ percentile of the standard normal distribution. By reformulating equation (38) we have

$$Pr\left(\hat{S}(x) - Z_{\frac{\alpha}{2}}\frac{\eta}{\sqrt{n}} \le S(x) \le \hat{S}(x) + Z_{\frac{\alpha}{2}}\frac{\eta}{\sqrt{n}}\right) = 1-\alpha.$$

Thus, an asymptotic $(1-\alpha)100\%$ Confidence Interval for $S(x)$ is

$$\hat{S}(x) \pm Z_{\frac{\alpha}{2}} \cdot \frac{\eta}{\sqrt{n}}, \tag{39}$$

which concludes the proof of the Proposition 3.4.

Since in practice we may not know p, we replace p with its estimate $\hat{p}=\frac{1}{\bar{X}}$ in the formula (39) for the asymptotic Confidence Interval.

4. Asymptotic distribution of the survival function of the BEG model.

Here, we find the asymptotic distribution of the estimated survival function of the BEG distribution with parameters β and p. We start this section by recalling some properties of the BEG model. We then present the ideas of the two methods we used for derivation of the asymptotic distribution of the estimated survival function. Once the ideas of the methods are described, we present the details of the derivation for both methods.

4.1. **Properties of the BEG model.** Recall the survival function of the BEG distribution (see, Kozubowski and Panorska, 2005). If $(X, N) \sim \mathcal{BEG}(\beta, p)$ then

$$S(x,n) = e^{-p\beta x}\left(1 - \sum_{k=0}^{n-1}(e^{-(1-p)\beta x})\frac{[(1-p)\beta x]^k}{k!}\right) + (1-p)^n\left(\sum_{k=0}^{n-1} e^{-\beta x}\frac{[\beta x]^k}{k!}\right)$$

for any real $x > 0$ and integer $n > 0$. Further EX=$(\beta p)^{-1}$, EN=p^{-1} and the covariance matrix of (X, N) is

$$\Sigma = \begin{bmatrix} \frac{1}{\beta^2 p^2} & \frac{1-p}{\beta p^2} \\ \frac{1-p}{\beta p^2} & \frac{1-p}{p^2} \end{bmatrix}. \tag{40}$$

Now let $(X_1, N_1).....(X_m, N_m)$ be a random sample from a $\mathcal{BEG}(\beta, p)$ distribution. The MLE of p and β are

$$\hat{\beta} = \frac{\bar{N_m}}{\bar{X_m}}, \hat{p} = \frac{1}{\bar{N_m}}, \tag{41}$$

where $\bar{X_m}$ and $\bar{N_m}$ are the sample means of $X_i's$ and $N_i's$, respectively.

The consistency and asymptotic normality of these MLE are proved in Kozubowski and Panorska, (2005). In particular, we have

$$\sqrt{m}\left[(\hat{\beta_m}, \hat{p_m}) - (\beta, p)\right] \xrightarrow[n\to\infty]{d} N(0, \Sigma_{MLE}), \tag{42}$$

where $N(0, \Sigma_{MLE})$ denotes a bivarate normal distribution with mean vector 0 and variance-covariance matrix Σ_{MLE}, which is given by

$$\Sigma_{MLE} = \begin{bmatrix} \beta^2 p & 0 \\ 0 & p^2(1-p) \end{bmatrix}. \tag{43}$$

and m is the sample size.

Idea of Method 1 for finding asymptotic distribution of $\hat{S}(x, n)$.

Since the survival function $S(x, n)$ is a continuous function of the parameters, $S(x.n) = g(x, n, \beta, p)$ as in equation(8), and the estimated survival function is the same function of the estimated parameters, $\hat{S}(x, n, \hat{\beta}, \hat{p}) = g(x, n, \hat{\beta}, \hat{p})$

$$\hat{S}(x, n, \hat{\beta}, \hat{p}) = e^{-\hat{p}\hat{\beta}x}\left(1 - \sum_{k=0}^{n-1} e^{-(1-\hat{p})\hat{\beta}x}\frac{[(1-\hat{p})\hat{\beta}x]^k}{k!}\right) + (1-\hat{p})^n \sum_{k=0}^{n-1} e^{-\hat{\beta}x}\frac{[\hat{\beta}x]^k}{k!} \tag{44}$$

for any real $x > 0$ and integer $n > 0$. Then by Theorem 2.2 we have,

$$\sqrt{m}\left[g(\hat{\beta_m}, \hat{p_m}) - g(\beta, p)\right] \xrightarrow{d} N(0, \eta^2). \tag{45}$$

where m in the above equation is the sample size and

$$\eta^2 = D\Sigma_{MLE}D', \tag{46}$$

where $D = \begin{bmatrix} \frac{\partial g}{\partial \beta} & \frac{\partial g}{\partial p} \end{bmatrix}$.

Equation (44) essentially says that $\hat{S}(x, n, \hat{\beta}, \hat{p})$ is asymptotically normal with mean equal to $S(x,n,\beta,p)$ and variance equal to η^2. The first procedure uses the above methodology and idea to find the asymptotic distribution, and the variance in particular for $\hat{S}(x,n)$.

Idea of Method 2 for finding asymptotic distribution of $\hat{S}(x)$.

The second method uses the same general methodology as Method 1, except $\bar{X}$ and $\bar{N}$ are used in place of $\hat{\beta_m}$ and $\hat{p_m}$. Let $\bar{Z_m} = (\bar{X}, \bar{N})$ and $\mu_m = (E\bar{X}, E\bar{N}) = ((\beta p)^{-1}, 1/p)$. Next, using the Central Limit Theorem, we know that, as the sample size $m \to \infty$,

$$\sqrt{m}(\bar{Z_m} - \mu_m) \xrightarrow{d} N(0, \Sigma), \tag{47}$$

where $N(0,\Sigma)$ is a bivariate normal distribution with mean vector zero, and covariance matrix Σ given by (40). Further, we note that the estimated survival function $\hat{S}(x,n)$ is a continuous function of $\bar{X}$ and $\bar{N}$, i.e. $\hat{S}(x,n,\bar{X},\bar{N}) = G(x,n,\bar{X},\bar{N})$. Then by Theorem 2.2, we have

$$\sqrt{m}\left[G(\bar{X},\bar{N}) - G(\mu)\right] \xrightarrow{d} N(0,\eta^2), \tag{48}$$

where m in the above equation is the sample size and

$$\eta^2 = D\Sigma D' \tag{49}$$

where $D = \begin{bmatrix} \frac{\partial G}{\partial y_1} & \frac{\partial G}{\partial y_2} \end{bmatrix} |_{(y_1,y_2)=\mu}$.

The second method, although using a different road, should produce the same asymptotic distribution of the estimated survival function for the BEG model as Method 1. We now turn to the derivation of the asymptotic distribution of $\hat{S}(x,n)$ using Method 1 and Method 2.

4.2. **Derivation of the asymptotic distribution of $\hat{S}(x,n)$ using Method 1.** We know from equation (8), that the survival function is

$$S(x,n) = e^{-p\beta x}\left(1 - \sum_{k=0}^{n-1}(e^{-(1-p)\beta x})\frac{[(1-p)\beta x]^k}{k!}\right) + (1-p)^n\left(\sum_{k=0}^{n-1} e^{-\beta x}\frac{[\beta x]^k}{k!}\right)$$
$$= e^{-p\beta x} - e^{-\beta x}\sum_{k=0}^{n-1}\frac{[\beta x]^k}{k!}[(1-p)^k - (1-p)^n]$$

Let $y_1 = \beta$ and $y_2 = p$ and $g(y_1, y_2)$=$S(x, n, y_1, y_2)$. Then the above equation can be written as,

$$g(y_1, y_2) = e^{-y_1 y_2 x} - e^{-y_1 x}\sum_{k=0}^{n-1}\frac{[y_1 x]^k}{k!}[(1-y_2)^k - (1-y_2)^n]. \tag{50}$$

Now differentating equation 50 with respect to y_1

$$\frac{\partial g(y_1,y_2)}{\partial y_1} = \frac{\partial e^{-y_1 y_2 x}}{\partial y_1} - \frac{\partial}{\partial y_1} e^{-y_1 x}\sum_{k=0}^{n-1}\frac{[y_1 x]^k}{k!}\left[(1-y_2)^k - (1-y_2)^n\right]$$
$$= -y_2 x e^{-y_1 y_2 x} - \left[-x e^{-y_1 x}\sum_{k=0}^{n-1}\frac{[y_1 x]^k}{k!}[(1-y_2)^k - (1-y_2)^n]\right.$$
$$\left. + e^{-y_1 x}\sum_{k=0}^{n-1}\frac{k {y_1}^{k-1} x^k}{k!}[(1-y_2)^k - (1-y_2)^n]\right]$$
$$= -y_2 x e^{-y_1 y_2 x} - \left[e^{-y_1 x}\sum_{k=0}^{n-1}\frac{[y_1 x]^k}{k!}[(1-y_2)^k - (1-y_2)^n]\left(-x + \frac{k}{y_1}\right)\right].$$

Hence, we get

$$\frac{\partial g(y_1 y_2)}{\partial y_1} = -y_2 x e^{-y_1 y_2 x} + e^{-y_1 x} \sum_{k=0}^{n-1} \left(\frac{{y_1}^{k-1} x^k}{k!} \mathbb{W}_k \right), \tag{51}$$

where

$$\mathbb{W}_k = [(1-y_2)^k - (1-y_2)^n](xy_1 - k). \tag{52}$$

Now, differentiating equation (50) with respect to y_2

$$\begin{aligned}
\frac{\partial g(y_1, y_2)}{\partial y_2} &= \frac{\partial e^{-y_1 y_2 x}}{\partial y_2} - \frac{\partial}{\partial y_2} e^{-y_1 x} \sum_{k=0}^{n-1} \frac{[y_1 x]^k}{k!} \left[(1-y_2)^k - (1-y_2)^n\right] \\
&= -y_1 x e^{-y_1 y_2 x} + e^{-y_1 x} \sum_{k=0}^{n-1} \left(\frac{{y_1}^k x^k}{k!} \left[k(1-y_2)^{k-1} - n(1-y_2)^{n-1}\right] \right) \\
&= -y_1 x e^{-y_1 y_2 x} + e^{-y_1 x} \sum_{k=0}^{n-1} \left(\frac{{y_1}^k x^k}{{y_2}^2 k!} \left[{y_2}^2 \left(k(1-y_2)^{k-1} - n(1-y_2)^{n-1}\right)\right] \right) \\
&= -y_1 x e^{-y_1 y_2 x} + e^{-y_1 x} \sum_{k=0}^{n-1} \left(\frac{{y_1}^k x^k}{{y_2}^2 k!} [\mathbb{Y}_k - k y_2 [(1-y_2)^k - (1-y_2)^n] \right. \\
&\left. + x y_1 y_2 [(1-y_2)^k - (1-y_2)^n]] \right).
\end{aligned}$$

Straight forward calculations from the above steps will yield

$$\frac{\partial g(y_1, y_2)}{\partial y_2} = -y_1 x e^{-y_1 y_2 x} + e^{-y_1 x} \sum_{k=0}^{n-1} \left(\frac{{y_1}^k x^k}{{y_2}^2 k!} \left[\mathbb{Y}_k + y_2 \mathbb{W}_k\right] \right), \tag{53}$$

where

$$\begin{aligned}
\mathbb{Y}_k &= -x y_1 y_2 [(1-y_2)^k - (1-y_2)^n] + k y_2 [(1-y_2)^k - (1-y_2)^n] + \\
& \quad {y_2}^2 [k(1-y_2)^{k-1} - n(1-y_2)^{n-1}]
\end{aligned} \tag{54}$$

and $\mathbb{W}_k$ is defined in (52).

Letting $A = \frac{\partial g}{\partial y_1}$ and $B = \frac{\partial g}{\partial y_2}$ and subsituting $y_1 = \beta$ and $y_2 = p$ in equation (51) and (53), we have

$$D = [\mathbb{A} \;\; \mathbb{B}],$$

where

$$\mathbb{A} = -pxe^{-\beta px} + e^{-\beta x} \sum_{k=0}^{n-1} \left(\frac{\beta^{k-1} x^k}{k!} \mathbb{W}_k \right)$$

and

$$\mathbb{B} = -\beta xe^{-\beta px} + e^{-\beta x} \sum_{k=0}^{n-1} \left(\frac{\beta^k x^k}{p^2 k!} \left[\mathbb{Y}_k + p\mathbb{W}_k \right] \right).$$

Now, the computation of the asymptotic variance η^2 can be done using equation (46). The computation is done in two steps. The first step is to compute $D \cdot \Sigma_{MLE}$, where Σ_{MLE} is given in equation (43). The second step finishes the computation of η^2 as $D \cdot \Sigma_{MLE} \cdot D'$. We start with step 1.

$$\begin{aligned} D \cdot \Sigma_{MLE} &= [\mathbb{A} \quad \mathbb{B}] \quad \begin{bmatrix} \beta^2 p & 0 \\ 0 & p^2(1-p) \end{bmatrix} \\ &= \begin{bmatrix} -\beta^2 p^2 xe^{-p\beta x} + pe^{-\beta x} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^k \mathbb{W}_k}{k!} \\ -\beta xp^2(1-p)e^{-p\beta x} + (1-p)e^{-\beta x} \sum_{k=0}^{n-1} \frac{\beta^k x^k}{k!} [\mathbb{Y}_k + p\mathbb{W}_k] \end{bmatrix}'. \end{aligned}$$

Now for the final step, we multiply the value of $D \cdot \Sigma_{MLE}$ by D'.

$$D.\Sigma_{MLE}.D^{/} = \begin{bmatrix} -\beta^2 p^2 x e^{-p\beta x} + e^{-\beta x} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^k p \mathbb{W}-k}{k!} \\ -\beta x p^2 (1-p) e^{-p\beta x} + (1-p) e^{-\beta x} \sum_{k=0}^{n-1} \frac{\beta^k x^k}{k!} [\mathbb{Y}_k + p\mathbb{W}_k] \end{bmatrix}^{/} \quad [\mathbb{A} \quad \mathbb{B}]^{/},$$

$$= \begin{bmatrix} \mathbb{C} \\ \mathbb{D}] \end{bmatrix}^{/} \quad [\mathbb{A} \quad \mathbb{B}]^{/}$$

where

$$\mathbb{C} = -\beta^2 p^2 x e^{-p\beta x} + p e^{-\beta x} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^k \mathbb{W}_k}{k!}$$

and

$$\mathbb{D} = -\beta x p^2 (1-p) e^{-p\beta x} + (1-p) e^{-\beta x} \sum_{k=0}^{n-1} \frac{\beta^k x^k}{k!} [\mathbb{Y}_k + p\mathbb{W}_k].$$

Since the computation is very long, we break it into two parts. The first part (AC) multiplies the first row of $D \cdot \Sigma_{MLE}$ by first row of $\mathbb{A}$ and the second part (BD) multiplies the second row of $D \cdot \Sigma_{MLE}$ by $\mathbb{B}$. Thus,

$$AC = \beta^2 p^3 x^2 e^{-2\beta p x} - 2 e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1} p^2}{k!} \mathbb{W}_k + e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k} p}{(k!)^2} \mathbb{W}_k^2.$$

The computation of BD follows below:

$$
\begin{aligned}
BD &= \beta^2 x^2 p^2 (1-p) e^{-2p\beta x} - 2(1-p)e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1}}{k!} [\mathbb{Y}_k + p\mathbb{W}_k] \\
&+ (1-p)e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{p^2 k!^2} [\mathbb{Y}_k + p\mathbb{W}_k]^2 \\
&= \beta^2 x^2 p^2 (1-p) e^{-2p\beta x} - 2(1-p)e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1}}{k!} \mathbb{Y}_k \\
&- 2(1-p)e^{-\beta x(1-p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1}}{k!} p\mathbb{W}_k \\
&+ (1-p)e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{p^2 (k!)^2} [\mathbb{Y}_k^2 + 2p\mathbb{Y}_k \mathbb{W}_k + p^2 \mathbb{W}_k^2].
\end{aligned}
$$

Now, adding AC and BD, which yield η^2, we have

$$\begin{aligned}
\eta^2 = \quad & \beta^2 p^3 x^2 e^{-2\beta p x} - 2e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1} p^2}{k!} \mathbb{W}_k + e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k} p}{(k!)^2} \mathbb{W}_k^2 \\
& + \beta^2 x^2 p^2 (1-p) e^{-2p\beta x} - 2(1-p) e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1}}{k!} \mathbb{Y}_k \\
& - 2p(1-p) e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1}}{k!} \mathbb{W}_k \\
& + (1-p) e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{p^2 (k!^2)} \mathbb{Y}_k^2 + 2(1-p) e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{p(k!^2)} \mathbb{Y}_k \mathbb{W}_k \\
& + (1-p) e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{(k!)^2} \mathbb{W}_k^2 . \\
= \quad & \beta^2 x^2 p^2 e^{-2p\beta x} - 2e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1}}{k!} p \mathbb{W}_k - 2(1-p) e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1}}{k!} \mathbb{Y}_k \\
& + (1-p) e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{p^2 (k!)^2} \mathbb{Y}_k^2 + e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{(k!)^2} \mathbb{W}_k^2 + 2(1-p) e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{p(k!)^2} \mathbb{Y}_k \mathbb{W}_k .
\end{aligned}$$

where $\mathbb{W}_k$ and $\mathbb{Y}_k$ are defined in equation 52 and 54 respectively.

The above procedure is one way to obtain the asymptotic variance of the survival function of the BEG model. Now we follow an alternate procedure, Method 2, to obtain the asymptotic variance of this survival function.

4.3. **Derivation of the asymptotic distribution of $\hat{S}(x, n)$ using Method 2.** In method 2, we use equation (41) and (44) to express the survival function $\hat{S}(x, n)$

as a function of $\bar{X}$ and $\bar{N}$, to get

$$\hat{S}(x,n,\bar{X}_m,\bar{N}_m) = e^{-\frac{x}{\bar{X}_m}} - e^{-\frac{\bar{N}_m x}{\bar{X}_m}} \sum_{k=0}^{n-1} \frac{[\frac{\bar{N}_m x}{\bar{X}_m}]^k}{k!} \left[(1-\frac{1}{\bar{N}_m})^k - (1-\frac{1}{\bar{N}_m})^n\right].$$

Now we let $y_1 = \bar{X}_m$, y_2=$\bar{N}_m$ and $\hat{S}(x,n,\bar{X}_m,\bar{N}_m)$= $g(y_1,y_2)$. Then the above equation will transform to

$$g(y_1,y_2) = e^{-\frac{x}{y_1}} - e^{-\frac{y_2 x}{y_1}} \sum_{k=0}^{n-1} \frac{[\frac{y_2 x}{y_1}]^k}{k!} \left[(1-\frac{1}{y_2})^k - (1-\frac{1}{y_2})^n\right]. \tag{55}$$

Next differentating equation (55) w.r.t. y_1, we get

$$\begin{aligned}\frac{\partial g(y_1,y_2)}{\partial y_1} &= \frac{\partial e^{-\frac{x}{y_1}}}{\partial y_1} - \frac{\partial}{\partial y_1} e^{-\frac{y_2 x}{y_1}} \sum_{k=0}^{n-1} \frac{[\frac{y_2 x}{y_1}]^k}{k!} \left[(1-\frac{1}{y_2})^k - (1-\frac{1}{y_2})^n\right] \\ &= e^{-\frac{x}{y_1}}(\frac{x}{{y_1}^2}) - \left[e^{-\frac{y_2 x}{y_1}} \sum_{k=0}^{n-1} \frac{[y_2 x {y_1}^{-1}]^k}{k!} [(1-\frac{1}{y_2})^k - (1-\frac{1}{y_2})^n] \left(\frac{y_2 x}{{y_1}^2} - \frac{k}{y_1}\right)\right].\end{aligned}$$

Computing the value of $\frac{\partial g(y_1,y_2)}{\partial y_1}$ at μ_z=$((\beta p)^{-1}, p^{-1})$, we get

$$\frac{\partial g(y_1,y_2)}{\partial y_1} = e^{-x\beta p} \cdot x\beta^2 p^2 - \left[e^{-x\beta} \sum_{k=0}^{n-1} \frac{[\beta x]^k}{k!} [(1-p)^k - (1-p)^n] \left(xp\beta^2 - kp\beta\right)\right]$$

After simplification, the above equation becomes

$$\frac{\partial g(y_1,y_2)}{\partial y_1} = e^{-x\beta p} \cdot x\beta^2 p^2 - e^{-x\beta} \sum_{k=0}^{n-1} \frac{p\beta^{k+1} x^k}{k!} \mathbb{W}_k, \tag{56}$$

where

$$\mathbb{W}_k{=}[(1-p)^k-(1-p)^n](x\beta-k),$$

is the same as $\mathbb{W}_k$ defined in the equation (52) by replacing $\beta = y_1$ and $p = y_2$. Differentating equation (55) with respect to y_2, we get

$$\begin{aligned}
\frac{\partial g(y_1,y_2)}{\partial y_2} &= \frac{\partial e^{-\frac{x}{y_1}}}{\partial y_2} - \frac{\partial}{\partial y_2} e^{-\frac{y_2 x}{y_1}} \sum_{k=0}^{n-1} \frac{[\frac{y_2 x}{y_1}]^k}{k!} \left[(1-\frac{1}{y_2})^k - (1-\frac{1}{y_2})^n\right] \\
&= -e^{-\frac{y_2 x}{y_1}}(-\frac{x}{y_1}) \sum_{k=0}^{n-1} \frac{[\frac{y_2 x}{y_1}]^k}{k!} [(1-\frac{1}{y_2})^k - (1-\frac{1}{y_2})^n] \\
&- e^{-\frac{y_2 x}{y_1}} \sum_{k=0}^{n-1} \frac{k {y_2}^{k-1} x^k}{{y_1}^k k!} [(1-\frac{1}{y_2})^k - (1-\frac{1}{y_2})^n] \\
&- e^{-\frac{y_2 x}{y_1}} \sum_{k=0}^{n-1} \frac{[\frac{y_2 x}{y_1}]^k}{k!} \left(\frac{k(1-\frac{1}{y_2})^{k-1}}{y_2^2} - \frac{n(1-\frac{1}{y_2})^{n-1}}{y_2^2}\right) \\
&= -e^{-\frac{y_2 x}{y_1}} \sum_{k=0}^{n-1} \frac{[\frac{y_2 x}{y_1}]^k}{k!} \left[\frac{-x}{y_1}[(1-\frac{1}{y_2})^k - (1-\frac{1}{y_2})^n]\right] \\
&- e^{-\frac{y_2 x}{y_1}} \sum_{k=0}^{n-1} \frac{[\frac{y_2 x}{y_1}]^k}{k!} \frac{k}{y_2} [(1-\frac{1}{y_2})^k - (1-\frac{1}{y_2})^n] \\
&- e^{-\frac{y_2 x}{y_1}} \sum_{k=0}^{n-1} \frac{[\frac{y_2 x}{y_1}]^k}{k!} \frac{1}{{y_2}^2} [k(1-\frac{1}{y_2})^{k-1} - n(1-\frac{1}{y_2})^{n-1}].
\end{aligned}$$

Computing the value of $\frac{\partial g(y_1,y_2)}{\partial y_2}$ at $\mu_z{=}((\beta p)^{-1}, p^{-1})$, we get

(57) $$\frac{\partial g(y_1,y_2)}{\partial y_2} = -e^{-\beta x} \sum_{k=0}^{n-1} \frac{\beta^k x^k}{k!} \mathbb{Y}_k.$$

where is the same as $\mathbb{Y}_k$ defined in equation (54) by replacing $\beta = y_1$ and $p = y_2$. Further, letting

$$A = e^{-x\beta p} \cdot x\beta^2 p^2 - e^{-x\beta} \textstyle\sum_{k=0}^{n-1} \frac{p\beta^{k+1} x^k}{k!} \mathbb{W}_k,$$

and

$$B{=}{-}e^{-\beta x}\textstyle\sum_{k=0}^{n-1}\frac{\beta^k x^k}{k!}\mathbb{Y}_k,$$

we get (see equation (56) and (57))

$$D{=}[A \quad B].$$

Finally, we compute the asymptotic variance of the survival function using equation (47). We do the computation in two steps. In the first step we compute the vector matrix $D \cdot \Sigma$, and in the second step we finish by multiplying the value obtained from the first step by D'.

We begin with the first step,

$$D \cdot \Sigma = [A \quad B] \quad \begin{bmatrix} \frac{1}{\beta^2 p^2} & \frac{1-p}{\beta p^2} \\ \frac{1-p}{\beta p^2} & \frac{1-p}{p^2} \end{bmatrix}.$$

Now let $C = \begin{bmatrix} \frac{1}{\beta^2 p^2} \\ \frac{1-p}{\beta p^2} \end{bmatrix}$ and $E = \begin{bmatrix} \frac{1-p}{\beta p^2} \\ \frac{1-p}{p^2} \end{bmatrix}$.

The computation of DC follows here

$$DC = xe^{-x\beta p} - \frac{1}{p}e^{-x\beta}\sum_{k=0}^{n-1}\frac{\beta^{k-1}x^k}{k!}\mathbb{W}_k - \frac{(1-p)}{p^2}e^{-\beta x}\sum_{k=0}^{n-1}\frac{\beta^{k-1}x^k}{k!}\mathbb{Y}_k. \tag{58}$$

The computation of DE follows below

$$DE = (1-p)x\beta e^{-x\beta p} - \frac{(1-p)}{p}e^{-x\beta}\sum_{k=0}^{n-1}\frac{\beta^k x^k}{k!}\mathbb{W}_k - \frac{(1-p)}{p^2}e^{-x\beta}\sum_{k=0}^{n-1}\frac{\beta^k x^k}{k!}\mathbb{Y}_k. \tag{59}$$

Now for the final step we multiply the vector matrix $[DC \quad DE]$ by D' to get η^2 as defined in (49), that is we do the following multiplication.

$$\begin{aligned}\eta^2 &= [DC \quad DE] \quad [D]' \\ &= [DC \quad DE] \quad [A \quad B].'\end{aligned}$$

The compution of the first part follows below.

$$\begin{aligned} DC \cdot A \;\; = \;\; & x^2\beta^2p^2e^{-2x\beta p} - e^{-x\beta(1+p)}\sum_{k=0}^{n-1}\frac{\beta^{k+1}x^{k+1}p}{k!}\mathbb{W}_k - e^{-x\beta(1+p)}\sum_{k=0}^{n-1}\frac{\beta^{k+1}x^{k+1}p}{k!}\mathbb{W}_k \\ & + e^{-2x\beta}\sum_{k=0}^{n-1}\frac{\beta^{2k}x^{2k}}{(k!)^2}\mathbb{W}_k^2 - (1-p)e^{-\beta x(1+p)}\sum_{k=0}^{n-1}\frac{\beta^{k+1}x^{k+1}}{k!}\mathbb{Y}_k \\ & + \frac{(1-p)}{p}e^{-2\beta x}\sum_{k=0}^{n-1}\frac{\beta^{2k}x^{2k}}{(k!)^2}\mathbb{Y}_k\mathbb{W}_k. \end{aligned}$$

Now, we compute the second part of the vector product of $DC \cdot B$. The computation follows below.

$$\begin{aligned} DE \cdot B \;\; = \;\; & -(1-p)e^{-\beta x(1+p)}\sum_{k=0}^{n-1}\frac{\beta^{k+1}x^{k+1}}{k!}\mathbb{Y}_k + \frac{(1-p)}{p}e^{-2\beta x}\sum_{k=0}^{n-1}\frac{\beta^{2k}x^{2k}}{(k!)^2}\mathbb{Y}_k\mathbb{W}_k \\ & + \frac{(1-p)}{p^2}e^{-2\beta x}\sum_{k=0}^{n-1}\frac{\beta^{2k}x^{2k}}{(k!)^2}\mathbb{Y}_k^2. \end{aligned}$$

Finally, we add $DC \cdot A$ and $DE \cdot B$ to get the asymptotic variance η^2.

$$\begin{aligned}\eta^2 &= \beta^2 x^2 p^2 e^{-2p\beta x} - 2e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1}}{k!} p\mathbb{W}_k - 2(1-p)e^{-\beta x(1+p)} \sum_{k=0}^{n-1} \frac{\beta^{k+1} x^{k+1}}{k!} \mathbb{Y}_k \\ &+(1-p)e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{p^2 (k!)^2} \mathbb{Y}_k^2 + e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{(k!)^2} \mathbb{W}_k^2 + 2(1-p)e^{-2\beta x} \sum_{k=0}^{n-1} \frac{\beta^{2k} x^{2k}}{p(k!)^2} \mathbb{Y}_k \mathbb{W}_k.\end{aligned} \tag{60}$$

Note that we got the same value of a asymptotic variance as we got using Method 1. We summarize the developments above in the following proposition.

Proposition 4.1. *Let* $(X_1, N_1), \cdots, (X_m, N_m)$ *be i.i.d from* $\mathcal{BEG}(\beta, p)$. *Let* $\hat{S}(x,n)$ *given by (44) be estimate of the survival function of the BEG vector. Then* $\hat{S}(x,n)$ *is asymptotically normal, that is for every* x *and positive integer* n,

$$\sqrt{m}\left(\hat{S}(x,n) - S(x,n)\right) \xrightarrow[n\to\infty]{d} N(0, \eta^2),$$

where η^2 *is given in the equation (60).*

5. Monte Carlo Illustration

We will illustrate convergence results as derived and discussed in Section (3). Here, we show the change in the distribution of $\hat{S}_n(x)$ for (1) increasing sample sizes and (2) for fixed sample size and increasing number of sumulations. We do the study for exponential and geometric distributions. The results will be summarized in tables and graphs.

5.1. Simulation results for the survival function of an exponential distribution.

5.1. Simulation results for the survival function of an exponential distribution. As discussed above, to show the change in the distribution of $\hat{S}_n(x)$ for increasing sample size, a program was written which generated a sample of size n from an exponential distribution. We chose fixed values for β=2 and x=1 and generated k=1000 random samples of size n. The goal was to see the convergence of the survival function to normality (in the distribution). The sample sizes used for the illustration were changing from 10 through 30, 50, 100, 300, 500 to 1000.
For each sample size, we computed the following:

(1) Estimated value of the survival function at x=1: $\hat{S}(1)$. Note that the true value is S(1)=$e^{-\beta x} = e^{-2} = 0.135335$.
(2) Sample variance of the observations.
(3) Theoretical value of η^2 for that n.
(4) p-value for Anderson-Darling (A-D) test of normality.

All the values are presented in Table 1 for different sample sizes. The samples are described graphically in Figure 1. The graphs shows histogram of the samples overlayed with the normal model, a probability plot of the data set with the normal model and a box plot.

Both the quantitative and graphical analysis illustrates:

(1) As the sample size increases, the distribution of $\hat{S}(x)$ approches normal law.

(2) As the sample size increases, the empirical value of the survival function is closer to the true value and the sample variance is closer to the asymptotic variance of $\hat{S}(x)$.

n	10	30	50	100	300	500	1000
$\hat{S}$	0.13533	0.13436	0.13292	0.13648	0.13524	0.13511	0.135244
$S.Variance$	0.006208	0.0022877	0.0014102	0.0007082	0.0002301	0.0001360	0.00000765
η^2	0.007336	0.0024420	0.0014652	0.0007326	0.0002442	0.0001465	0.00000732
p value	<0.001	<0.048	0.071	>0.250	>0.250	>0.250	>0.250
5% Sign. level	Not Normal	Not Normal	Normal	Normal	Normal	Normal	Normal

Table 1. Computed values of the estimated survival function at x=1 (row 2), sample variance (row 3), asymptotic variance (row 4), p-value for Anderson-Darling test of normality (row 5) and decision on 5% significance level in Anderson-Darling test (row 6).

Figure 3 clarifies the short illustration of the Table 1.

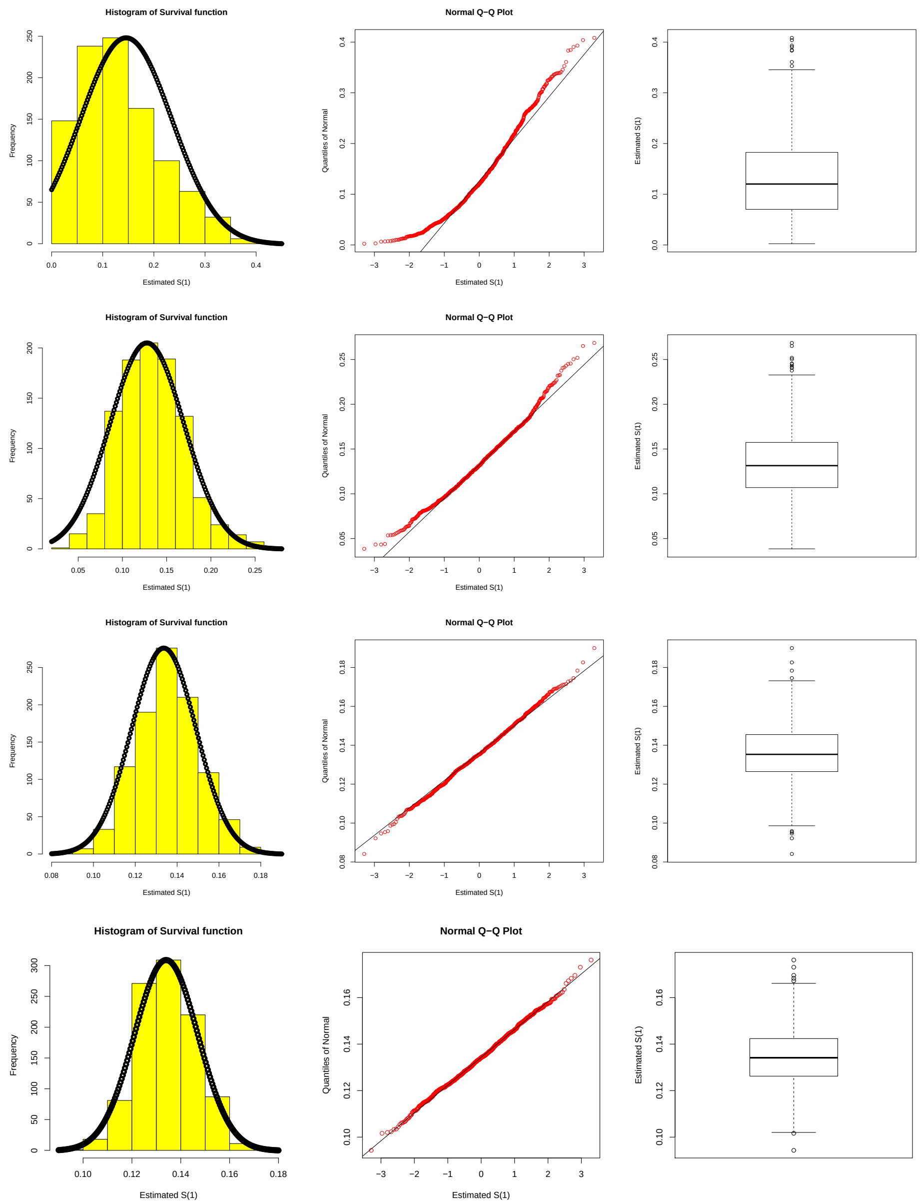

FIGURE 3. Histogram with overlayed normal PDF (left panel), normal probability plot with best fit line (center panel) and box plot (right panel) of the estimated survival function of an exponential distribution for the sample sizes varying from n=10 (top row), n=50 (second row), n=300 and n=500 (last row). The best fit line is the least square fit line added to aid visualization of linearity of the probability plots.

Hence from Table 1 and Figure 3, we see that for large sample size, we have the convergence in the distribution for $\hat{S}(x)$. The next study shows that if we have a large sample size n, then even a relatively small number of samples k will provide normal distribution of the estimated survival function. For this study, we chose n= 500 and k varying from 10 to 500. The quantitative results are summarized in Table 2 and graphical illustration is in Figure 4. We kept β=2 and x=1 for this simulation and tested normality using A-D test. We made decisions on 5% significance level. Note(Table 2), that even small number of simulations, that is small sample size of 10 estimated $\hat{S}(1)$ passes A-D normality test. Further, the sample variances of $\hat{S}(1)$ are very close to the asymptotic variance of $\hat{S}(1)$ even for the small sample from the distribution of $\hat{S}(1)$. Same closeness is true regarding the true and estimated values of S(1).

Figure 2 clarifies the short illustration of the Table 2.

k	10	30	50	100	300	500
$\hat{S}$	0.13578	0.13753	0.13726	0.13636	0.13376	0.13557
$S.Variance$	0.0001479	0.0001731	0.0001340	0.0001723	0.0001561	0.0001408
η^2	0.0001465	0.0001465	0.0001465	0.0001465	0.0001465	0.0001465
p value	>0.25	>0.25	>0.25	>0.25	>0.25	
5% Sign. level	Normal	Normal	Normal	Normal	Normal	Normal

Table 2. Computed values of the estimated survival function at x=1 (row 2), sample variance (row 3), asymptotic variance (row 4), p-value for Anderson-Darling test of normality (row 5) and decision on 5% significance level in Anderson-Darling test (row 6).

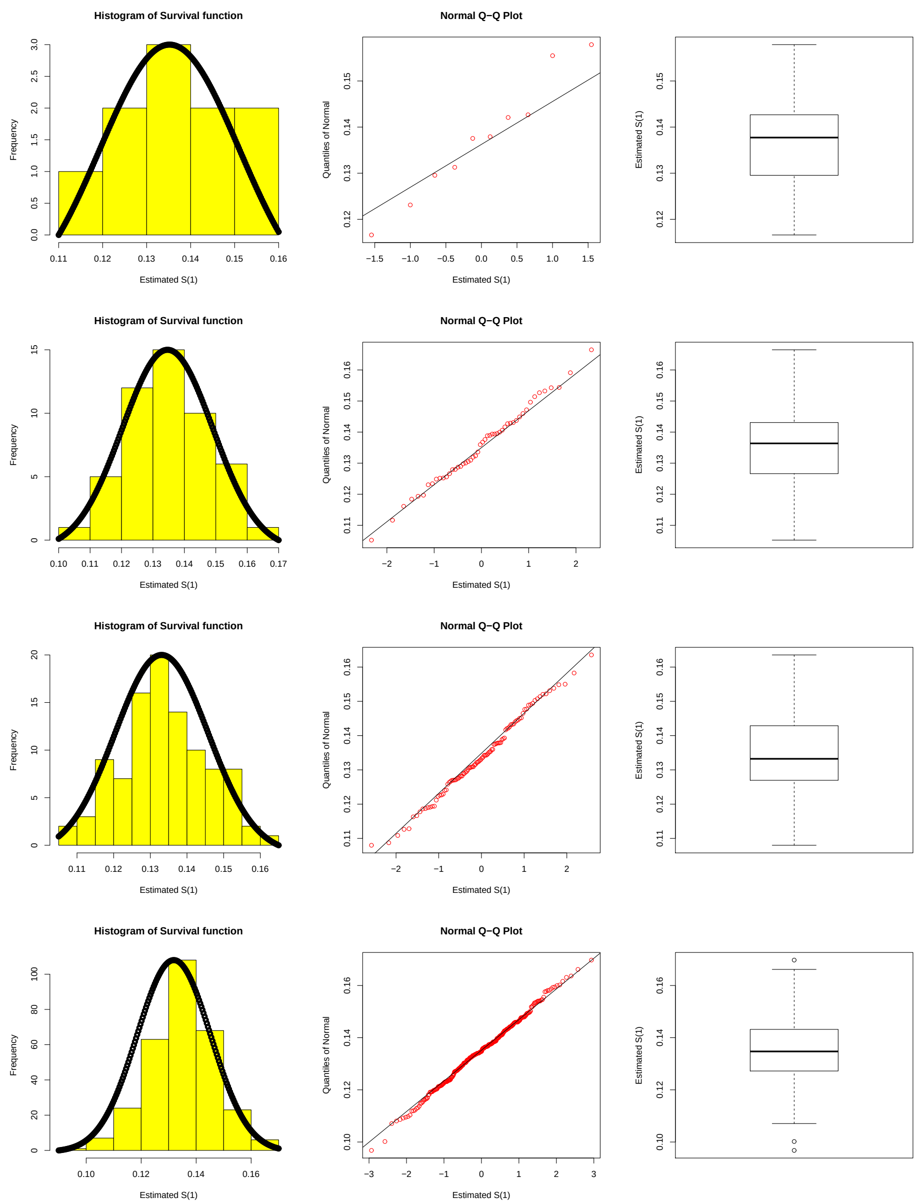

FIGURE 4. Histogram with overlayed normal PDF (left panel), normal probability plot with best fit line (center panel) and box plot (right panel) of the estimated survival function of an exponential distribution for fixed sample size n=500 for the varying samples from k=10 (top row), k=50 (second row), k=100 and k=300 (last row). The best fit line is the least square fit line added to aid visualization of linearity of the probability plots.

5.2. **Sumilation results for the survival function of a geometric distribution.** The illustration for the geometric distribution follows the same procedure as exponential distribution. We show the change in the distribution of $\hat{S}(x)$ for increasing sample size for which a program was written that generated a sample of size n from a geometric distribution. We chose fixed values for $p = 0.5$ and x=2 and generated k=1000 random samples of size n. The goal was to see the convergence of the survival function to normality (in distribution). The sample sizes used for this illustration were for changing from 10 through 30, 50, 100, 300, 500 to 1000.

For each sample size, we computed the following:

(1) Estimated value of the survival function at x=2; $\hat{S}(2)$. Note that the true value is S(2)= $(1-p)^2$= 0.25.

(2) Sample variance of the observation.

(3) Theoretical value of η^2.

(4) p-value for A-D test for normality.

All the values are presented in Table 3 for different sample sizes. The samples are described graphically in Figure 5. The graphs show histogram overlayed with the normal model, a probability plot of the data set with normal model and a box plot. Both the quantitative and graphical analysis illustrates:

(1) As the sample size increases, the distribution of $\hat{S}(x)$ approches normal law.

(2) As the sample size increases, the empirical value of the survival function is closer to the true value, and the sample variance is closer to the asymptotic variance of $\hat{S}(x)$.

n	10	30	50	100	300	500	1000
$\hat{S}$	0.23682	0.24522	0.24656	0.24878	0.24952	0.24939	0.24982
$S.Variance$	0.011369	0.040430	0.002474	0.001294	0.0004591	0.000244	0.00011
η^2	0.012500	0.004166	0.002500	0.001250	0.0004166	0.000250	0.00012
p value	<0.007	<0.0215	>0.250	>0.250	>0.250	>0.250	>0.250
5% Sign. level	Not Normal	Not Normal	Normal	Normal	Normal	Normal	Normal

TABLE 3. Computed values of the estimated survival function at x=2 (row 2), sample variance (row 3), asymptotic variance (row 4), p-value for Anderson-Darling test of normality (row 5) and decision on 5% significance level in Anderson-Darling test (row 6).

Figure 5 clarifies the short illustration of the Table 3.

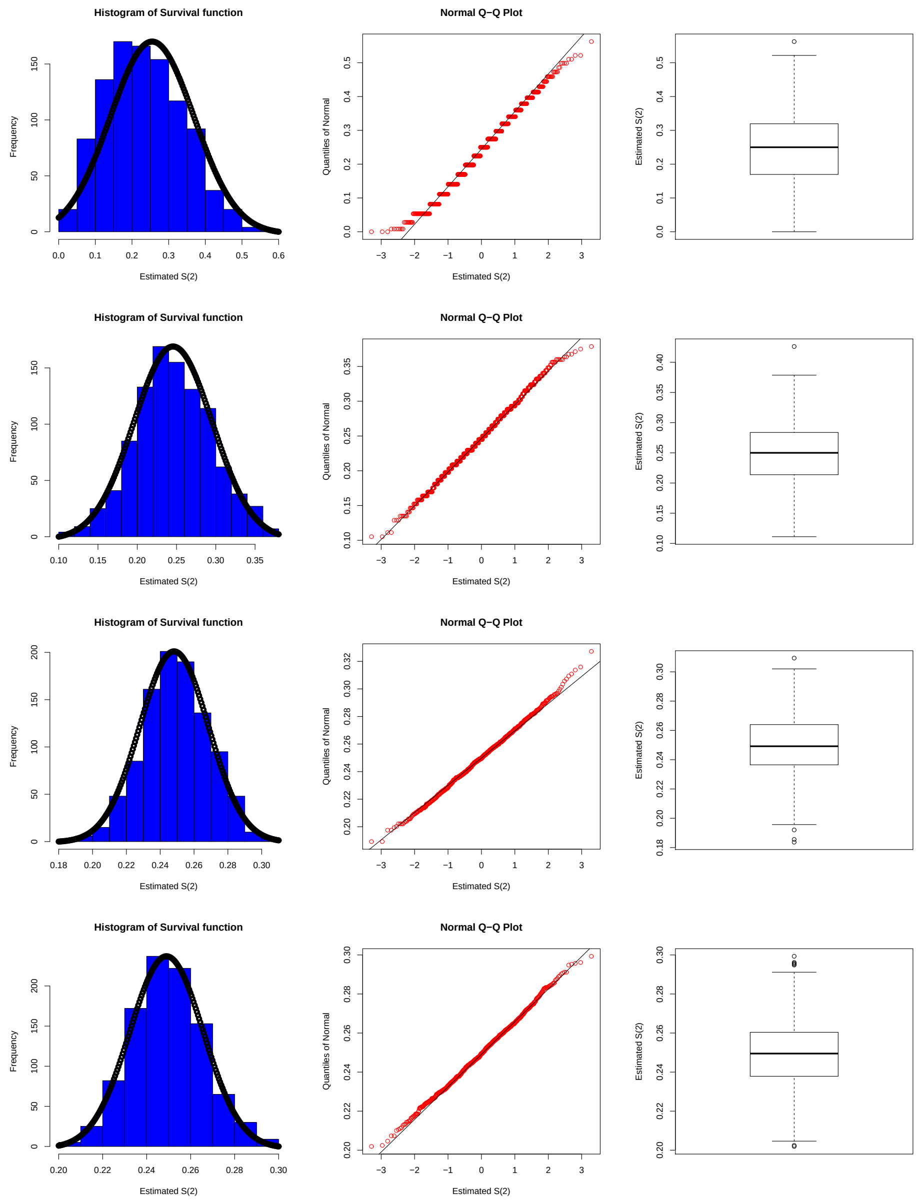

FIGURE 5. Histogram with overlayed normal PDF (left panel), normal probability plot with best fit line (center panel) and box plot (right panel) of the estimated survival function of a geometric distribution for a sample size varying from n=10 (top row), n=50(second row), n=300 and n=500 (last row). The best fit line is the least square fit line added to aid visualization of linearity of the probability plots.

Hence from Table 3 and from Figure 5, we see that as the sample size increases, we see convergence of $\hat{S}(x)$ the distribution to normal law. The next study shows that if we have a large sample size n, then even a realtively small number of samples k will provide normal distribution of the estimated survival function. For this study, we chose n= 500 and k varying from 10 to 500. The quantitative results are summarized in Table 4 and graphical illustration is in Figure 6. We kept $p = 0.5$ and x=2 for this simulations and tested normality using Anderson-Darling (A-D) test. We made decisions on 5% significance level. Note, that even small number of simulations, that is small sample size of 10 estimated $\hat{S}(2)$ passes A-D normality test. Further, the sample variances of $\hat{S}(2)$ are very close to the asymptotic variance of $\hat{S}(2)$ even for the small sample from the distribution of $\hat{S}(2)$. Same closeness is true regarding the true and estimated values of S(2).

Figure 6 clarifies the short illustration of Table 4.

k	10	30	50	100	300	500
$\hat{S}$	0.24086	0.25189	0.25182	0.24729	0.24902	0.25037
$S.Variance$	0.0001587	0.0002736	0.0002970	0.0002653	0.0002594	0.0002551
η^2	0.0002500	0.0002500	0.002500	0.0002500	0.0002500	0.0002500
p value	>0.250	>0.250	>0.250	>0.250	>0.250	>0.250
5% Sign. level	Normal	Normal	Normal	Normal	Normal	Normal

Table 4. Computed values of the estimated survival function at x=2 (row 2), sample variance (row 3), asymptotic variance (row 4), p-value for Anderson-Darling test of normality (row 5) and decision on 5% significance level in Anderson-Darling test (row 6).

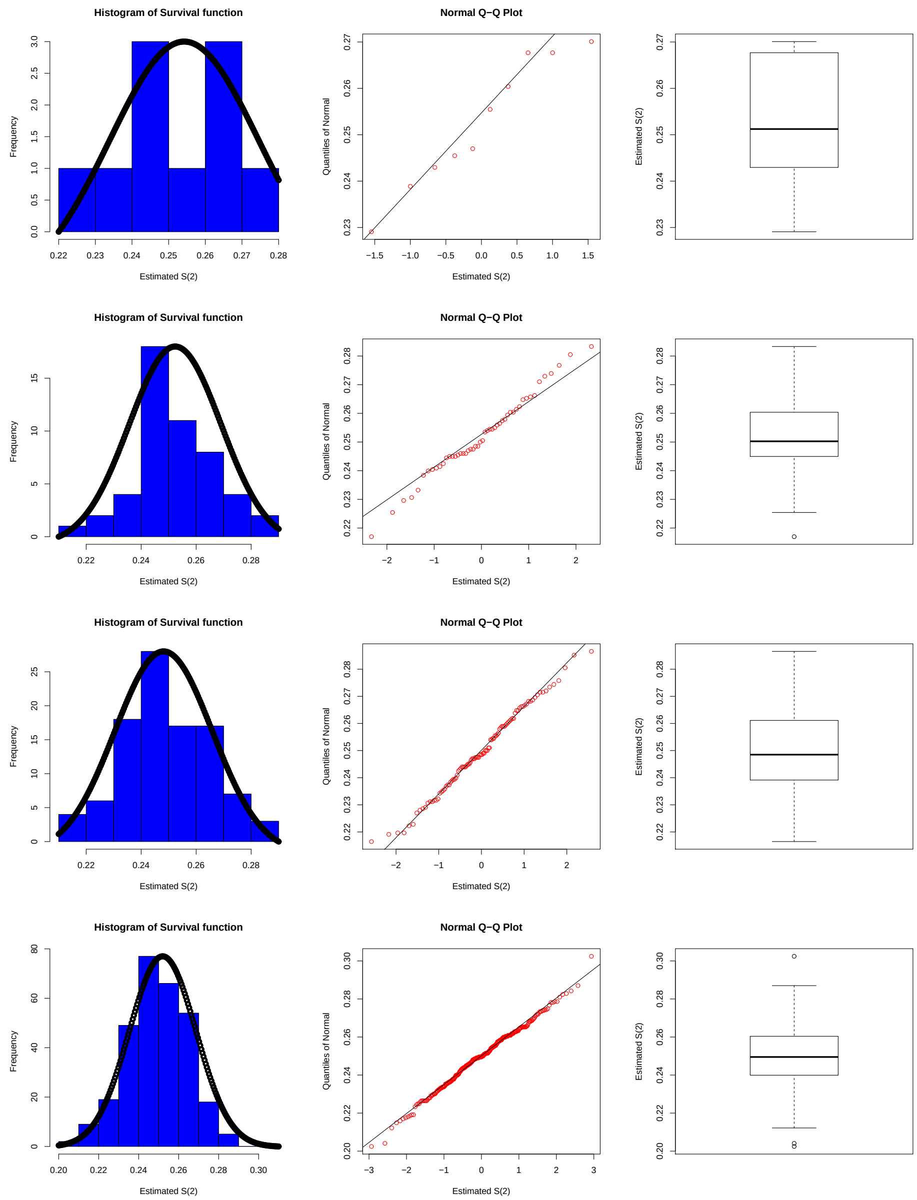

FIGURE 6. Histogram with overlayed normal PDF (left panel), normal probability plot with best fit line (center panel) and box plot (right panel) of the estimated survival function of a geometric distribution for a sample size varying from n=10 (top row), n=50 (second row), n=100 andn=300 (last row). The best fit line is the least square fit line added to aid visualization of linearity of the probability plots.

6. Appendix

The appendix contains $\mathbb{R}$ code used for the simulations described in Section 5. The function exponential takes the following inputs:

(1) b= β, the exponential parameter.
(2) x= the argument for the survival function.
(3) n= sample size.
(4) k= number of simulations.

As output it returns:

(1) Estimated survival function $\hat{S}(x)$ as the mean values obtained from the k samples.
(2) True value of S(x).
(3) Sample variance of $\hat{S}(x)$.
(4) Asymptotic variance of $\hat{S}(x)$.
(5) Histogram, Normal probability plot, Box plot.

The $\mathbb{R}$ code for the exponential distribution is given below.

6.1. Numeric R code simulations for Exponential Distribution.

```
exponential<-function(b,x,n,k){
m<-c(1:k)
  for(i in 1:k){
y<-rexp(n,b)
m[i]<-exp(-x/mean(y))
}
m.bar<-mean(m)
m.var<-var(m)
```

```
sdev<-sqrt(m.var)
asy.mean<-exp(-b*x)
asy.var<-x^2*b^2/(n*exp(2*b*x))
output<-c(m.bar,asy.mean,m.var,asy.var)
output
windows()
boxplot(m,xlab="",ylab="Estimated S(1)")
windows()
qqnorm(m,xlab="Estimated S(1)",ylab="Quantiles of Normal",col="red")
par(new=T)
qqline(m)
windows()
fr<-min(m)
ma<-max(m)
h<-hist(m,xlab="Estimated S(1)",main="Histogram of Survival function"
,col="yellow")
par(new=T)
plot(seq(from=fr,to=ma,length=200),dnorm(seq(from=fr,to=ma,length=200),
mean=m.bar,sd=sdev),
axes=F,xlab="",ylab="",lty=4,lwd=4)
}
```

6.2. **Numeric R code simulations for Geometric Distribution.** Similar code was written for geometric distribution in $\mathbb{R}$. Here also, we give the similar input as in exponential case. The function "geometric" takes $p = 0.5$, (geometric parameter) x=

the argument for the survival function, n= sample size and k=number of simulations. As output it returns:

(1) Estimated survival function $\hat{S}(x)$ as the mean values obtained from the k samples.

(2) True value of S(x).

(3) Sample variance of $\hat{S}(x)$.

(4) Asymptotic variance of $\hat{S}(x)$.

(5) Histogram, Normal probability plot, Box plot.

The $\mathbb{R}$ code for the geometric is given below.

```
geometric<-function(p,x,n,k){
m<-c(1:k)
  for(i in 1:k){
y<-rgeom(n,p)+1
m[i]<-(1-1/mean(y))^x
}
m.bar<-mean(m)
m.var<-var(m)
sdev<-sqrt(m.var)
asy.mean<-(1-p)^x
asy.var<-x^2*p^2*(1-p)^(2*x-1)/n
output<-c(m.bar,asy.mean,m.var,asy.var)
output
windows()
boxplot(m,xlab="",ylab="Estimated S(2)")
windows()
```

```
qqnorm(m,xlab="Estimated S(2)",ylab="Quantiles of Normal",col="red")
par(new=T)
qqline(m)
windows()
fr<-min(m)
ma<-max(m)
h<-hist(m,xlab="Estimated S(2)",main="Histogram of Survival function"
,col="blue")
par(new=T)
plot(seq(from=fr,to=ma,length=200),dnorm(seq(from=fr,to=ma,length=200),
mean=m.bar,sd=sdev),
axes=F,xlab="",ylab="",lty=4,lwd=4)
}
```

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