

University of Nevada, Reno

**An Analysis of a Strategic Market Game Using Gold as
Money and Ornament**

A thesis submitted in partial fulfillment of the
requirements for the degree of Master of Science in
Mathematics

by

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THE GRADUATE SCHOOL

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Abstract

In Quint-Shubik (2012), the authors present and solve several models of a simple economy, the idea being to show how to game theoretically model the role of money and financial institutions. In particular, in a chapter entitled Markets with Gold, a simple two-good-plus-gold economy is presented, in which players may use the same gold both as a money and as a durable good (jewelry) which provides a stream of services. At any time, the gold is allowed only to be used for one of these functions.

In this thesis we consider a slightly different version of the model. There are now two different kinds of gold a monetary gold and a jewelry gold, with a conversion cost between them. We consider two models one without banking and one with banking. Using the theory of non-linear programming, we solve for Nash equilibrium strategies for these models.

Contents

Abstract	i
1 Introduction	1
1.1 Background	1
1.2 Model Breakdown	3
1.3 The Gold Model	8
2 Two Distinct Types of Gold	11
2.1 Analysis	13
2.1.1 Case 1: $\lambda = 0, \lambda^* = 0, v = 0, w = 0$	18

2.1.2	Case 2: $\lambda = 0, \lambda^* = 0, v > 0, w = 0$	20
2.1.3	Case 3: $\lambda = 0, \lambda^* = 0, v = 0, w > 0$	20
2.1.4	Case 4: $\lambda = 0, \lambda^* > 0, v = 0, w = 0$	21
2.1.5	Case 5: $\lambda = 0, \lambda^* > 0, v > 0, w = 0$	21
2.1.6	Case 6: $\lambda = 0, \lambda^* > 0, v = 0, w > 0$	22
2.1.7	Case 7: $\lambda > 0, \lambda^* = 0, v = 0, w = 0$	23
2.1.8	Case 8: $\lambda > 0, \lambda^* = 0, v > 0, w = 0$	24
2.1.9	Case 9: $\lambda > 0, \lambda^* = 0, v = 0, w > 0$	25
2.1.10	Case 10: $\lambda > 0, \lambda^* = 0, v = 0, w = 0$	27
2.1.11	Case 11: $\lambda > 0, \lambda^* = 0, v > 0, w = 0$	27
2.1.12	Case 12: $\lambda > 0, \lambda^* > 0, v = 0, w > 0$	28
3	Two Distinct Types of Gold with Banking	30
3.1	Types of Banks	30

3.2	The Gold Model with Banking	33
3.3	Analysis	36
3.3.1	Case 1: $\mu = 0, \lambda = 0, \lambda^* = 0, v = 0, w = 0$	41
3.3.2	Case 2: $\mu = 0, \lambda = 0, \lambda^* = 0, v > 0, w = 0$	42
3.3.3	Case 3: $\mu = 0, \lambda = 0, \lambda^* = 0, v = 0, w > 0$	43
3.3.4	Case 4: $\mu = 0, \lambda = 0, \lambda^* > 0, v = 0, w = 0$	44
3.3.5	Case 5: $\mu = 0, \lambda = 0, \lambda^* > 0, v > 0, w = 0$	44
3.3.6	Case 6: $\mu = 0, \lambda = 0, \lambda^* > 0, v = 0, w > 0$	45
3.3.7	Case 7: $\mu = 0, \lambda > 0, \lambda^* = 0, v = 0, w = 0$	46
3.3.8	Case 8: $\mu = 0, \lambda > 0, \lambda^* = 0, v > 0, w = 0$	47
3.3.9	Case 9: $\mu = 0, \lambda > 0, \lambda^* = 0, v = 0, w > 0$	48
3.3.10	Case 10: $\mu = 0, \lambda > 0, \lambda^* > 0, v = 0, w = 0$	49
3.3.11	Case 11: $\mu = 0, \lambda > 0, \lambda^* > 0, v > 0, w = 0$	50

3.3.12	Case 12: $\mu = 0, \lambda > 0, \lambda^* > 0, v = 0, w > 0$	50
3.3.13	Case 13: $\mu > 0, \lambda = 0, \lambda^* = 0, v = 0, w = 0$	52
3.3.14	Case 14: $\mu > 0, \lambda = 0, \lambda^* = 0, v > 0, w = 0$	52
3.3.15	Case 15: $\mu > 0, \lambda = 0, \lambda^* = 0, v = 0, w > 0$	53
3.3.16	Case 16: $\mu > 0, \lambda = 0, \lambda^* > 0, v = 0, w = 0$	53
3.3.17	Case 17: $\mu > 0, \lambda = 0, \lambda^* > 0, v > 0, w = 0$	53
3.3.18	Case 18: $\mu > 0, \lambda = 0, \lambda^* > 0, v = 0, w > 0$	54
3.3.19	Case 19: $\mu > 0, \lambda > 0, \lambda^* = 0, v = 0, w = 0$	54
3.3.20	Case 20: $\mu > 0, \lambda > 0, \lambda^* = 0, v > 0, w = 0$	56
3.3.21	Case 21: $\mu > 0, \lambda > 0, \lambda^* = 0, v = 0, w > 0$	58
3.3.22	Case 22: $\mu > 0, \lambda > 0, \lambda^* > 0, v = 0, w = 0$	59
3.3.23	Case 23: $\mu > 0, \lambda > 0, \lambda^* > 0, v > 0, w = 0$	60
3.3.24	Case 24: $\mu > 0, \lambda > 0, \lambda^* > 0, v = 0, w > 0$	60

4 Conclusion

List of Tables

2.1	Potential Bounds on Decision Variable Values	17
3.1	Potential Bounds on Decision Variable Values with $\mu = 0$	39
3.2	Potential Bounds on Decision Variable Values with $\mu > 0$	40

Chapter 1

Introduction

1.1 Background

This thesis is an extension of the work done and presented by Tom Quint, Ph.D., and Martin Shubik, Ph.D., in their strategic market game theory book entitled *Barley, Gold and Fiat: A Pure Theory of Money*. Their book focuses on the use of game theory to model the role of money and financial institutions in our economy. To begin, it is helpful to discuss the types of money that can be used in an economy. Those are (1) perishables, whose value lasts for a single period and are consumed at some point in time; (2) storable consumables, which can be consumed at the end of a period or

stored for later consumption or use; (3) durables, which hold value for several periods and give off a stream of services; and (4) fiat, which is a fictitious durable that has no service or consumption value but is valued through its mutual acceptance and use for transactions in a market (see Quint & Shubik, 2012).

In their book, Quint and Shubik begin by discussing and analyzing their basic model of a strategic market game (which is presented in the next section), and then introduce the different types of money in a progressive manner, beginning with storable consumables and continuing through durables and fiat. Their models also introduce a variety of scenarios, such as asymmetric endowments or models with different types of “rich large agents” (e.g. banks), and a series of equilibrium analyses are made given these varying circumstances.

This thesis stems from the ideas presented specifically in Chapter 6 of *Barley, Gold and Fiat: A Pure Theory of Money*. In that chapter, Quint and Shubik discuss the unique properties inherent in using gold as a form of money, namely that gold may have value in facilitating trade (i.e. it holds “transactional value”), and it has value as a good itself bringing a stream of services to the owner of it (i.e. it holds “ornamental value”). While performing their analysis of this model, Quint and Shubik bundle both aspects of gold as one and thus have only “one type of gold”. This thesis analyzes the model under the idea that there are “two types of gold”, either money or

ornament, and the values of each type are separate. This results in some modifications of the model and the introduction of new variables. This new model, along with the equilibriums under various constraints, are presented in Chapters 2 and 3.

1.2 Model Breakdown

We begin by presenting the "basic model" as presented by Tom Quint and Martin Shubik in Chapter 3 of their work titled *Barley, Gold and Fiat: A Pure Theory of Money*. There are two types of traders, Type *I* and Type *II*, with a continuum of traders of each type. The two types of traders each have their own kind of good, Good 1 for Type *I* and Good 2 for Type *II*. Trading takes place in a single period with the two trader types being symmetrically placed, having similar initial endowments that are expressed as $(a, 0, m)$ for Type *I* traders and $(0, a, m)$ for Type *II* traders, where the first component is the cumulative amount of Good 1 the traders have, the second component is the cumulative amount of Good 2 the traders have, and the third component is the cumulative amount of storable consumable money the traders have.

An individual Type I trader, say trader α , has a strategy of (b^α, q^α) , where b^α is the amount of money bid for Good 2, and q^α is the amount of Good 1 to put up for

trade. Similarly, Type II trader β would have the strategy $(\bar{b}^\beta, \bar{q}^\beta)$. We assume that all traders of one kind are identical and thus this continuum of Type I traders makes up the collective strategy of (b, q) where $b = \int_\alpha b^\alpha$ is the total amount bid for Good 2 summed across all the Type I traders, and similarly, $q = \int_\alpha q^\alpha$ is the total amount of Good I to put up for sale summed across all the Type I traders. A similar idea applies for the strategy for Type II traders, (\bar{b}, \bar{q}) . If we are particularly interested in the strategy for a specific player α , then we would just divide (b, q) through by the measure of the set of Type I traders. Thus from here on, we perform all analysis simply on Type *I* and Type *II* traders, rather than individual players.

Each trader type has a utility function expressed as $\varphi(x, y) + z$, where φ is concave and increasing. The function φ outputs the utility gained from consumption of x -amount of Good 1 and y -amount of Good 2. Thus, the (symmetric) efficient level of consumption is attained when each type of trader consumes $\frac{q}{2}$ of each kind of good. The variable z represents the amount of utility gained by excess commodity money once trade is complete, with the assumption that one unit of commodity money is valued at one unit of utility.

The intention of each type of trader is to maximize their total utility. We analyze the model under various conditions in order to determine what equilibrium strategies are for each trader type under a specific set of parameters. Methods of

Kuhn-Tucker Theory are applied, specifically the use of Lagrangian multipliers and boundary conditions.

This game is a strategic market game played out in the following order:

- (1) the players decide how much to bid on other traders' goods and how much of their own good to put up for sale, then
- (2) the prices are determined endogenously from the values chosen in step (1) (a discussion of the endogenous nature of these prices follows later) and then,
- (3) the amount of consumption good to be received based on prices and the cash income received from sales are determined for the players.

The strategy for Type *I* traders, as mentioned above, is based on what values to assign the decision variables b and q and is expressed as (b, q) , where b is the total amount of money to bid for Good 2 and q is the total amount of Good 1 to offer for trade. Traders of Type *II* have similar decisions and we denote their strategy as (\bar{b}, \bar{q}) where \bar{b} is the total amount to bid on Good 1 and \bar{q} is the total amount of Good 2 to offer for trade.

Our Type 1 optimization problem is:

$$\max_{b,q} \varphi\left(a - q, \frac{b}{p}\right) + m + pq - b \quad (1.1)$$

$$s.t. \quad m - b \geq 0 \tag{1.2}$$

$$0 \leq q \leq a \quad \text{and} \quad b \geq 0 \tag{1.3}$$

Equation (1.2) is the *cash flow constraint* and is associated with the Lagrangian multiplier λ . If there is slack in the constraint (1.2), the traders are not spending all of their money and we refer to this as *hoarding*. The Type *II* traders have a similar problem as follows:

$$\begin{aligned} \max_{\bar{b}, \bar{q}} \quad & \varphi \left(\frac{\bar{b}}{p}, a - \bar{q} \right) + m + \bar{p}\bar{q} - \bar{b} \\ s.t. \quad & m - \bar{b} \geq 0 \\ & 0 \leq \bar{q} \leq a \quad \text{and} \quad \bar{b} \geq 0 \end{aligned}$$

Due to the symmetry of the two trader-types, all work shown will be in terms of trader Type *I*, the problem for traders of Type *II* will be similar. Recall that in step 2) of the game mentioned above, the prices for goods are determined endogenously once the traders have made their decisions for b, q, \bar{b} , and \bar{q} in step 1). The prices are determined by the equations:

$$p = \frac{\bar{b}}{q} \tag{1.4}$$

$$\bar{p} = \frac{b}{\bar{q}} \tag{1.5}$$

We will refer to these as our *market balance equations*. While it seems that both p and \bar{p} are based on the decision variables b, q, \bar{b} , and \bar{q} , the decisions are made

by a continuum of players and thus no specific trader has direct control over the market prices. Therefore, the prices are determined endogenously and are treated as constants in our problems.

To solve the model, we need to find the values of $b, q, p, \bar{b}, \bar{q},$ and \bar{p} which will simultaneously optimize the player types problem while satisfying the market balance equations. Game theoretically, we will be finding a Nash Equilibrium of the game. The driving force behind the analysis of this model is the value of the parameter m . The amount of money available to the traders in conjunction with other parameters (to be introduced as the paper progresses) determines if the consumption levels of the two types of goods are at an efficient level for both trader types and whether or not the cash flow constraint is non-binding.

To solve the model, Quint and Shubik had the function $\varphi(x, y) = 2\sqrt{xy}$, which, given the initial endowments, meant that efficient trade was reached when the traders consumed $(\frac{a}{2}, \frac{a}{2})$ of Good 1 and Good 2 respectively. They solved this "basic model" and found two results. First, that when $m \geq \frac{a}{2}$ the traders were able to reach efficient trade, and second, when $m < \frac{a}{2}$ they were unable to reach efficient trade.

The collaborative work of Quint and Shubik have covered a number of models that progress from the basic model discussed above, to the models that introduce a

lender (i.e. a bank or “rich monied agent”), as well as varying the type of money used to facilitate trade. The work conducted for this thesis focuses on the durable money “gold” as opposed to the other presented money forms which are storable consumables and fiat money.

1.3 The Gold Model

By using gold as a trading commodity, new variables and parameters are introduced for analysis. This is due to the fact that gold can provide its owner with two types of services: (I) it can be used to facilitate trade of goods and (II) it can be used to bring utility as an ornamental good to the trader that possesses it. In Chapter Six, *Markets with Gold*, Quint and Shubik mention that the ability to be treated as an asset as well as material used for services is a unique quality when using physical durables (i.e. gold, silver, metals) as money. Because of the stream of services that can be rendered from these durables, it is important to make a distinction between whether one is using gold as an asset, or for its services. Due to that fact, when discussing the quantity of gold money, there are two different variables to be used, m to measure the asset in physical units and \dot{m} to measure the services in units per unit time. Although the variables differ, in this model, both service and asset are bundled

together, (i.e. the bundled variable is $[m, \dot{m}]$) since there is no means (presented hitherto) of separating the gold as an asset from its services¹. Thus when analyzing the game, $m = \dot{m}$. Further, \dot{b} is the amount of gold services bid by Type I traders for Good 2 and \dot{I} is the amount of gold services received for q units of Good 1 sold at a price p . Due to the bundled service and asset of gold, $\dot{b} = b$, and $\dot{I} = pq$.

In Quint and Shubik's gold model, one must either use gold strictly as an asset, or for its services, but not both simultaneously. Since gold may be used in different ways at different times, the trading period must be divided based on what function the gold is playing. So, k_2 is the proportion of the period that passes before goods are exchanged, $k_3 - k_2$ is the proportion of the period the gold is being transferred in the market from buyer to seller (thus neither benefits from it), and $1 - k_3$ is the remaining proportion of time after goods have been exchanged and the gold is used by the final holder. Once the period has ended, the gold each trader possesses has a per-unit salvage value we denote as Π . What follows is the new Trader Type I optimization problem:

$$\max_{b,q} \varphi(a - q, \frac{b}{p}) + k_2 \dot{m} + (k_3 - k_2)(\dot{m} - \dot{b}) + (1 - k_3)(\dot{m} - \dot{b} + \dot{I}) + \Pi(m - b + pq) \quad (1.6)$$

$$s.t. \quad m - b \geq 0 \quad (1.7)$$

¹In reality, it is possible for traders to rent out any gold they own for its services, but we exclude that option in our analysis

$$0 \leq q \leq a \quad \text{and} \quad b \geq 0 \tag{1.8}$$

Once again, we have a cash-flow constraint, equation (1.7), and since the consumption of goods remains unchanged, the arguments for φ are the same as our basic model. What Quint and Shubik found in their analysis (which was performed with $\Pi = 0$) was that varying the parameters k_2 and k_3 very often led to inefficient trade. Specifically, if $k_2 = k_3 = 0$, the services of gold are completely lost for traders and the amount of money needed is $m = a/2$; if $k_2 = k_3 = 1$, the period of time traders will get to enjoy the services of gold is before any trade takes place, and once trade is complete, there will be no time to enjoy gold services, thus price of goods are infinite and an infinite amount of gold is needed. When $k_2 \neq k_3$, there is a loss of utility in trade, or a “tax” for trading, and thus efficient trade is not achieved. It is also noted that when $k_2 = 0$, $k_3 = 1$, and $\Pi = 0$ then clearly no trade would happen at all because gold would be completely useless to the seller as a means of payment.

Chapter 2

Two Distinct Types of Gold

In the previous chapter, we discussed the work of Quint and Shubik analyzing one type of gold with two roles. However, they also propose the idea to look at gold in another light: as two different types of good each with separate functions. Realistically, we can look at “monetary gold” (gold as coin) which yields no services but acts strictly to facilitate trade, and “consumption gold” (gold as jewelry) which cannot facilitate trade but provides a service utility to the trader that owns it. Then there is a ”conversion cost” to turn the type of gold from one in to the other. Quint and Shubik present the model (which follows in the remaining paragraphs of this section) and solve a single simple case with a large amount of coin leading to efficient trade.

Further work is presented for the first time here in section 2.1.

Initial endowments must now take in to account how much of each form of gold traders have. Thus m is the amount the trader types have in coin and m^* is the amount they have in gold jewelry and so the Type *I* traders' initial endowments are $(a, 0, m, m^*)$ and Type *II* traders' are $(0, a, m, m^*)$. Conversion from one form of gold to another is allowed at a percentage penalty or cost which is modeled using a parameter $\alpha \in [0, 1]$. Thus, for example, x units of gold coin is converted to αx units of gold jewelry, and similarly, y units of gold jewelry converts to αy units of gold coin. The conversion of gold happens just once at the start of a game, before trade takes place.

Once trade is complete, any excess gold coin or jewelry has a salvage value, which is the worth per-unit at the end of the game for gold coin and jewelry, which are denoted as Π and Π^* respectively. Since services of gold jewelry can be utilized throughout the trading period as well as salvaged once trade is done, our per unit value of jewelry becomes $(1 + \Pi^*)$. With separate values for coin and jewelry, traders are faced with the decision of how much coin to convert to jewelry, or how much jewelry to convert to coin. The strategies for Type *I* and Type *II* traders becomes (b, q, v, w) and $(\bar{b}, \bar{q}, \bar{v}, \bar{w})$ respectively, where v and \bar{v} are the variables representing the amount of coin to convert to jewelry for traders of Type *I* and Type *II* respectively,

and w and \bar{w} are the amounts of jewelry to convert to coin.

The optimization problem for the Type *I* traders now becomes:

$$\max_{b,q,v,w} \varphi\left(a - q, \frac{b}{p}\right) + \Pi(m - b - v + \alpha w + pq) + (1 + \Pi^*)(m^* - w + \alpha v) \quad (2.1)$$

$$s.t. \quad m - b - v + \alpha w \geq 0 \quad (\lambda) \quad (2.2)$$

$$m^* - w + \alpha v \geq 0 \quad (\lambda^*) \quad (2.3)$$

$$0 \leq q \leq a \quad \text{and} \quad b, v, w \geq 0 \quad (2.4)$$

The problem for Type *II* traders is similar. Notice here the added constraint of (2.3), which we will call the jewelry constraint, and behaves similarly to the cash-flow constraint in that we can't end up with negative amounts of jewelry. This will have the Lagrangian multiplier λ^* .

2.1 Analysis

We solve the model in the case where

$$\varphi(x, y) = 2\sqrt{x \cdot y}$$

and focus on the problem for trader Type 1 (Type 2 is similar). This definition of φ is a commonly used function in economic analysis since it nicely models a decreasing

marginal utility.

Using Langrange Multipliers and the Kuhn-Tucker conditions, we begin by identifying the Langrange function to be

$$L = 2\sqrt{(a-q) \cdot \frac{b}{\bar{p}}} + \Pi(m-b-v+\alpha w+pq) + (1+\Pi^*)(m^*-w+\alpha v) - \lambda(-m+b+v-\alpha w) - \lambda^*(-m^*+w-\alpha v) \quad (2.5)$$

The first order Kuhn-Tucker conditions with respect to the variables b, q, v, w respectively are:

$$\sqrt{\frac{a-q}{b}} = \sqrt{\bar{p}} \cdot (\lambda + \Pi) \quad (2.6)$$

$$\sqrt{\frac{b}{a-q}} = \sqrt{\bar{p}} \cdot p \cdot \Pi \quad (2.7)$$

$$\lambda = \alpha\lambda^* + \alpha(1 + \Pi^*) - \Pi \quad (2.8)$$

$$\lambda = \frac{1}{\alpha}(1 + \Pi^* + \lambda^*) - \Pi \quad (2.9)$$

The Kuhn-Tucker conditions based on the cash-flow and jewelry constraints are:

$$\lambda = 0 \text{ and } m - b - v + \alpha w \geq 0$$

$$\text{or} \quad (2.10)$$

$$\lambda \geq 0 \text{ and } m - b - v + \alpha w = 0$$

as well as

$$\begin{aligned} \lambda^* = 0 \text{ and } m^* - w + \alpha v &\geq 0 \\ \text{or} & \\ \lambda^* \geq 0 \text{ and } m^* - w + \alpha v &= 0 \end{aligned} \tag{2.11}$$

Further, we have:

$$b, v, w \geq 0 \text{ and } 0 \leq q \leq a$$

along with our balance equations (1.4) and (1.5).

It is important to note that the values of v and w have the relationship that either $(v = 0 \text{ and } w = 0)$, $(v = 0 \text{ and } w > 0)$, or $(v > 0 \text{ and } w = 0)$. The case where $v > 0 \text{ and } w > 0$ cannot occur since it is obvious that it would not be optimal to convert both coin to jewelry and jewelry to coin in the same period. Also, if $v = 0$, we disregard equation (2.8) which is the derivative of the Lagrangian function with respect to v , and similarly, when $w = 0$, we disregard equation (2.9) which is the derivative with respect to w .

There are two areas for α to lie, that is $0 \leq \alpha \leq \min \left\{ \frac{\Pi}{(1+\Pi^*)}, \frac{(1+\Pi^*)}{\Pi} \right\}$, where we would say that the cost of gold conversion is relatively high, or $\min \left\{ \frac{\Pi}{(1+\Pi^*)}, \frac{(1+\Pi^*)}{\Pi} \right\} \leq \alpha \leq 1$ when we can say that the cost of gold conversion is relatively low.

The initial values of m and m^* , the market salvage values for gold coin, Π ,

and gold jewelry, Π^* , as well as the cost of conversion, α , determine whether we will change gold coin to jewelry, gold jewelry to coin, or neither, which will in turn tell us if we have a tight cash-flow constraint ($\lambda > 0, \lambda^* = 0$), a tight jewelry constraint ($\lambda = 0, \lambda^* > 0$), both tight cash-flow and jewelry constraints ($\lambda > 0, \lambda^* > 0$) or both loose cash flow and jewelry constraints ($\lambda = 0, \lambda^* = 0$). Since there are three options for the combination of values of w and v (the amount of gold to convert from one form of gold to the other) and there are four combinations of having tight or loose constraints, it appears there are 12 potential cases of equilibrium strategies. However, some combinations of values for λ , λ^* , v , and w just cannot happen. Table (2.1) gives an outline to the combinations of these values and which cases are and are not possible

The computations and equilibrium values are found in the remaining subsections. The results for 2.3.1 Case 1 are very similar to the results in 2.3.6 Case 6. This is because in Case 1, we do not convert any gold jewelry to coin, or coin to jewelry, so the problem becomes decoupled into one where we buy, sell, and consume goods using just gold coin, and the value of gold jewelry becomes added bonus that scales up the utility and takes no part in the optimization of everything else. In this light, the gold is now a type of fiat (i.e. it is valued only as currency that facilitates trade and can do nothing more) and we may apply the analysis of fiat models in Quint-Shubik

Table 2.1: Potential Bounds on Decision Variable Values

	$v = 0, w = 0$	$v > 0, w = 0$	$v = 0, w > 0$
$\lambda = 0, \lambda^* = 0$	Possible (Case 1)	Not Possible (Case 2)	Not Possible (Case 3)
$\lambda = 0, \lambda^* > 0$	Not Possible (Case 4)	Not Possible (Case 5)	Possible (Case 6)
$\lambda > 0, \lambda^* = 0$	Possible (Case 7)	Possible (Case 8)	Possible (Case 9)
$\lambda > 0, \lambda^* > 0$	Not Possible (Case 10)	Not Possible (Case 11)	Possible (Case 12)

(Chapter 7) to analyze the "coin currency" part of the model. The model in Case 6 has a similar effect in that we are essentially analyzing a case that uses fiat. This is because we change all our jewelry to gold coin to begin with, thus it is as if we started with $m + \alpha m^*$ units of coin and jewelry once again plays no part in optimizing (except to fill our pocket with more money at the start of the period), so once again, we may apply the analysis from Quint-Shubik's fiat model. Both these cases reach efficient trade.

All other cases will be inefficient except for Case 8, which will only be efficient when $\alpha = \frac{\Pi}{1+\Pi^*}$. In this case, we are exchanging gold coin to jewelry, which implies

the value of jewelry is greater than coin. So, as $\alpha \rightarrow 1$, i.e. the cost to convert coin to jewelry becomes free, our consumption levels become less and less efficient since the traders find more value in having gold jewelry than goods. The value of gold jewelry also causes inefficient trade in Case 9. Cases 7 and 12 are inefficient due to a lack of coin.

2.1.1 Case 1: $\lambda = 0, \lambda^* = 0, v = 0, w = 0$

This equilibrium strategy case results from having a “large” amount of gold coin, m , and a “large” amount of gold jewelry, m^* , as well as a relatively costly gold conversion rate, which means $0 \leq \alpha \leq \min \left\{ \frac{\Pi}{(1+\Pi^*)}, \frac{(1+\Pi^*)}{\Pi} \right\}$. Since it is relatively expensive to convert gold and we have a large amount of coin, there is no need to transfer any jewelry to coin thus $w = 0$. Similarly, we have a large amount of jewelry so we would not spend a lot of coin to convert to a small amount of jewelry and $v = 0$.

With the initial endowment of coin being large and the fact that we are not converting any gold, we can assume that our cash flow and budget constraints (2.2) and (2.3) are loose thus $\lambda = 0$ and $\lambda^* = 0$. The following simplification of (2.6) follows:

$$\sqrt{\frac{a-q}{b}} = \sqrt{\bar{p}} \cdot \Pi \quad (2.12)$$

(2.7) and (2.12) give us the following simplification:

$$\sqrt{\bar{p}} \cdot \Pi = \frac{1}{\sqrt{\bar{p}} \cdot p \cdot \Pi} \rightarrow \bar{p} \cdot p = \frac{1}{\Pi^2}$$

Since the problems for both trader types are similar, and the markets for both Good 1 and Good 2 are isomorphic, all our solutions will be symmetric ¹, thus:

$$\bar{p} = p = \frac{1}{\Pi} \tag{2.13}$$

Equations (2.12) and (2.13) together with our balance equation (1.5) and symmetry imply

$$\frac{a - q}{pq} = \bar{p} \cdot \Pi^2 \rightarrow a - q = p^2 \Pi^2 q \rightarrow a - q = q \rightarrow a = 2q \rightarrow q = \frac{a}{2}$$

So $q = \bar{q} = \frac{a}{2}$. Thus efficient trade will be reached. Balance equations (1.4) and (1.5) tell us $b = \bar{b} = pq = \frac{a}{2\Pi}$. Using equations (2.10) and (2.11) and substituting in our values for b, v , and w , we say that $m \geq \frac{a}{2\Pi}$ and $m^* \geq 0$ is when the initial endowments of gold coin and jewelry are “large”, and must hold for this case to be valid.

Note that $m^* \geq 0$ is “large” says that any amount of gold jewelry is a large amount. This makes sense since there is sufficient coin to facilitate trade, thus any positive amount of jewelry is an added “bonus” and will not need to be converted to coin to purchase goods.

¹This symmetry argument carries through for all cases in this thesis, and I might have, from time to time, freely interchanged from b to \bar{b} , p to \bar{p} , etc.

2.1.2 Case 2: $\lambda = 0, \lambda^* = 0, v > 0, w = 0$

$v > 0$ means that we are converting coin to jewelry, and this will only happen if the value of gold jewelry is worth relatively more than gold coin, $\Pi < 1 + \Pi^*$. However, this also implies that there is no benefit to having any excess coin left after the period ends, so we would want to drive our cash constraint to zero. This is done by converting almost all the gold coin to jewelry saving just enough for trade, which gets spent, and thus would leave nothing left over. Therefore, $\lambda = 0$ would not happen and we have no case here.

2.1.3 Case 3: $\lambda = 0, \lambda^* = 0, v = 0, w > 0$

$w > 0$ means we are converting jewelry to coin, and this would happen for one of two reasons. One, if the value of coin is relatively higher than jewelry, $\Pi > 1 + \Pi^*$ or two, if we wanted to change a minimal amount of coin to facilitate trade because the value of jewelry is more than coin. If gold jewelry is converted to coin for the first reason, then since jewelry cannot facilitate trade, and it is worth less than coin, there is no reason to keep any around thus we would convert all of it to coin and drive the constraint to zero. This means $\lambda^* = 0$ would not make sense and we have no case here. If it is the second reason, then coin is worth less than jewelry and we would

only want to have enough to facilitate trade and have nothing left over, thus $\lambda = 0$ won't happen. In either scenario, this case does not work.

2.1.4 Case 4: $\lambda = 0, \lambda^* > 0, v = 0, w = 0$

Here there is no conversion from coin to jewelry or jewelry to coin. Since we can't spend jewelry, and we aren't transferring jewelry to coin, there is no reason for our jewelry constraint to be driven to zero. Thus, it will never happen that $\lambda^* > 0$ when $w = 0$.

2.1.5 Case 5: $\lambda = 0, \lambda^* > 0, v > 0, w = 0$

$w = 0$ implies that we are converting coin to jewelry, and this will only happen if the value of gold jewelry is worth more than gold coin, $\Pi < 1 + \Pi^*$. However, if we are converting some gold coin to jewelry, there is no reason that our jewelry constraint will be tight thus $\lambda^* > 0$ is not possible and there is no case here. It cannot happen that $\lambda^* > 0$ when $w = 0$.

2.1.6 Case 6: $\lambda = 0, \lambda^* > 0, v = 0, w > 0$

Here $w > 0$ and $\lambda^* > 0$ implies that the salvage value of gold coin is (relatively) greater than gold jewelry, $1 + \Pi^* < \Pi$. This is verified by equation (2.11) which simplifies to $w = m^*$, thus it is optimal to convert all jewelry to coin. $\lambda = 0$ implies that we have enough gold coin to facilitate trade meaning m is large. The bounds on m being large will be found later.

Replacing λ with 0, equation (2.9) reduces to:

$$(1 + \Pi^*) + \lambda^* = \alpha\Pi \rightarrow \lambda^* = \alpha\Pi - (1 + \Pi^*)$$

Since $\lambda^* > 0$, then this case is only valid so long as $\frac{1+\Pi^*}{\Pi} < \alpha < 1$ holds. Equation (2.6) and (2.7) yield:

$$\frac{1}{\sqrt{\bar{p}\Pi}} = \sqrt{\bar{p}p\Pi} \rightarrow \bar{p}p = \frac{1}{\Pi}$$

By symmetry, we have $p = \bar{p} = \frac{1}{\Pi}$ which we substitute in to equation (2.7) and use the balance condition to solve for q :

$$\sqrt{\frac{\bar{q}\bar{p}}{a-q}} = \sqrt{\bar{p}p\Pi} \rightarrow \frac{\bar{q}}{a-q} = p^2\Pi^2 \rightarrow \frac{q}{a-q} = 1 \rightarrow q = \frac{a}{2}$$

By symmetry we have $\bar{q} = q = \frac{a}{2}$ which tells use that efficient trade will be reached.

It follows from the values of p and q and the boundary conditions that $b = \bar{b} = \frac{a}{2\Pi}$

The boundary of this case is that $m + \alpha m^* \geq \frac{a}{2\Pi}$ must hold in order to guarantee

the cash-flow constraint is satisfied.

Note that this case reduces to 2.3.1 Case 1.

2.1.7 Case 7: $\lambda > 0, \lambda^* = 0, v = o, w = 0$

The cost to convert from one form of gold to another must be high (i.e. α is small) since $v = w = 0$. Also, our cash-flow constraint for coin is tight, $\lambda > 0$ and the cash-flow constraint for jewelry is loose, $\lambda^* = 0$ which implies we started with little coin and some jewelry (since no conversions happened) . Using equations (2.10) and (2.11) and substituting values of v and w we find that $b = m$ and $m^* \geq 0$. By symmetry $\bar{b} = m$. To solve for p and \bar{p} we use equations (2.6) and (2.7):

$$\bar{p}(\lambda + \Pi)^2 = \frac{1}{\bar{p}p^2\Pi^2} \rightarrow \bar{p}^2p^2 = \frac{1}{(\lambda + \Pi)^2\Pi^2} \rightarrow \bar{p}p = \frac{1}{(\lambda + \Pi)\Pi} \rightarrow \bar{p} = p = \frac{1}{\sqrt{(\lambda + \Pi)\Pi}}$$

Balance equations tell us that $q = \bar{q} = \frac{b}{p} = m\sqrt{(\lambda + \Pi)\Pi}$. Substituting these values for $b, q,$ and p into equation (2.7) squared, we get :

$$\begin{aligned} \frac{b}{a-q} &= \frac{m}{a-m\sqrt{(\lambda+\Pi)\Pi}} = \frac{\Pi^2}{[(\lambda+\Pi)\Pi]^{3/2}} \rightarrow \\ &\rightarrow m[(\lambda + \Pi)\Pi]^{3/2} = \Pi^2 \left[a - m\sqrt{(\lambda + \Pi)\Pi} \right] \quad (2.14) \end{aligned}$$

In order to simplify analysis, we let $x = \sqrt{(\lambda + \Pi)\Pi}$ thus:

$$\begin{aligned} mx^3 &= \Pi^2 a - m\Pi^2 x \rightarrow mx^3 = \Pi^2 a - m\Pi^2 x \\ &\rightarrow mx^3 + m\Pi^2 x = \Pi^2 a \rightarrow m(x^3 + \Pi^2 x) = \Pi^2 a \quad (2.15) \end{aligned}$$

Since we know that $\Pi^2 a > 0$, we can say that $\lambda \geq 0 \leftrightarrow x \geq \Pi$. If we let $x = \Pi$, then (2.15) reduces to $m = \frac{a}{2\Pi}$ which is when this case meets the boundaries of the efficient trade scenario that occurred in 2.3.1 Case 1 and 2.3.6 Case 6.

2.1.8 Case 8: $\lambda > 0, \lambda^* = 0, v > 0, w = 0$

Since $v > 0$, this means the salvage value of coin is less than jewelry. Having excess coin is not beneficial since it is worth less than jewelry, so we convert some of it to jewelry leaving just enough to facilitate trade, thus $\lambda > 0$ and $\lambda^* = 0$. Equation (2.10) implies that $v = m - b$, where we solve for b shortly. Equation (2.8) becomes:

$$\lambda = \alpha(1 + \Pi^*) - \Pi$$

and so to keep $\lambda > 0$, $\frac{\Pi}{1 + \Pi^*} < \alpha < 1$ must hold for this case to be valid. Replacing λ in equation (2.6) and simplifying we obtain:

$$\sqrt{\frac{a - q}{b}} = \sqrt{\bar{p}} \cdot \alpha \cdot (1 + \Pi^*) \quad (2.16)$$

Equations (2.7) and (2.16) together with symmetry tell us

$$\sqrt{\bar{p}}\alpha(1 + \Pi^*) = \frac{1}{\sqrt{\bar{p}p\Pi}} \rightarrow p = \bar{p} = \sqrt{\frac{1}{\alpha\Pi(1 + \Pi^*)}}.$$

Substituting this value for p and \bar{p} in the square of (2.16) and using our balance equation $b = \bar{p}\bar{q}$, we find $q = \bar{q} = \frac{a\Pi}{\alpha(1+\Pi^*)+\Pi}$ and $b = \bar{b} = pq = \frac{a\Pi}{[\alpha(1+\Pi^*)+\Pi]\sqrt{\alpha\Pi(1+\Pi^*)}}$.

Equation (2.11) implies that the boundary of this case is $m^* + \alpha m \geq \frac{a\Pi\alpha}{[\alpha(1+\Pi^*)+\Pi]\sqrt{\alpha\Pi(1+\Pi^*)}}$ and it must hold in order to guarantee the cash-flow constraint is satisfied.

Note that if $\alpha = \frac{\Pi}{1+\Pi^*}$ then $b = \bar{b} = \frac{a}{2\Pi}$ and $q = \bar{q} = \frac{a}{2}$ and efficient trade will be achieved. If α is closer to 1 however, efficient trade will not be achieved. Economically, this makes sense since there will be more value to turning coin into jewelry rather than spending it on goods.

2.1.9 Case 9: $\lambda > 0, \lambda^* = 0, v = 0, w > 0$

We are exchanging jewelry to coin ($v = 0, w > 0$), but do not wish to drive the jewelry constraint to zero, ($\lambda^* = 0$) nor do we wish to have any excess coin ($\lambda > 0$). This can only happen when the value of jewelry is greater than coin, i.e. $\Pi < 1 + \Pi^*$, but there is still a desire to convert some jewelry to facilitate trade.

Simplifying equation (2.9) we get an expression for λ which is

$$\lambda = \frac{1}{\alpha}(1 + \Pi^*) - \Pi$$

and since $\lambda > 0$, $\alpha < \frac{1+\Pi^*}{\Pi}$ must hold. But $\Pi < 1 + \Pi^*$, so this will always be true.

Substituting this value for λ in to the square of equation (2.6) we get

$$\frac{a - q}{b} = \bar{p} \left(\frac{(1 + \Pi^*)}{\alpha} - \Pi + \Pi \right)^2 = \bar{p} \frac{(1 + \Pi^*)^2}{\alpha^2} \quad (2.17)$$

Combining this with the inverse square of equation (2.10) and symmetry, we get

$$\begin{aligned} \bar{p} \frac{(1 + \Pi^*)^2}{\alpha^2} &= \frac{1}{\bar{p} p^2 \Pi^2} \rightarrow \bar{p}^2 p^2 = \frac{\alpha^2}{(1 + \Pi^*)^2 \Pi^2} \rightarrow \bar{p} p = \frac{\alpha}{\Pi(1 + \Pi^*)} \\ &\rightarrow p = \bar{p} = \sqrt{\frac{\alpha}{\Pi(1 + \Pi^*)}} \end{aligned}$$

Substituting this back in to equation (2.17), we use our balance conditions and symmetry to solve for q (and thus \bar{q} .)

$$\begin{aligned} \frac{a - q}{b} &= \frac{a - q}{\bar{p}\bar{q}} = \bar{p} \frac{(1 + \Pi^*)^2}{\alpha^2} \rightarrow \frac{a - q}{q} = \bar{p}^2 \frac{(1 + \Pi^*)^2}{\alpha^2} = \frac{\alpha}{\Pi(1 + \Pi^*)} \frac{(1 + \Pi^*)^2}{\alpha^2} = \frac{(1 + \Pi^*)}{\alpha\Pi} \\ &\rightarrow \frac{a}{q} = 1 + \frac{(1 + \Pi^*)}{\alpha\Pi} \rightarrow q = \bar{q} = \frac{a\alpha\Pi}{(1 + \Pi^*) + \alpha\Pi} \end{aligned}$$

Since our balance equations tell us $b = \bar{p}\bar{q}$ we get

$$b = \bar{b} = \sqrt{\frac{\alpha}{\Pi(1 + \Pi^*)}} \cdot \frac{a\alpha\Pi}{(1 + \Pi^*) + \alpha\Pi}$$

Equation (2.10) tells us that $m = b - \alpha w \rightarrow \alpha w = b - m$ which means that the amount of gold jewelry to convert to coin is exactly enough for the traders to bid b

for goods. Thus for this case to hold, since $w > 0$, $b - m > 0 \rightarrow b > m$ thus this case is bounded by $m < \sqrt{\frac{\alpha}{\Pi(1+\Pi^*)}} \cdot \frac{a\alpha\Pi}{(1+\Pi^*)+\alpha\Pi}$.

If we suppose that conversion from gold coin to jewelry is free, i.e. $\alpha = 1$, and observe what happens to q we see that

$$q = \frac{a\Pi}{(1 + \Pi^*) + \Pi} < \frac{a\Pi}{\Pi + \Pi} = \frac{a}{2}$$

where the inequality stems from the fact that $\Pi < (1 + \Pi^*)$. We do not reach efficient trade no matter what the cost of conversion is, and this is simply due to the fact that gold jewelry is worth more than buying and selling goods.

2.1.10 Case 10: $\lambda > 0, \lambda^* = 0, v = 0, w = 0$

Just as in Case 4, we cannot have a solution where $\lambda^* > 0$ when $w = 0$.

2.1.11 Case 11: $\lambda > 0, \lambda^* = 0, v > 0, w = 0$

Just as in Case 4, we cannot have a solution where $\lambda^* > 0$ when $w = 0$.

2.1.12 Case 12: $\lambda > 0, \lambda^* > 0, v = 0, w > 0$

This case happens when we have very little gold (in either form) to begin with. $w > 0$ and $\lambda^* > 0$ implies that the value of gold coin is worth more than jewelry and we will convert what jewelry we have leaving none left over. This is verified by equation (2.11) which reduces to $w = m^*$. Even though we are converting all our jewelry to coin, $\lambda > 0$ can still happen if we had little money to start with, and little jewelry to convert and contribute to purchasing goods. Thus after trade, we will expect to have no money left thus $\lambda > 0$ makes sense.

We substitute $w = m^*$ in to equation (2.10) to find that $b = \bar{b} = m + \alpha m^*$.

Equation (2.9) gives us $\lambda = \frac{1}{\alpha}(1 + \Pi^* + \lambda^*) - \Pi$ which we substitute in to the square of equation (2.6).

$$\frac{a - q}{b} = \bar{p} \frac{((1 + \Pi^*) + \lambda^*)^2}{\alpha^2} \quad (2.18)$$

Setting this equal to the square of the reciprocal of equation (2.7) and taking in to account symmetry, we get

$$\begin{aligned} \bar{p} \frac{((1 + \Pi^*) + \lambda^*)^2}{\alpha^2} &= \frac{1}{p^3 \Pi^2} \rightarrow p^4 = \frac{\alpha^2}{\Pi^2 ((1 + \Pi^*) + \lambda^*)^2} \rightarrow p^2 = \frac{\alpha}{\Pi ((1 + \Pi^*) + \lambda^*)} \\ &\rightarrow p = \bar{p} = \sqrt{\frac{\alpha}{\Pi ((1 + \Pi^*) + \lambda^*)}} \end{aligned}$$

Substituting this back in to equation (2.18) with our balance equation and symmetry we solve for q :

$$\begin{aligned} \frac{a-q}{qp} &= \bar{p} \frac{((1+\Pi^*)+\lambda^*)^2}{\alpha^2} \rightarrow \frac{a-q}{q} = p^2 \frac{((1+\Pi^*)+\lambda^*)^2}{\alpha^2} = \\ \frac{\alpha}{\Pi((1+\Pi^*)+\lambda^*)} \cdot \frac{((1+\Pi^*)+\lambda^*)^2}{\alpha^2} &= \frac{(1+\Pi^*)+\lambda^*}{\alpha\Pi} \rightarrow \frac{a}{q} = 1 + \frac{(1+\Pi^*)+\lambda^*}{\alpha\Pi} \\ \rightarrow q = \bar{q} &= \frac{a\alpha\Pi}{\alpha\Pi + (1+\Pi^*) + \lambda^*} \end{aligned}$$

To solve for λ^* , we express a second equation for b by $\bar{p}\bar{q}$ and setting this equal to our first expression of $b = m + \alpha m^*$.

$$\begin{aligned} b &= \sqrt{\frac{\alpha}{\Pi((1+\Pi^*)+\lambda^*)}} \cdot \frac{a\alpha\Pi}{\alpha\Pi + (1+\Pi^*) + \lambda^*} = m + \alpha m^* \\ \rightarrow [(\alpha^2\Pi^3 + \Pi^3)(\alpha m^* + m)^2] \lambda^{*3} &+ [(3\Pi(1+\Pi^*) + 2\alpha\Pi^2)(\alpha m^* + m)^2] \lambda^{*2} \\ &+ [(4\alpha\Pi^2(1+\Pi^*) + 3\Pi(1+\Pi^*)^2)(\alpha m^* + m)^2] \lambda^* \\ + [(\Pi(1+\Pi^*)^3 + 2\alpha\Pi^2(1+\Pi^*)^2 + \alpha^2\Pi^3(1+\Pi^*)) (\alpha m^* + m)^2 - a^3\alpha^3\Pi^2] &= 0 \end{aligned} \tag{2.19}$$

The trinomial given in equation (2.19) can then be solved either by computer, or by using a formula for finding the roots of a trinomial. Once that is found, that may be substituted in our expressions of λ, b, q, v , and w above, thus completing the model.

Chapter 3

Two Distinct Types of Gold with Banking

3.1 Types of Banks

The world of banking is a complicated and unpredictable one, thus, to facilitate the analysis of trading models that allow lending, we grossly over-simplify the way banks are set-up and function in the real economy. For our purposes, we define banks to be “rich” players with large amounts of money and no goods, whose decision variables include how much money to lend out to traders in a single period at an endogenous

interest rate. We categorize the banks into two main types: Private banks and Central banks.

Private banks operate with the intention to maximize their own utility functions. If the private bank is *corporate*, then its only interest is in maximizing its monetary profit, since as a corporation, it cannot consume any goods and thus has no interest in obtaining any. However, if it is a private bank that is *individually-owned* (or family owned), then it derives utility from both profit and the consumption of goods.

Central Banks, or government run banks, operate with the intention to maximize some function of social welfare. Often its strategy is to lend all its money at an interest rate that will allow them to break even at the end of the period with no gain while allowing the traders to reach equilibrium levels in trade and consumption (they may also be referred to as an Altruistic Bank). Alternatively, the Central Bank may be run as a profit maximizing bank where the profit may be used to fund something on behalf of the people/government, for example, to support a country's king or military, in which case it is acts similarly to a corporate bank.

For the purposes of this work, we will look at the banks as being Altruistic. For analysis on other forms of banking, interested parties are encouraged to explore the models presented in Dr. Quint and Dr. Shubik's work, *Barley, Gold and Fiat: A*

Pure Theory of Money.

To begin development of the models, let the total amount of money in the economy be given by M . In the basic model, the Type I traders have initial endowments of $(a, 0, m)$ and Type II traders begin with endowments of $(0, a, m)$, both defined as in section 1.2. Thus, since the bank have only money as their initial endowments, they begin with $(0, 0, M - 2m)$. The decision variable for the bank is how much money it decides to lend out to the traders, and we denote that variable as g . The endogenous interest rate is given as:

$$1 + \rho = \frac{d + \bar{d}}{g} \tag{3.1}$$

per period where d and \bar{d} are the decision variables corresponding to how much money Type I and Type II traders owe the bank when the period is done. The interest rate acts such that when players borrow x units, they must pay back the x units plus $\rho \cdot x$ units worth of interest to the bank, or in other words, $(1 + \rho) \cdot x$ units are paid to the bank at the end of the period. We add equation (3.1) to our list of *balance equations* which includes (1.4) and (1.5).

For a truly altruistic bank, it will lend all of its money out in order to do what it can to facilitate trade, thus, it is treated as a dummy player whose decision variable will always be $g = M - 2m$ thus g is in fact, the constant G .

The profit-maximizing bank's problem looks like this:

$$\max_g M - 2m + \rho g \quad (3.2)$$

$$s.t. \quad M - 2m - g \geq 0 \quad (3.3)$$

$$g \geq 0 \quad (3.4)$$

Here the objective function is the total amount of money the bank has, $M - 2m$ plus the interest the bank pockets by lending g units of money, which is ρg . The first constraint is the bank's lending constraint that restricts the bank from lending out more than it actually has.

3.2 The Gold Model with Banking

Until now, the models were analyzed with the only available money being the initial endowments of the traders. Here, we introduce another player, the Bank, who is endowed with a large amount of money and has the ability to lend a certain amount of it to the traders. For the gold model, we assume that the bank only has interest in gold coin, and that it gains no utility from gold jewelry. Thus, it only lends gold coin, and traders may only pay back the loan in gold coin. Gold may only be converted from one form to another by the traders once in the period, and that happens before the trading of goods takes place.

As in Chapter 2, each trader type is endowed initially with $(a, 0, m, m^*)$ for Type I and $(0, a, m, m^*)$ for Type II. The strategic variables b, q, v , and w ($\bar{b}, \bar{q}, \bar{v}$, and \bar{w} for Type II) remain and we add another variable d (or \bar{d}) which is the amount of money the traders will owe the bank at the end of the period. To clarify, traders of Type I are not borrowing d units from the bank, but they are actually borrowing $\frac{d}{1+\rho}$, so for the period, they will owe $(1+\rho) \cdot \frac{d}{1+\rho} = d$ units (similar for traders of Type II).

The game is played out in the following order:

- (1) The bank decides how much money to lend the traders.
- (2) The traders decide how much money to borrow from the bank.
- (3) The interest rates are determined endogenously from the values chosen in steps (1) and (2).
- (4) The traders decide how much gold coin to convert to gold jewelry, or how much gold jewelry to convert to coin.
- (5) The traders decide how much to bid on other traders' goods and how much of their own good to put up for sale, then
- (6) the prices are determined endogenously from the values chosen in step (4) and

then,

- (7) the amount of consumption good to be received based on prices and the cash income received from sales are determined for the players.
- (8) Finally, the traders pay back the amount owed to the bank

This is a dynamic game, and the bank is the first to make their move. The traders' then simultaneously make their decisions based on the decision of the bank, thus treating the bank's decision variable like it were a parameter. This game can be expressed as an extensive form game, where the traders' know what the bank's decision is prior to making their decisions, and we are finding a perfect equilibrium by first solving for the traders' strategies as a function of g , then solving for the bank's strategy.

With the added decision variable d , the Type I traders' problem is now:

$$\max_{b,q,v,w,d} \varphi(a - q, \frac{b}{\bar{p}}) + \Pi(m - b - v + \alpha w + pq + \frac{d}{1 + \rho} - d) + (1 + \Pi^*)(m^* - w + \alpha v) \quad (3.5)$$

$$s.t. \quad m - b - v + \alpha w + \frac{d}{1 + \rho} \geq 0 \quad (\lambda) \quad (3.6)$$

$$m^* - w + \alpha v \geq 0 \quad (\lambda^*) \quad (3.7)$$

$$m - b + pq + \alpha w - v - d + \frac{d}{1 + \rho} \geq 0 \quad (\mu) \quad (3.8)$$

$$0 \leq q \leq a \quad \text{and} \quad b, v, w \geq 0 \quad (3.9)$$

Equation (3.5) is similar to our original gold model maximizing function except that we have added the amount borrowed and the amount to be paid back in gold coin. The jewelry and goods consumption portion of that equation remains the same. Equation (3.6) is our *cash-flow constraint* with the added lending amounts and is still associated with the Lagrangian multiplier λ . Equation (3.7) is our *jewelry constraint* which remains unchanged and has the associated multiplier λ^* . Equation (3.8) is a new constraint we refer to as the *budget constraint* which is now necessary because once trading is complete, there must be enough money left over to repay the bank. The associated multiplier for the budget constraint will be μ .

3.3 Analysis

To perform the analysis, once again we define the function

$$\varphi(x, y) = 2\sqrt{x \cdot y}$$

and focus on the problem for trader Type 1 (Type 2 is similar.) We assume that the initial endowments are still $(a, 0, m, m^*)$ for Type I traders.

Using Lagrange Multipliers and Kuhn-Tucker conditions, we begin by identi-

fyng the new Langrange function to be:

$$\begin{aligned}
L = & 2\sqrt{(a-q) \cdot \frac{b}{\bar{p}}} + \Pi \left(m - b - v - \alpha w + pq + \frac{d}{1+\rho} - d \right) \\
& + (1 + \Pi^*)(m^* - w + \alpha v) - \lambda \left(-m + b + v - \alpha w - \frac{d}{1+\rho} \right) \\
& - \lambda^*(-m^* + w - \alpha v) - \mu \left(-m + b - pq - \alpha w + v + d - \frac{d}{1+\rho} \right)
\end{aligned} \tag{3.10}$$

The first order Kuhn-Tucker conditions with respect to the variables b, q, v, w, d respectively are:

$$\sqrt{\frac{a-q}{b}} = \sqrt{\bar{p}} \cdot (\lambda + \Pi + \mu) \tag{3.11}$$

$$\sqrt{\frac{b}{a-q}} = \sqrt{\bar{p}} \cdot p \cdot (\Pi + \mu) \tag{3.12}$$

$$\lambda = \alpha\lambda^* + \alpha(1 + \Pi^*) - \Pi - \mu \tag{3.13}$$

$$\lambda = \frac{1}{\alpha}(1 + \Pi^* + \lambda^*) - \Pi - \mu \tag{3.14}$$

$$\lambda = \rho(\Pi + \mu) \tag{3.15}$$

The Kuhn-Tucker conditions based on the cash-flow, jewelry, and budget constraints are:

$$\begin{aligned}
\lambda = 0 \text{ and } m - b - v + \alpha w + \frac{d}{1+\rho} \geq 0 \\
\text{or}
\end{aligned} \tag{3.16}$$

$$\lambda \geq 0 \text{ and } m - b - v + \alpha w + \frac{d}{1+\rho} = 0$$

as well as

$$\lambda^* = 0 \text{ and } m^* - w + \alpha v \geq 0$$

or

(3.17)

$$\lambda^* \geq 0 \text{ and } m^* - w + \alpha v = 0$$

as well as

$$\mu = 0 \text{ and } m - b + pq - v + \alpha w + \frac{d}{1 + \rho} - d \geq 0$$

or

(3.18)

$$\mu \geq 0 \text{ and } m - b + pq - v + \alpha w + \frac{d}{1 + \rho} - d = 0$$

Further, we have:

$$b, v, w, d \geq 0 \text{ and } 0 \leq q \leq a$$

along with our balance equations (1.4), (1.5), and (3.1).

Just as in Chapter 2, there are combinations of values the decision variables (v, w) and multipliers $(\mu, \lambda, \lambda^*)$ can take. The possible combinations of v and w are once again, $(v = 0, w = 0)$, $(v > 0, w = 0)$, and $(v = 0, w > 0)$. The values for λ and λ^* are $(\lambda = 0, \lambda^* = 0)$, $(\lambda = 0, \lambda^* > 0)$, $(\lambda > 0, \lambda^* = 0)$, and $(\lambda > 0, \lambda^* > 0)$ and lastly, either $\mu = 0$ or $\mu > 0$. So in total, we have 24 potential cases of strategies which are all presented in tables (3.1) and (3.2) with whether or not that specific combination of values is possible.

Table 3.1: Potential Bounds on Decision Variable Values with $\mu = 0$

	$v = 0, w = 0$	$v > 0, w = 0$	$v = 0, w > 0$
$\lambda = 0, \lambda^* = 0$	Possible (Case 1)	Not Possible (Case 2)	Not Possible (Case 3)
$\lambda = 0, \lambda^* > 0$	Not Possible (Case 4)	Not Possible (Case 5)	Possible (Case 6)
$\lambda > 0, \lambda^* = 0$	Possible (Case 7)	Not Possible (Case 8)	Possible (Case 9)
$\lambda > 0, \lambda^* > 0$	Not Possible (Case 10)	Not Possible (Case 11)	Possible (Case 12)

The analyses for each of the cases are found in the remaining subsections. What was found was that although introducing a bank enables traders to have access to more money, we still have just two cases (3.3.1 Case 1 and 3.3.6 Case 6) where efficient trade occurs. Just as in Chapter 2, in Case 1, we have sufficient amount of money and thus we are not interested in exchanging any for jewelry (or we're not able to). Since we are not exchanging one kind of gold to another, the problem is decoupled just as in Chapter 2, and we have a model that is essentially the fiat model in Quint-Shubik's Chapter 7, thus their analysis can be applied in these cases. In Case 6, all gold jewelry is converted to gold coin from the start and once that happens, the problem is now a problem where we have gold taking on only one form, fiat, and once again we can use the analysis from Quint-Shubik.

Another observation in these cases is that anytime the cash flow constraint is

Table 3.2: Potential Bounds on Decision Variable Values with $\mu > 0$

	$v = 0, w = 0$	$v > 0, w = 0$	$v = 0, w > 0$
$\lambda = 0, \lambda^* = 0$	Not Possible (Case 13)	Not Possible (Case 14)	Not Possible (Case 15)
$\lambda = 0, \lambda^* > 0$	Not Possible (Case 16)	Not Possible (Case 17)	Not Possible (Case 18)
$\lambda > 0, \lambda^* = 0$	Possible (Case 19)	Possible (Case 20)	Possible (Case 21)
$\lambda > 0, \lambda^* > 0$	Not Possible (Case 22)	Not Possible (Case 23)	Possible (Case 24)

loose, i.e. $\lambda = 0$ in equation 3.16, the interest rate ρ is zero. Mathematically, this stems from equation 3.15, but intuitively, this makes sense because the cash flow constraint will be loose only if there is enough money in the market to facilitate trade. If there is enough money to facilitate trade, there is no necessity to borrow from the bank, which will cost traders if $\rho > 0$. Thus the only way their money will be used is if they charge no interest on borrowing by the traders, which is fine since we have altruistic bank.

3.3.1 Case 1: $\mu = 0, \lambda = 0, \lambda^* = 0, v = 0, w = 0$

Since $w = 0$, we are not converting any jewelry to coin, so naturally, $\lambda^* = 0$ must also hold true since the jewelry is not going anywhere. $\lambda = 0$ implies that we had sufficient money to facilitate trade and have some left over. We will find the boundary for m later. We begin by solving for our variables d and \bar{d} . Equation (3.1) along with symmetry tells us that

$$1 + \rho = \frac{d + \bar{d}}{G} \rightarrow 1 + \rho = \frac{2d}{G} \rightarrow d = \bar{d} = \frac{G(1 + \rho)}{2}$$

The values of λ and μ substituted in to equations (3.11) and (3.12) yield

$$\sqrt{\frac{a - q}{b}} = \sqrt{\bar{p}} \cdot \Pi \quad (3.19)$$

and

$$\sqrt{\frac{b}{a - q}} = \sqrt{\bar{p}} \cdot p \cdot \Pi \quad (3.20)$$

Setting the first equation equal to the inverse of the second and taking in to account symmetry we get

$$\sqrt{\bar{p}} \cdot \Pi = \frac{1}{\sqrt{\bar{p}} \cdot p \cdot \Pi} \rightarrow \bar{p}p = \frac{1}{\Pi^2} \rightarrow p = \bar{p} = \frac{1}{\Pi}$$

¹This value of d and \bar{d} holds true no matter what case we're in and will be carried through the remaining cases in this chapter.

Using our balance equation and substituting this value of \bar{p} in to the square of equation (3.11), we get a value for q as follows

$$\frac{a-q}{pq} = \bar{p}\Pi^2 \rightarrow \frac{a-q}{q} = \bar{p}p\Pi^2 \rightarrow \frac{a-q}{q} = 1 \rightarrow a-q = q \rightarrow a = 2q \rightarrow q = \bar{q} = \frac{a}{2}$$

Substituting these values for p and q in to our balance equation we obtain the following value for b and \bar{b}

$$b = \bar{b} = pq = \frac{a}{2\Pi}$$

Since $\lambda = 0$, equation (3.16) gives us a bound of m which is

$$m - b + \frac{d}{1+\rho} \geq 0 \rightarrow m \geq \frac{a}{2\Pi} - \frac{G}{2}$$

So this is what is meant by a sufficient amount of money. This case results in "efficient trade" where the traders are consuming half of one kind of good and half of the other kind of good.

3.3.2 Case 2: $\mu = 0, \lambda = 0, \lambda^* = 0, v > 0, w = 0$

$v > 0$ and $w = 0$ means that we are converting coin to jewelry, and this will only happen if the value of gold jewelry is worth relatively more than gold coin, $\Pi < 1 + \Pi^*$. However, this also implies that there is no benefit to having any excess coin left after the period ends, so we would want to drive our cash constraint to zero. This is done

by converting almost all the gold coin to jewelry saving just enough for trade, which gets spent, and thus leaving nothing left over. Therefore, $\lambda = 0$ would not happen and we have no case here regardless of the value of μ .

3.3.3 Case 3: $\mu = 0, \lambda = 0, \lambda^* = 0, v = 0, w > 0$

$v = 0$ and $w > 0$ means we are converting jewelry to coin, and this would happen for one of two reasons. One, if the value of coin is relatively higher than jewelry, $\Pi > 1 + \Pi^*$ or two, if we wanted to change a minimal amount of coin to facilitate trade because the value of jewelry is more than coin. If gold jewelry is converted to coin for the first reason, then since jewelry cannot facilitate trade, and it is worth less than coin, there is no reason to keep any around thus we would convert all of it to coin and drive the constraint to zero. This means $\lambda^* = 0$ would not make sense and we have no case here. If it is the second reason, then coin is worth less than jewelry and we would only want to have enough to facilitate trade and have nothing left over, thus $\lambda = 0$ won't happen. In either scenario, this case does not work.

3.3.4 Case 4: $\mu = 0, \lambda = 0, \lambda^* > 0, v = 0, w = 0$

This case will not happen for the exact reasons it could not happen in Cases 4 and 5 of Chapter 2. There is no conversion from coin to jewelry or jewelry to coin. Since we can't spend jewelry, and we aren't transferring jewelry to coin, there is no reason for our jewelry constraint to be driven to zero. Thus, it will never happen that $\lambda^* > 0$ when $w = 0$.

3.3.5 Case 5: $\mu = 0, \lambda = 0, \lambda^* > 0, v > 0, w = 0$

This case will not happen for the exact reasons it could not happen in Cases 4, 5, 10, and 11 of Chapter 2. There is no conversion from coin to jewelry or jewelry to coin. Since we can't spend jewelry, and we aren't transferring jewelry to coin, there is no reason for our jewelry constraint to be driven to zero. Thus, it will never happen that $\lambda^* > 0$ when $w = 0$.

Another important observation to note is that since $\mu = 0$ it is implied that after we pay back the bank, we will have gold coin left-over. However, since $v > 0$, it is implied that the value of gold jewelry is more than gold coin, so it would never be optimal to end up with left over gold coin after paying back the bank. Thus, it can't

happen that $\mu = 0$ when $v > 0$.

3.3.6 Case 6: $\mu = 0, \lambda = 0, \lambda^* > 0, v = 0, w > 0$

Since $w > 0$, we are exchanging jewelry to coin. $\lambda^* > 0$ means that we're driving our jewelry constraint to zero which implies that the value of gold jewelry is less than coin, i.e. $1 + \Pi^* < \Pi$. Equation (3.17) simplifies $m^* = w$ which verifies all this. Since $\lambda = 0$, equation (3.15) implies $\rho = 0$ since $\Pi > 0$. Our value of $d = \bar{d} = \frac{G(1+\rho)}{2} = \frac{G}{2}$ is found from equation (3.1). The work for solving for b, \bar{b}, q , and \bar{q} is identical to 3.3.1 Case 1, since the values of λ and μ are the same. Thus

$$q = \bar{q} = \frac{a}{2} \quad \text{and} \quad b = \bar{b} = \frac{a}{2\Pi}$$

Since $\lambda^* > 0$, we also have that $w = m^*$ by equation (3.17). Also, we obtain a value for λ^* by equation (3.14) and substituting in $\lambda = 0$ and $\mu = 0$ which is

$$0 = \frac{1}{\alpha}\lambda^* + \frac{1}{\alpha}(1 + \Pi^*) - \Pi \rightarrow \lambda^* = \alpha\Pi - (1 + \Pi^*)$$

Since $\lambda = 0$, equation (3.16) gives us a bound on this case which is

$$m - b + \alpha w - d \geq 0 \rightarrow m + \alpha m^* \geq \frac{a}{2\Pi} - \frac{G}{2}$$

Thus, as long as $m + \alpha m^*$ satisfies the condition above, there will be sufficient money to facilitate efficient trade.

It is good to point out here that it makes sense that in this case the bank does not charge any interest on lending out money (i.e. $\rho = 0$). Since there is enough money held by the traders to begin with, and there's no reason to have any jewelry in the end, there is no necessity in borrowing money to facilitate trade, and thus if it cost money to borrow, the traders would not do so.

3.3.7 Case 7: $\mu = 0, \lambda > 0, \lambda^* = 0, v = 0, w = 0$

It is implied that since $v = 0$ and $w = 0$, the cost of converting gold is relatively high, thus α is small. Since we aren't converting any jewelry, the jewelry constraint will not be driven to zero thus $\lambda^* = 0$. Equation (3.15) gives us that $\lambda = \rho\Pi$. If we substitute this value of λ in to equation (3.11) we get

$$\sqrt{\frac{a-q}{b}} = \sqrt{\bar{p}} \cdot \Pi(1+\rho) \quad (3.21)$$

Setting this equation equal to the reciprocal of equation (3.12) and taking in to account symmetry, we can obtain an expression for p and \bar{p} as follows

$$\sqrt{\bar{p}} \cdot \Pi(1+\rho) = \frac{1}{\sqrt{\bar{p}} \cdot p\Pi} \rightarrow \bar{p}p = \frac{1}{(1+\rho)\Pi^2} \rightarrow p = \bar{p} = \frac{1}{\Pi\sqrt{1+\rho}}$$

We can substitute this value of \bar{p} in to the square of equation (3.21) and use our balance equation to solve for q as follows

$$\frac{a-q}{pq} = \bar{p}\Pi^2(1+\rho)^2 \rightarrow \frac{a-q}{q} = p^2\Pi^2(1+\rho)^2 \rightarrow \frac{a}{q} = 2+\rho \rightarrow q = \bar{q} = \frac{a}{2+\rho}$$

We can now obtain an expression for b and \bar{b}

$$b = \bar{b} = pq = \frac{a}{\Pi\sqrt{1+\rho}(2+\rho)} \quad (3.22)$$

Equation (3.1) gives $d = \bar{d} = \frac{G(1+\rho)}{2}$.

Since $\lambda > 0$, equation (3.16) gives us a second expression for b .

$$m - b + \frac{d}{1+\rho} \rightarrow b = \bar{b} = m + \frac{G}{2}$$

We can set the two found expressions for b equal to one another to get

$$m + \frac{G}{2} = \frac{a}{\Pi\sqrt{1+\rho}(2+\rho)}$$

Which can be solved for ρ using computational methods.

Equation (3.18) and $\mu = 0$ gives us our bound for m which is

$$m - b + pq + \frac{d}{1+\rho} - d > 0 \rightarrow m > \frac{G(1+\rho)}{2} - \frac{G}{2} = \rho \left(\frac{G}{2} \right)$$

3.3.8 Case 8: $\mu = 0, \lambda > 0, \lambda^* = 0, v > 0, w = 0$

Since $\mu = 0$ it is implied that after we pay back the bank, we will have gold coin left-over. However, since $v > 0$, it is implied that the value of gold jewelry is more than gold coin, so it would never be optimal to end up with left over gold coin after paying back the bank. Thus, it can't happen that $\mu = 0$ when $v > 0$.

3.3.9 Case 9: $\mu = 0, \lambda > 0, \lambda^* = 0, v = 0, w > 0$

Since $w > 0$ but $\lambda^* = 0$, it is implied we want to convert some jewelry to coin, but not all of it, thus jewelry must be worth more than coin, $\Pi < 1 + \Pi^*$. We can quickly obtain expressions for λ by equations (3.14) and (3.15) and setting these equal to one another we find a value for $(1 + \rho)$

$$\rho\Pi = \frac{(1 + \Pi^*)}{\alpha} - \Pi \rightarrow (1 + \rho)\Pi = \frac{(1 + \Pi^*)}{\alpha} \rightarrow (1 + \rho) = \frac{(1 + \Pi^*)}{\alpha\Pi}$$

We can substitute this in to the balance equation (3.1) to get a value for d and \bar{d} as follows

$$\frac{(1 + \Pi^*)}{\alpha\Pi} = \frac{d + \bar{d}}{G} \rightarrow d = \bar{d} = \frac{G(1 + \Pi^*)}{2\alpha\Pi}$$

The following sequence of inequalities occurs based on the fact that $\alpha < 1$ and $\Pi < 1 + \Pi^*$

$$\frac{G(1 + \Pi^*)}{2\alpha\Pi} > \frac{G(1 + \Pi^*)}{2\alpha(1 + \Pi^*)} = \frac{G}{2\alpha} > \frac{G}{2}$$

Thus the bank will be getting paid back more than $\frac{G}{2}$ which makes sense since they are charging an interest rate on the money lent to the players, i.e. ρ is positive.

Equation (3.12) squared and our balance equation imply the following:

$$\frac{pq}{a - q} = p^3\Pi^2 \rightarrow q = (a - q)p^2\Pi^2 \rightarrow q(1 + p^2\Pi^2) = p^2\Pi^2a \rightarrow q = \frac{p^2\Pi^2a}{1 + p^2\Pi^2}$$

Substituting in for p and simplifying, we obtain $q = \bar{q} = \frac{a}{2 + \rho}$. The values of p and q substituted in to our balance equations gives us that $b = \bar{b} = \frac{a}{(2 + \rho)\Pi\sqrt{1 + \rho}}$. We obtain

an expression for w by equation (3.16) and substituting in for d and $1 + \rho$ which gives:

$$m - b + \alpha w + \frac{d}{1 + \rho} = 0 \rightarrow \alpha w = b - m - \frac{G}{2}$$

which means that the amount of gold jewelry we wish to convert to coin will be the amount needed to bid the amount $b = \frac{a}{(2+\rho)\Pi\sqrt{1+\rho}}$ after we've accounted for how much we have initially, m , and the maximum amount we can borrow from the bank, $\frac{G}{2}$.

Equation (3.17) gives us a bound on this case as follows:

$$m^* - w \geq 0 \rightarrow \alpha m^* - \alpha w = \alpha m^* - b + m + \frac{G}{2} \geq 0 \rightarrow m + \alpha m^* \geq \frac{a}{(2 + \rho)\Pi\sqrt{1 + \rho}} - \frac{G}{2}$$

Also, by equation (3.14), $\frac{1+\Pi^*}{\Pi} \geq \alpha$ must also hold, but since $\alpha \leq 1$, this will always be true. So this case is not restricted by the value of α .

3.3.10 Case 10: $\mu = 0, \lambda > 0, \lambda^* > 0, v = 0, w = 0$

This case will not happen for the exact reasons it could not happen in Cases 4, 5, 10, and 11 of Chapter 2. There is no conversion from coin to jewelry or jewelry to coin. Since we can't spend jewelry, and we aren't transferring jewelry to coin, there is no reason for our jewelry constraint to be driven to zero. Thus, it will never happen that $\lambda^* > 0$ when $w = 0$.

3.3.11 Case 11: $\mu = 0, \lambda > 0, \lambda^* > 0, v > 0, w = 0$

This case will not happen for the exact reasons it could not happen in Cases 4, 5, 10, and 11 of Chapter 2. There is no conversion from coin to jewelry or jewelry to coin. Since we can't spend jewelry, and we aren't transferring jewelry to coin, there is no reason for our jewelry constraint to be driven to zero. Thus, it will never happen that $\lambda^* > 0$ when $w = 0$.

Another important observation to note is that since $\mu = 0$ it is implied that after we pay back the bank, we will have gold coin left-over. However, since $v > 0$, it is implied that the value of gold jewelry is more than gold coin, so it would never be optimal to end up with left over gold coin after paying back the bank. Thus, it can't happen that $\mu = 0$ when $v > 0$.

3.3.12 Case 12: $\mu = 0, \lambda > 0, \lambda^* > 0, v = 0, w > 0$

Here we have converted jewelry to coin $w > 0$ and driven our jewelry constraint to zero, i.e. $\lambda^* > 0$. This also implies through equation (3.17) that $w = m^*$, meaning we convert all our gold jewelry to gold coin thus $1 + \Pi^* < \Pi$. We can use equations (3.14) and (3.15) to obtain two expression for λ and set them equal to one another

to find λ^* as follows

$$\begin{aligned} \frac{1}{\alpha}[(1 + \Pi^*) + \lambda^*] - \Pi &= \rho\Pi \rightarrow \frac{1}{\alpha}[(1 + \Pi^*) + \lambda^*] = \rho\Pi + \Pi \\ \rightarrow (1 + \Pi^*) + \lambda^* &= \alpha\Pi(1 + \rho) \rightarrow \lambda^* = \alpha\Pi(1 + \rho) - (1 + \Pi^*) \end{aligned}$$

Since $\lambda^* > 0$, $1 + \rho > \frac{1 + \Pi^*}{\alpha\Pi}$ must hold for this case to be valid. Our balance equation (3.1) gives $d = \bar{d} = \frac{G(1 + \rho)}{2}$. Equation (3.16) and $\lambda > 0$ imply

$$m - b + \alpha w + \frac{d}{1 + \rho} = 0 \rightarrow b = \bar{b} = m + \alpha m^* + \frac{G}{2} \quad (3.23)$$

Substituting equation (3.15) in for λ of equation (3.11) we get

$$\sqrt{\frac{a - q}{b}} = \sqrt{\bar{p}} \cdot \Pi(1 + \rho) \quad (3.24)$$

Setting this equation equal to the reciprocal of equation (3.12) and taking in to account symmetry, we can obtain an expression for p and \bar{p} as follows

$$\sqrt{\bar{p}} \cdot \Pi(1 + \rho) = \frac{1}{\sqrt{\bar{p}} \cdot p\Pi} \rightarrow \bar{p}p = \frac{1}{(1 + \rho)\Pi^2} \rightarrow p = \bar{p} = \frac{1}{\Pi\sqrt{1 + \rho}}$$

We can substitute this value of \bar{p} in to the square of equation (3.24) and use our balance equation to solve for q .

$$\frac{a - q}{pq} = \bar{p}\Pi^2(1 + \rho)^2 \rightarrow \frac{a - q}{q} = p^2\Pi^2(1 + \rho)^2 \rightarrow \frac{a}{q} = 2 + \rho \rightarrow q = \bar{q} = \frac{a}{2 + \rho}$$

We now obtain a second expression for b and \bar{b} which is

$$b = \bar{b} = pq = \frac{a}{\Pi\sqrt{1 + \rho}(2 + \rho)} \quad (3.25)$$

Equation (3.23) and (3.25) yield the following equality

$$m + \alpha m^* + \frac{G}{2} = \frac{a}{\Pi\sqrt{1 + \rho}(2 + \rho)}$$

This equality can be solved for ρ using computational methods.

Equation (3.18) and $\mu = 0$ gives us our bound for m and m^* which is

$$m - b + pq + \alpha w + \frac{d}{1 + \rho} - d > 0 \rightarrow m + \alpha m^* > \frac{G(1 + \rho)}{2} - \frac{G}{2} = \rho \left(\frac{G}{2} \right)$$

This case is similar to 3.3.7 Case 7, and if we look at our initial amount of gold coin to be $m + \alpha m^*$, this problem is exactly the same (minus the bonus utility from having gold jewelry).

3.3.13 Case 13: $\mu > 0, \lambda = 0, \lambda^* = 0, v = 0, w = 0$

Since, $\lambda = 0$, equation (3.16) tells us that $m - b - v + \alpha w + \frac{d}{1 + \rho} > 0$. However, $\mu > 0$ so equation (3.18) tells us that $m - b - v + \alpha w + \frac{d}{1 + \rho} - d = 0$ but since $d \geq 0$, this equation will never be satisfied. Thus any case where $\lambda = 0$ and $\mu > 0$ is impossible.

3.3.14 Case 14: $\mu > 0, \lambda = 0, \lambda^* = 0, v > 0, w = 0$

Since, $\lambda = 0$, equation (3.16) tells us that $m - b - v + \alpha w + \frac{d}{1 + \rho} > 0$. However, $\mu > 0$ so equation (3.18) tells us that $m - b - v + \alpha w + \frac{d}{1 + \rho} - d = 0$ but since $d \geq 0$, this

equation will never be satisfied. Thus any case where $\lambda = 0$ and $\mu > 0$ is impossible.

3.3.15 Case 15: $\mu > 0, \lambda = 0, \lambda^* = 0, v = 0, w > 0$

Since, $\lambda = 0$, equation (3.16) tells us that $m - b - v + \alpha w + \frac{d}{1+\rho} > 0$. However, $\mu > 0$ so equation (3.18) tells us that $m - b - v + \alpha w + \frac{d}{1+\rho} - d = 0$ but since $d \geq 0$, this equation will never be satisfied. Thus any case where $\lambda = 0$ and $\mu > 0$ is impossible.

3.3.16 Case 16: $\mu > 0, \lambda = 0, \lambda^* > 0, v = 0, w = 0$

Since, $\lambda = 0$, equation (3.16) tells us that $m - b - v + \alpha w + \frac{d}{1+\rho} > 0$. However, $\mu > 0$ so equation (3.18) tells us that $m - b - v + \alpha w + \frac{d}{1+\rho} - d = 0$ but since $d \geq 0$, this equation will never be satisfied. Thus any case where $\lambda = 0$ and $\mu > 0$ is impossible.

3.3.17 Case 17: $\mu > 0, \lambda = 0, \lambda^* > 0, v > 0, w = 0$

Since, $\lambda = 0$, equation (3.16) tells us that $m - b - v + \alpha w + \frac{d}{1+\rho} > 0$. However, $\mu > 0$ so equation (3.18) tells us that $m - b - v + \alpha w + \frac{d}{1+\rho} - d = 0$ but since $d \geq 0$, this equation will never be satisfied. Thus any case where $\lambda = 0$ and $\mu > 0$ is impossible.

3.3.18 Case 18: $\mu > 0, \lambda = 0, \lambda^* > 0, v = 0, w > 0$

Since, $\lambda = 0$, equation (3.16) tells us that $m - b - v + \alpha w + \frac{d}{1+\rho} > 0$. However, $\mu > 0$ so equation (3.18) tells us that $m - b - v + \alpha w + \frac{d}{1+\rho} - d = 0$ but since $d \geq 0$, this equation will never be satisfied. Thus any case where $\lambda = 0$ and $\mu > 0$ is impossible.

3.3.19 Case 19: $\mu > 0, \lambda > 0, \lambda^* = 0, v = 0, w = 0$

Since $w = 0$, $\lambda^* = 0$ must also be true. The fact that $\lambda > 0$ and $\mu > 0$, equations (3.16) and (3.18) become

$$m - b + \frac{d}{1 + \rho} = 0 \quad (3.26)$$

and

$$m + \frac{d}{1 + \rho} - d = 0 \quad (3.27)$$

which implies that $b = \bar{b} = d = \bar{d} = \frac{G(1+\rho)}{2}$ where the final equality here comes from the balance equation (3.1). If we substitute d in for b in equation (3.26) we get a second expression $d = \frac{(1+\rho)m}{\rho}$.² When we set our two expressions of d equal to one another we get a value for ρ as follows

$$\frac{m(1 + \rho)}{\rho} = \frac{G(1 + \rho)}{2} \rightarrow \rho = \frac{2m}{G}$$

²This expression is only valid so long as $m > 0$. If $m = 0$, then $d = 0$ and our balance equation (3.1) would imply that our interest rate $\rho < 0$ which is not possible.

Equation (3.15) gives us that $\lambda = \rho(\Pi + \mu)$ and if we substitute this in to equation (3.11) we get

$$\sqrt{\frac{a-q}{b}} = \sqrt{\bar{p}}(1+\rho)(\Pi+\mu) \quad (3.28)$$

Setting this equal to the reciprocal of equation (3.12) we get an expression for p and \bar{p} as follows

$$\begin{aligned} \sqrt{\bar{p}}(1+\rho)(\Pi+\mu) &= \frac{1}{\sqrt{\bar{p}p}(\Pi+\mu)} \rightarrow \bar{p}p = \frac{1}{(1+\rho)(\Pi+\mu)^2} \\ \rightarrow p = \bar{p} &= \frac{1}{\sqrt{1+\rho}(\Pi+\mu)} \end{aligned}$$

Using the balance condition $b = pq$ and substituting the value of p above in the square of equation (3.28) we are able to find an expression for q and \bar{q} .

$$\begin{aligned} \frac{a-q}{pq} = \bar{p}(1+\rho)^2(\Pi+\mu)^2 &\rightarrow \frac{a-q}{q} = p^2(1+\rho)^2(\Pi+\mu)^2 = 1+\rho \\ \rightarrow \frac{a}{q} = 2+\rho &\rightarrow q = \bar{q} = \frac{a}{2+\rho} \end{aligned}$$

We get a second equation for b (and thus d) by our balance equation $b = \bar{p}\bar{q}$ which is

$$b = \frac{1}{\sqrt{1+\rho}(\Pi+\mu)} \cdot \frac{a}{2+\rho}$$

Setting our two expressions of b equal to one another, we can solve for μ as follows

$$b = \frac{G(1+\rho)}{2} = \frac{a}{\sqrt{1+\rho}(2+\rho)(\Pi+\mu)} \rightarrow \mu = \frac{2a}{G(1+\rho)^{3/2}(2+\rho)} - \Pi$$

If the traders had no money to begin with, i.e. $m = 0$, our values would change. First of all, equation (3.27) would imply $\rho = 0$ (this is because $d = 0$ cannot happen, see

footnote 1 above.) However, if $\rho = 0$, equation (3.15) implies $\lambda = 0$ and that removes us from this case. Thus this case is bounded below by $m > 0$. Further, it is also only valid so long as $\mu > 0$ so if we substitute $\rho = \frac{2m}{G}$ our expression for μ gives us an upper bound on m which is

$$\begin{aligned} \frac{2a}{G(1+\rho)^{3/2}(2+\rho)} - \Pi &= \frac{2a}{G(1+\frac{2m}{G})^{3/2}(2+\frac{2m}{G})} - \Pi \geq 0 \\ &\rightarrow (G+2m)^{3/2}(G+m) \leq \frac{G^{3/2}a}{\Pi} \end{aligned}$$

3.3.20 Case 20: $\mu > 0, \lambda > 0, \lambda^* = 0, v > 0, w = 0$

Here, since $v > 0$, it is implied that jewelry is valued more than coin, and since we will have some jewelry, clearly our constraint will not be driven to zero and $\lambda^* = 0$ must be true. We can quickly get expressions for λ from equations (3.13) and (3.15) which can be used to solve for μ as follows:

$$\lambda = \alpha(1+\Pi^*) - \Pi - \mu = \rho(\Pi + \mu) \rightarrow \alpha(1+\Pi^*) = (1+\rho)(\Pi + \mu) \rightarrow \mu = \frac{\alpha(1+\Pi^*)}{(1+\rho)} - \Pi$$

If we substitute this expression for μ in to equation (3.15), we get $\lambda = \frac{\rho\alpha(1+\Pi^*)}{1+\rho}$.

Equations (3.11) and (3.12) along with symmetry can be used to solve for p and \bar{p} .

$$\begin{aligned} \sqrt{\bar{p}}(\lambda + \Pi + \mu) &= \frac{1}{\sqrt{\bar{p}}p(\Pi + \mu)} \rightarrow \bar{p}p = \frac{1}{(\lambda + \Pi + \mu)(\Pi + \mu)} \\ &= \frac{1}{\left(\frac{\rho\alpha(1+\Pi^*)}{1+\rho} + \frac{\alpha(1+\Pi^*)}{1+\rho}\right) \left(\frac{\alpha(1+\Pi^*)}{1+\rho}\right)} = \frac{1+\rho}{\alpha^2(1+\Pi^*)^2} \rightarrow p = \bar{p} = \frac{\sqrt{1+\rho}}{\alpha(1+\Pi^*)} \end{aligned}$$

If we substitute for λ , μ , and \bar{p} in to the square of equation (3.11) and use our balance condition $b = \bar{p}\bar{q}$, we get

$$\begin{aligned} \frac{a-q}{\bar{p}\bar{q}} &= \bar{p} \left(\frac{\rho\alpha(1+\Pi^*) + \alpha(1+\Pi^*)}{1+\rho} \right) = \bar{p}\alpha(1+\Pi^*) \rightarrow \frac{a-q}{q} = \bar{p}^2\alpha(1+\Pi^*) \\ &\rightarrow \frac{a-q}{q} = \frac{1+\rho}{\alpha(1+\Pi^*)}\alpha(1+\Pi^*) = 1+\rho \rightarrow \frac{a}{q} = 2+\rho \rightarrow q = \bar{q} = \frac{a}{2+\rho} \end{aligned} \quad (3.29)$$

Symmetry and our balance equation give us

$$b = \bar{b} = pq = \frac{a\sqrt{1+\rho}}{\alpha(2+\rho)(1+\Pi^*)}$$

Since $\lambda > 0$ and $\mu > 0$, equations (3.16) and (3.17) imply $b = \bar{b} = d = \bar{d}$ and we know from equation (3.1) that $d = \frac{G(1+\rho)}{2}$. These two different expressions of b yield the following expression that can be solved for ρ using computational methods.

$$\frac{G(1+\rho)}{2} = \frac{a\sqrt{(1+\rho)}}{\alpha(2+\rho)(1+\Pi^*)} \rightarrow \sqrt{1+\rho}(2+\rho) = \frac{2a}{\alpha G(1+\Pi^*)}$$

Substituting $b = \frac{G(1+\rho)}{2}$ in to equation (3.16) we get $v = m - \rho \left(\frac{G}{2}\right)$. Using equation (3.17), we are able to get a lower bound on this case which is $m^* + \alpha m \geq \alpha \rho \left(\frac{G}{2}\right)$

Also, this case is only valid so long as μ is nonnegative which is

$$\mu = \frac{\alpha(1+\Pi^*)}{(1+\rho)} - \Pi > 0 \rightarrow \frac{\Pi(1+\rho)}{1+\Pi^*} < \alpha < 1$$

must hold.

3.3.21 Case 21: $\mu > 0, \lambda > 0, \lambda^* = 0, v = 0, w > 0$

In this case, we are considering what happens when we change some, but not all, gold jewelry to gold coin (which is why even though $w > 0, \lambda^* = 0$ still holds.) This will only happen when the value of gold jewelry is more than gold coin. Equations (3.14) and (3.15) give us an expression for λ which can be used to solve for μ .

$$\lambda = \frac{(1 + \Pi^*)}{\alpha} - (\Pi + \mu) = \rho(\Pi + \mu) \rightarrow \frac{(1 + \Pi^*)}{\alpha} = (1 + \rho)(\Pi + \mu) \rightarrow \mu = \frac{(1 + \Pi^*)}{\alpha(1 + \rho)} - \Pi$$

Substituting this in to equation (3.15), we get $\lambda = \frac{\rho(1 + \Pi^*)}{\alpha(1 + \rho)}$. Equations (3.11) and (3.12) with symmetry gives us an expression for p and \bar{p} as follows

$$\begin{aligned} \sqrt{\bar{p}}(\lambda + \Pi + \mu) &= \frac{1}{\sqrt{\bar{p}p}(\Pi + \mu)} \rightarrow \bar{p}p = \frac{1}{(\lambda + \Pi + \mu)(\Pi + \mu)} = \frac{\alpha^2(1 + \rho)}{(1 + \Pi^*)^2} \\ &\rightarrow p = \bar{p} = \frac{\alpha\sqrt{1 + \rho}}{(1 + \Pi^*)} \end{aligned}$$

Substituting values of μ and p in to the square of equation (3.12) and taking in to account symmetry and $b = \bar{p}\bar{q}$, we get

$$\begin{aligned} \frac{\bar{p}\bar{q}}{a - q} &= \bar{p}p^2 \left(\frac{\alpha\sqrt{1 + \rho}}{1 + \Pi^*} \right)^2 \rightarrow \frac{q}{a - q} = p^2 \cdot \frac{\alpha^2(1 + \rho)}{(1 + \Pi^*)^2} = 1 + \rho \rightarrow \frac{a - q}{q} = \frac{1}{1 + \rho} \\ &\rightarrow \frac{a}{q} = \frac{2 + \rho}{1 + \rho} \rightarrow q = \bar{q} = \frac{a(1 + \rho)}{2 + \rho} \end{aligned} \tag{3.30}$$

Since $\lambda > 0$ and $\mu > 0$, equations (3.16) and (3.18) imply $b = \bar{b} = d = \bar{d}$ and we know from equation (3.1) that $d = \frac{G(1 + \rho)}{2}$. We also have a second expression for

$d = \bar{d} = b = \bar{b} = pq$ so if we set these equal to one another we obtain

$$\frac{G(1 + \rho)}{2} = \frac{a\alpha\sqrt{1 + \rho}(1 + \rho)}{(2 + \rho)(1 + \Pi^*)} \rightarrow \frac{\sqrt{1 + \rho}}{2 + \rho} = \frac{G(1 + \Pi^*)}{2a\alpha}$$

which can be solved for ρ using computational methods.

Equation (3.16) can also be used to solve for w substituting in $b = d = \frac{G(1 + \rho)}{2}$

which gives

$$m - \frac{G(1 + \rho)}{2} + \alpha w + \frac{G}{2} = 0 \rightarrow \alpha w = \frac{\rho G}{2} - m \rightarrow w = \frac{\rho G - 2m}{2\alpha}$$

Using equation (3.17) we obtain a bound on the values of m and m^* which is

$$m^* - w > 0 \rightarrow m^* - \frac{\rho G - 2m}{2\alpha} > 0 \rightarrow m + \alpha m^* > \rho \left(\frac{G}{2} \right)$$

This case is valid so long as $\mu > 0$ is satisfied, thus

$$\mu = \frac{1 + \Pi^*}{\alpha(1 + \rho)} - \Pi > 0 \rightarrow 0 < \alpha < \frac{1 + \Pi^*}{\Pi(1 + \rho)}$$

must also hold.

3.3.22 Case 22: $\mu > 0, \lambda > 0, \lambda^* > 0, v = 0, w = 0$

This case will not happen for the exact reasons it could not happen in Cases 4, 5, 10, and 11 of Chapter 2. There is no conversion from coin to jewelry or jewelry to coin.

Since we can't spend jewelry, and we aren't transferring jewelry to coin, there is no reason for our jewelry constraint to be driven to zero. Thus, it will never happen that $\lambda^* > 0$ when $w = 0$.

3.3.23 Case 23: $\mu > 0, \lambda > 0, \lambda^* > 0, v > 0, w = 0$

This case will not happen for the exact reasons it could not happen in Cases 4, 5, 10, and 11 of Chapter 2. There is no conversion from coin to jewelry or jewelry to coin. Since we can't spend jewelry, and we aren't transferring jewelry to coin, there is no reason for our jewelry constraint to be driven to zero. Thus, it will never happen that $\lambda^* > 0$ when $w = 0$.

3.3.24 Case 24: $\mu > 0, \lambda > 0, \lambda^* > 0, v = 0, w > 0$

Here, $w > 0$ and $\lambda^* > 0$ imply that we change all our jewelry to coin and thus the value of coin must be more than jewelry. Equation (3.17) verifies this since $\lambda^* > 0$ means $w = m^*$. Equation (3.1) tells us that $d = \bar{d} = \frac{G(1+\rho)}{2}$. Just as in 3.3.19 Case 19, since $\lambda > 0$ and $\mu > 0$, equations (3.16) and (3.18) imply that $b = \bar{b} = d = \bar{d}$. Equations (3.14) and (3.15) give us two equations for λ which we set equal to one

another and obtain the following expression for λ^*

$$\lambda = \rho(\Pi + \mu) = \frac{1}{\alpha}(1 + \Pi^* + \lambda^*) - \Pi - \mu \rightarrow \lambda^* = \alpha(1 + \rho)(\Pi + \mu) - (1 + \Pi^*)$$

To solve for p and \bar{p} we begin by substituting equation (3.15) for λ in to equation (3.11) and simplifying.

$$\sqrt{\frac{a-q}{b}} = \sqrt{\bar{p}}(\rho(\Pi + \mu) + (\Pi + \mu)) = \sqrt{\bar{p}}(1 + \rho)(\Pi + \mu) \quad (3.31)$$

Setting this equal to the reciprocal of equation (3.12) and taking in to account symmetry, we solve for p and \bar{p} .

$$\sqrt{\bar{p}}(1 + \rho)(\Pi + \mu) = \frac{1}{\sqrt{\bar{p}p}(\Pi + \mu)} \rightarrow \bar{p}p = \frac{1}{(1 + \rho)(\Pi + \mu)^2} \rightarrow p = \bar{p} = \frac{1}{\sqrt{1 + \rho}(\Pi + \mu)}$$

We now use our balance equation $b = \bar{p}\bar{q}$ and substitute in the value for \bar{p} in to the square of equation (3.31) to solve for q and \bar{q}

$$\begin{aligned} \frac{a-q}{\bar{p}q} &= \bar{p}(1 + \rho)^2(\Pi + \mu)^2 \rightarrow \frac{a-q}{q} = \bar{p}^2(1 + \rho)^2(\Pi + \mu)^2 = \frac{(1 + \rho)^2(\Pi + \mu)^2}{(1 + \rho)(\Pi + \mu)^2} = 1 + \rho \\ &\rightarrow \frac{a}{q} = 2 + \rho \rightarrow q = \bar{q} = \frac{a}{2 + \rho} \end{aligned}$$

The balance equation $b = \bar{p}\bar{q}$ gives us a second expression for b which we can set equal to one another and find an expression for μ

$$\begin{aligned} b = \bar{p}\bar{q} &= \frac{a}{\sqrt{1 + \rho}(2 + \rho)(\Pi + \mu)} = \frac{G(1 + \rho)}{2} \rightarrow \frac{2a}{G(1 + \rho)^{3/2}(2 + \rho)} = \Pi + \mu \\ &\rightarrow \mu = \frac{2a}{G(1 + \rho)^{3/2}(2 + \rho)} - \Pi \end{aligned}$$

We get a third expression for b from equation (3.16) as follows

$$m - b + \alpha w + \frac{d}{1 + \rho} = m - b + \alpha m^* + \frac{G}{2} = 0 \rightarrow b = m + \alpha m^* + \frac{G}{2}$$

We can set this equal to $b = \frac{G(1+\rho)}{2}$ in order to solve for ρ .

$$m + \alpha m^* + \frac{G}{2} = \frac{G(1 + \rho)}{2} \rightarrow \rho = \frac{2(m + \alpha m^*)}{G}$$

This case is valid so long as μ is nonnegative, thus the following inequality

$$\begin{aligned} \mu = \frac{2a}{G(1 + \rho)^{3/2}(2 + \rho)} - \Pi > 0 &\rightarrow \frac{2a}{G(1 + \rho)^{3/2}(2 + \rho)} > \Pi \rightarrow \frac{2a}{G\Pi} > (1 + \rho)^{3/2}(2 + \rho) \\ &\rightarrow \frac{a}{G\Pi} > \left(1 + \frac{2(m + \alpha m^*)}{G}\right)^{3/2} \left(1 + \frac{m + \alpha m^*}{G}\right) \end{aligned}$$

must hold. Note that since $b = m + \alpha m^* + \frac{G}{2}$, the traders are essentially spending all their money, plus half of what the bank has available. So $b + \bar{b} = 2m + 2\alpha m^* + G$ means that all the possible gold coin in the market is being spent on purchasing goods.

Chapter 4

Conclusion

In their book *Barley, Gold and Fiat: A Pure Theory of Money*, Tom Quint and Martin Shubik presented a "basic model" in an economy where traders used storable consumables as a form of money. They found that the traders were able to achieve efficient trade when they had $m \geq \frac{a}{2}$ units of money, and that this model behaved very similar to a model that used fiat, which they discussed and analyzed in Chapter 7 of their book. When they changed the type of money to gold that was only used in one form, they found that more often than not, traders were not able to reach efficient trade, due to the value in the services that gold provided as an ornament. They then proposed the idea to have gold in two forms, and see if allowing the traders to convert

from one form to the other would result in more efficient trading. This thesis is the work behind that idea, and what was found was that when gold was able to be in two forms, money or ornament, efficient trade occurred more often than if the gold could only take on a single form. In fact, there were instances that the two-types-of-gold model behaved similarly to the fiat model, and the analysis from the fiat model in Quint-Shubik's work could be applied. So, the model with two types of gold is the bridge that connects the model with one type of gold to the model that uses fiat as money in Quint-Shubik's *Barley, Gold and Fiat: A Pure Theory of Money*.