University of Nevada, Reno

# Combinatorial Knot Floer Homology

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

by

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ABSTRACT

Knot Floer Homology $\widehat{HFK}(K)$ of a knot $K$ in $S^3$ was defined by Peter Ozsváth and Zoltan Szabó in 2003. It has since become powerful invariant for the study of properties of knots. The definition of the Knot Floer Homology groups initially involved counting holomorphic disks in symmetric products of Riemann surfaces, which made them difficult to compute.

In 2006, in a paper titled "A combinatorial Description of Knot Floer Homology", the authors Ciprian Manolescu, Peter Ozsváth, and Sucharit Sarkar discovered an algorithm for purely combinatorial description of knot Floer homology, making its computation, in principle, fully accessible. This thesis defense will describe their combinatorial algorithm along with examples and applications.

# Contents

CHAPTER 1

# Introduction

Knot theory is the study of embeddings of circles in space. It is a part of topology that studies the properties of deformable spaces that remain unchanged by continuous transformations that can be undone.

Chapter 2 begins with giving a concrete definition of a knot/link followed by some examples. Soon after, the Seifert's surface is introduced which then leads into a discussion on knot invariants, namely, the knot signature and the Alexander-Conway polynomial. The remaining chapters give a description of the knot Floer homology followed by a computational example and its applications.

Heegaard Floer homology is an invariant due to Peter Ozsváth and Zoltán Szabó of closed, oriented three-manifolds equipped with a $\text{Spin}^c$ structure. This invariant is computed using a Heegaard diagram of the space via a construction that is analogous to Lagrangian Floer homology [**10**]. The Heegaard Floer homology extends to give an invariant denoted $\widehat{HFK}(Y, L)$ for null-homologous knots/links $L$ in a closed, oriented three-manifold $Y$.

For any null-homologous knot $K$ in a three-dimensional manifold $Y$ with Seifert surface $F$, one can associate to it the Abelian groups $\widehat{HFK}(Y, K, [F], i)$ where $i \in \mathbb{Z}$. The knot Floer homology $\widehat{HFK}(Y, K) \cong \bigoplus_{i \in \mathbb{Z}} \widehat{HFK}(Y, K, [F], i)$ is a finitely generated Abelian group [**6**].

The knot Floer homology holds several topological information about the knot. For instance, it can detect fibered knots in $S^3$. Additionally, Ozsváth and Szabó proved that the topmost filtration level of nonzero $\widehat{HFK}(S^3, K)$ is exactly the genus of the knot [**7**][**6**]. Furthermore, the knot Floer homology $\widehat{HFK}(S^3, K)$ is related to the Alexander polynomial of $K$ and is a strictly stronger invariant than the Alexander polynomial [**8**].

Several authors, with Sucharit Sarkar being one of the initial authors, have given combinatorial constructions of knot Floer homology of the knot grid diagram presentations of knots in $S^3$ [**4**]. There is an analogous construction for oriented links [**5**].

Chapters 3-6 describes this combinatorial construction. These chapters explain how to present a knot diagram in a grid on a coordinate plane. Then these chapters work towards defining the chain complex of the grid diagram and the boundary operator. Chapter 6 states an important theorem on computing the knot Floer homology of a given knot. Lastly, Chapter 7 uses the applications of $\widehat{HFK}(S^3, K)$ mentioned above to make genus bounds and fibration conclusions about certain knots.

CHAPTER 2

# Knots/Links

This chapter introduces various definitions and theorems pertaining to knots/links **[1][11][3]**.

## 2.1. Definitions

DEFINITION 2.1.1 (Embedding). A function $f : \bigsqcup_{i=1}^{n} S^1_i \to S^3$ (or $\mathbb{R}^3$) is called an embedding if it is injective and of class at least $C^1$.

DEFINITION 2.1.2 (Isotopic embedding). Two embeddings $f, g : \bigsqcup_{i=1}^{n} S^1_i \to S^3$ are called isotopic if there exists a map $F : \bigsqcup_{i=1}^{n} S^1 \times [0,1] \to S^3$ of class at least $C^1$ such that for $t \in S^1$

(i) $F(t,0) = f(t)$,

(ii) $F(t,1) = g(t)$,

(iii) $F|_{S^1\times\{s\}} : \bigsqcup_{i=1}^{n} S^1 \to S^3$ is an embedding.

PROPOSITION 2.1.1. Isotopy forms an equivalence relation.

PROOF. The proof of reflexivity, symmetry, and transitivity is as follows:

(i) (Reflexivity) $f$ is isotopic to $f$.

In this case, the $C^1$ map $F : S^1 \times [0,1] \to S^3$ can be defined as $F(t,s) = f(t)$ and $F(t,1) = f(t)$. Since $F(t,s) = f(t)$ for $t \in S^1$ and $s \in [0,1]$, $F|_{S^1\times\{s\}} = f(t)$ and $F|_{S^1\times\{s\}}$ is an embedding. Thus $f$ is isotopic to $f$.

(ii) (Symmetry) If $f$ is isotopic to $g$ then $g$ is isotopic to $f$.

Since $f$ is isotopic to $g$, there exists a $C^1$ map $F : S^1 \times [0,1] \to S^3$ such that $F(t,0) = f(t)$ and $F(t,1) = g(t)$ with the property that $F|_{S^1\times\{s\}}$ is an embedding. Next, define a $C^1$ map $G : S^1 \times [0,1] \to S^3$ by $G(t,s) = F(t,1-s)$ for $t \in S^1$ and $s \in [0,1]$. Then $G(t,0) = F(t,1) = g(t)$ and $G(t,1) = F(t,0) = f(t)$. Observe that $G|_{S^1\times\{s\}}$ is an embedding since it is equal to the embedding $F|_{S^1\times\{1-s\}}$. This concludes that $g$ is isotopic to $f$.

(iii) (Transitivity) If $f$ is isotopic to $g$ and $g$ is isotopic to $h$ then $f$ is isotopic to $h$.

Since $f$ is isotopic to $g$, there exists a $C^1$ map $F : S^1\times[0,1] \to S^3$ such that $F(t,0) = f(t)$, $F(t,1) = g(t)$, and $F|_{S^1\times\{s\}}$ is an embedding. Since $g$ is isotopic to $h$, there exists a $C^1$ map $G : S^1 \times [0,1] \to S^3$ such that $G(t,0) = g(t)$, $G(t,1) = h(t)$, and $G|_{S^1\times\{s\}}$ is an embedding. Then $F$ and $G$ together yield a map $H : S^1\times[0,1] \to S^3$

defined by

$$H(t,s) = \begin{cases} F(t,2s) & 0 \leq s \leq \frac{1}{2} \\ G(t,2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

such that $H(t,0) = F(t,0) = f(t)$ and $H(t,1) = G(t,1) = h(t)$. Lastly,

$$H|_{S^1\times\{s\}} = \begin{cases} F|_{S^1\times\{2s\}} & 0 \leq s \leq \frac{1}{2} \\ G|_{S^1\times\{2s-1\}} & \frac{1}{2} \leq s \leq 1 \end{cases}$$

is an embedding since $F|_{S^1\times\{2s\}}$ and $G|_{S^1\times\{2s-1\}}$ are both embeddings. (It is hard to see that $H$ can be perturbed to class $C^1$.) Thus $f$ is isotopic to $h$.

□

Definition 2.1.3 (Knot/Link). A link is an isotopy class of an embedding $f : (\coprod_{i=1}^{n} S_i^1) \to S^3$ (or $\mathbb{R}^3$). When $n = 1$, the link is called a knot. An orientation on a knot/link is an orientation of their domain.

Definition 2.1.4 (Split link). A split link is a union of two links seperated by disjoint spheres.

Definition 2.1.5 (Mirror image of a knot/link). Let $f : (\coprod_{i=1}^{n} S_i^1) \to S^3$ be a map representing a knot/link $K$ and let $r : S^3 \to S^3$ be a reflection across the hyperplane. The knot/link $r \circ f$ is called the mirror of $K$.

Definition 2.1.6 (Reverse of a knot/link). If $L$ is an oriented knot/link then the knot/link with the opposite orientation, denoted $-L$, is called the reverse of $L$.

Definition 2.1.7 (Amphicheiral knot). A knot $K$ is said to be amphicheiral if it is isotopic to its mirror image $\overline{K}$.

## 2.2. Examples of Knots

It is common place to represent a knot/link by a diagram, that is, a projection of the image of the embedding of the knot in $S^3$ onto a two dimensional plane. As such each knot/link can be represented by many different diagrams but each diagram corresponds to a unique knot/link. Going forward, all knots/links will be represented with its diagrams.

If a knot/link is oriented, its orientation will be indicated with arrows on the knot/link diagram, and such diagrams will be referred to as oriented diagrams. Furthermore, if all overcrossings are replaced with undercrossings and vice-versa in a diagram, the resulting diagram represents the mirror knot/link of the knot/link represented by the original diagram.

Some of the common examples of knots are the unknot, figure-8 knot, and $(2, n)$-torus knots denoted by $T_{2,n}$ (odd $n$). The torus knot $T_{2,n}$ ($n$ odd) is said to be right-handed when $n > 0$ and left-handed when $n < 0$. In particular, $T_{2,3}$ is called a right-handed trefoil and $T_{2,-3}$ is called a left-handed trefoil. Furthermore, $T_{2,n}$ becomes a link when $n$ is even. More examples of links include the unlink and Hopf link.

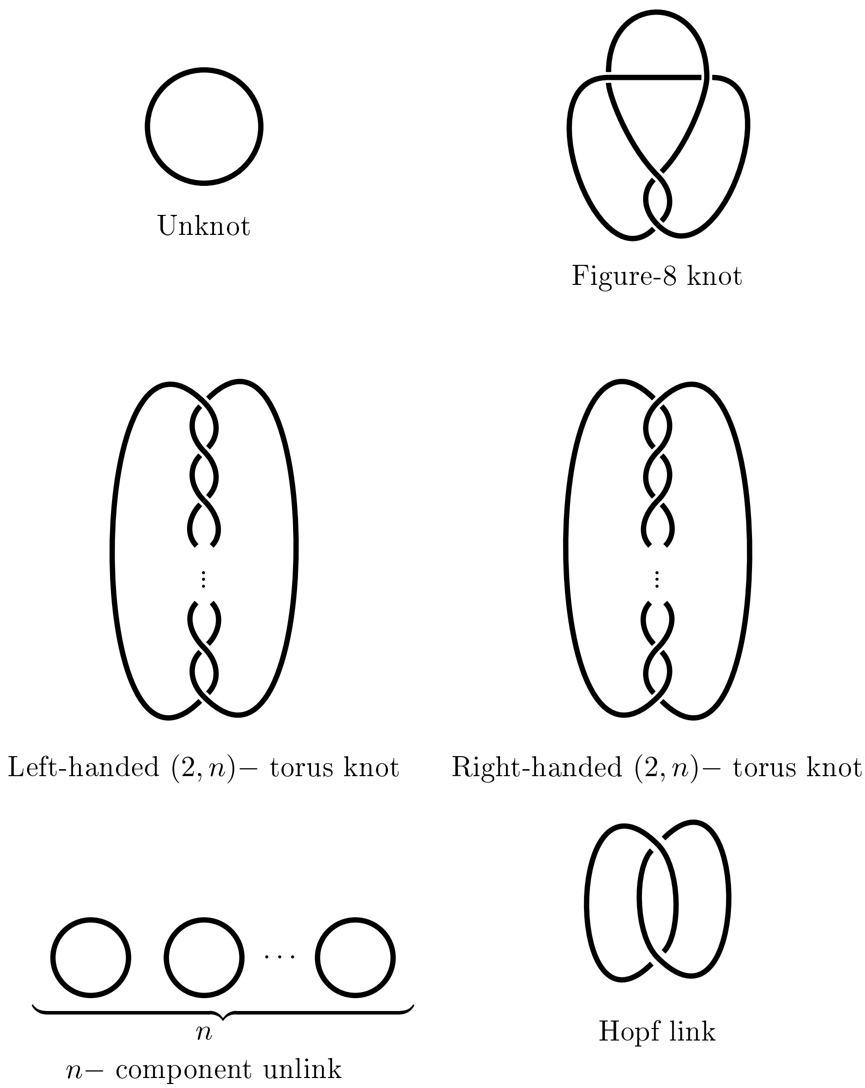



FIGURE 2.2.1. Examples of knots and links.

Example 2.2.1. An unlink is a split link since each one of its components can be embedded in a sphere and the union of these spheres is disjoint.

Example 2.2.2. The Hopf link is not a split link since each component can not be seperated by disjoint spheres. A proof of this is given in Section 2.5.

## 2.3. Seifert's Algorithm

Any knot/link in $S^3$ can be thought of as a boundary of some orientable surface embedded in $S^3$. This section describes how to obtain such a surface for a given knot/link.

As shown in Figure 2.3.1, $K_+$, $K_-$ and $K_0$ are three oriented knots that are identical except near a small neighborhood of a crossing. Notice that interchanging the under- and overcrossing of $K_+$ produces $K_-$ and vice-versa . This change is called switching a crossing. Eliminating the crossing of either $K_+$ or $K_-$ by swapping the incoming strand of the knot with the outgoing strand and vice-versa produces $K_0$. Such a change is called smoothing a crossing [**1**].

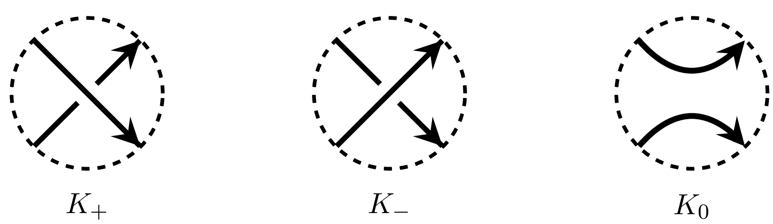



Figure 2.3.1. Three oriented diagrams which differ only inside a small neighborhood.

Definition 2.3.1 (Seifert surface). A Seifert surface for an oriented knot/link $L$ in $S^3$ is a connected compact oriented surface contained in $S^3$ that has $L$ as its oriented boundary [**3**].

Algorithm (Seifert's algorithm). A Seifert surface is obtained from the following algorithm called the Seifert's algorithm [**11**].

(1) Fix an orientation in the diagram of the knot/link.
(2) At each crossing, eliminate the over- and undercrossings by smoothing them.
(3) This produces a disjoint collection of oriented circles in the plane called Seifert circles. Each Seifert circle bounds a disk in the plane. Although these disks are

nested, they can be made disjoint by pushing their interiors slightly off the plane in a perpendicular fashion, starting with the innermost ones and working outward. Moreover, these disks can be assigned a + and − side according to the convention that the oriented boundary runs counterclockwise as seen from the + side.

(4) Connect these disks together at the old crossings with half-twisted strips. The orientation of the twists should correspond to the orientation of the crossings that they replace.

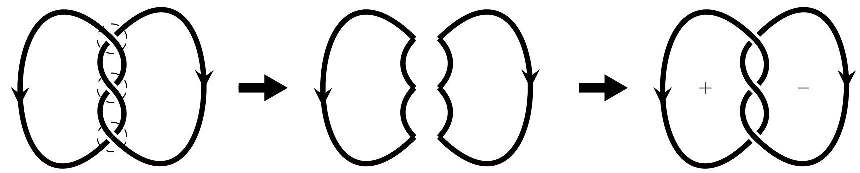
(5) This produces a surface whose boundary is the original knot/link.

It remains to show that a Seifert surface is orientable. If a strip connects two disks at the same height perpendicular to the plane then they must have different signs on their upper surfaces. Thus, the signs can be extended naturally across the half-twisted strips. If a strip connects two disks at different heights then they will same signs on their upper surfaces, and the half-twist brings the upper side of the lower disk to match with the upper side of the upper disk. Therefore, the surface is orientable. An orientable surface is two-sided [**1**].

Definition 2.3.2 (Genus of an orientable surface). Let $F$ be a connected compact orientable surface with $\partial F \cong \bigsqcup_{i=1}^{n} S^1$. Then the genus $g = g(F)$ is defined as the integer with $H_1(F) \cong \mathbb{Z}^{2g}$.

Definition 2.3.3 (Genus of a knot/link). The genus of a knot/link $L$ is the smallest genus of any Seifert surface of $L$.

Example 2.3.1. The sequence of diagrams in Figure 2.3.2 shows how to obtain a Seifert surface of a left-handed trefoil. Consider each crossing of the knot as shown in Figure (a). Eliminate these crossings to obtain two disjoint oriented Seifert circles as shown in Figure (b). These circles can be thought of as bounding two disks. At the old crossings, connect these disks with half-twisted strips whose orientation corresponds to the orientation of the old crossings to obtain a Seifert surface as shown in Figure (c). Observe that this is a two-sided surface with the + and − sides as shown in Figure (c).



(a) (b) (c)

Figure 2.3.2. A Seifert surface of a left-handed trefoil.

Example 2.3.2. A Seifert surface of a right-handed trefoil can be obtained similarly and is described by the sequence of diagrams in Figure 2.3.3.

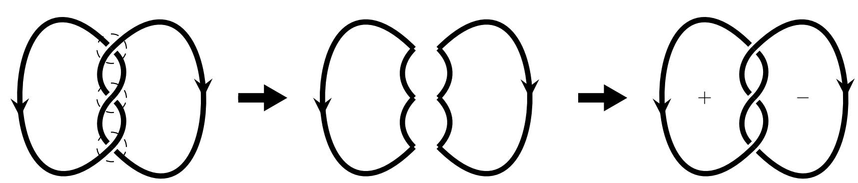


Figure 2.3.3. A Seifert surface of a right-handed trefoil.

Example 2.3.3. Example 2.3.1 can be extended to a general left-handed torus knot $T_{2,n}$ ($n < 0$ and $n$ is odd). Its Seifert surface is given in Figure 2.3.4.

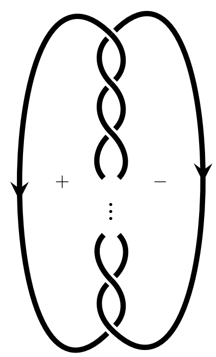



FIGURE 2.3.4. A Seifert surface of a left-handed torus knot.

EXAMPLE 2.3.4. Similarly, Example 2.3.2 can be generalized to a right-handed torus knot $T_{2,n}$ ($n > 0$ and $n$ is odd) whose Seifert surface is given in Figure 2.3.5.

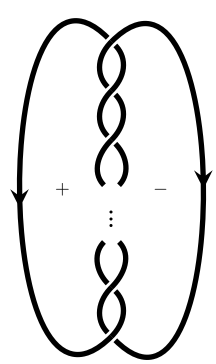



FIGURE 2.3.5. A Seifert surface of a right-handed torus knot.

EXAMPLE 2.3.5. The sequence of diagrams in Figure 2.3.6 shows how to obtain a Seifert surface of a figure-8 knot.

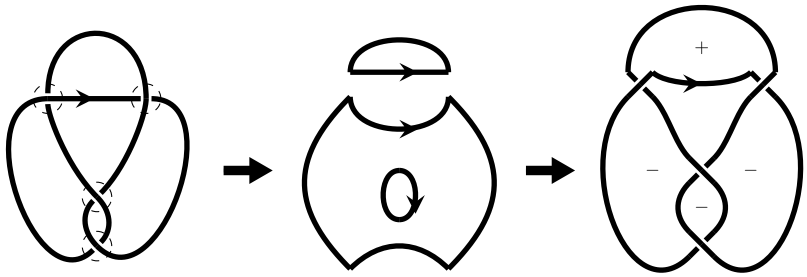



FIGURE 2.3.6. A Seifert surface of a figure-8 knot.

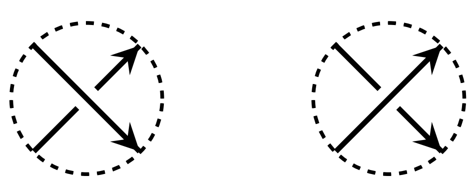
## 2.4. Signature of Knots

The signature of a knot is a numerical invariant which can be calculated explicitly. The discussion in this section will lead to a more complicated polynomial invariant of the knot/link.

DEFINITION 2.4.1 (Linking number). Let $D$ be an oriented diagram of a 2-component link $L_1 \cup L_2$. Let $c$ be a crossing in $D$ and define

$$\epsilon(c) = \begin{cases} +1 & \text{if } c \text{ is a positive crossing,} \\ -1 & \text{if } c \text{ is a negative crossing,} \end{cases}$$

as shown in Figure 2.4.1. Let $D_i$ denote the component of $D$ corresponding to $L_i$. The crossings of $D$ are of three types: $D_1$ with itself, $D_2$ with itself, and $D_1$ with $D_2$. We shall concentrate on the last type which will be denoted by $D_1 \cap D_2$. The linking number of $D_1$ with $D_2$ is defined to be $\operatorname{lk}(D_1, D_2) = \frac{1}{2} \sum_{c \in D_1 \cap D_2} \epsilon(c)$ [**1**].



+1 −1

FIGURE 2.4.1. Positive and negative crossings.

Definition 2.4.2 (Winding number). The winding number of a knot diagram $D$ with respect to a point $p$ in the plane but not on the knot diagram is the linking number $\mathrm{lk}(D, U_p)$ where $U_p$ is the unknot formed by a segment perpendicular to the plane and passing through $p$, with its endpoints connected by an arc away from the knot diagram $D$. The orientation of $U_p$ points in the positive direction of the $z$-axis.

Definition 2.4.3 (Seifert form). Let $F$ be a Seifert surface of a given knot/link $K$. Let $b : F \times [-1, 1] \to S^3 - K$ be an embedding such that $b(F \times \{0\}) = F$ and $b(F \times \{1\})$ lies on the positive side of $F$. Any subset $G \subset F$ can be lifted out of the surface on either the positive or negative side, so write $G^+ = b(G \times \{1\})$ and $G^- = b(G \times \{-1\})$ respectively. Define a map $\phi : H_1(F) \times H_1(F) \to \mathbb{Z}$ by $(x, y) \mapsto \mathrm{lk}(x, y^+)$ where$x$, $y$ are loops representing 1-cycles of $H_1(F)$. This map is called the Seifert form [**1**].

Definition 2.4.4 (Seifert matrix). Let $\{a_1, a_2, \cdots, a_{2g}\}$ be a basis for $H_1(F) \cong \mathbb{Z}^{2g}$ where $g$ is the genus of $F$. Then the Seifert matrix $V = [v_{i,j}]$ is the $2g \times 2g$ integer matrix with entries $v_{i,j} = \mathrm{lk}(a_i, a_j^+)$ for all $i, j = \{1, 2, \cdots, 2g\}$.

Definition 2.4.5 (Signature of a knot). The signature of a knot $K$, denoted $\sigma(K)$, is the number of positive eigenvalues of the matrix $V + V^T$ minus the number of negative eigenvalues of the matrix $V + V^T$.

Theorem 2.4.1. For any knot $K$, $\sigma(\overline{K}) = -\sigma(K)$. Consequently, all amphicheiral knots have signature zero [**1**].

Proof. Let $D$ be the knot diagram of the knot $K$ and let $F$ be a Seifert surface for $D$. Then $\overline{K}$ is obtained by reflecting $K$ across a hyperplane and let $\overline{F}$ be the image of $F$ under this reflection. Switching all the crossings in $D$ yields the mirror image $\overline{K}$ of the knot. This results in switching the signs of all linking numbers used in the calculation of the entries of the Seifert matrix $V$. Hence the Seifert matrix for $\overline{K}$ is $-V$. Thus $\sigma(\overline{K}) = -\sigma(K)$. Since all amphicheiral knots are isotopic to its mirror image, their signatures are equal to each other which is only possible if the signatures are zero. □

Example 2.4.1. The Seifert surface of a right-handed trefoil $K$ found in Example 2.3.2 is redrawn in Figure 2.4.2 with bases $a_1$, $a_2$ of $H_1(K)$ oriented counterclockwise.

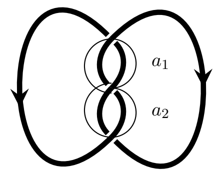



FIGURE 2.4.2. A Seifert surface of a right-handed trefoil with bases $a_1$, $a_2$ of $H_1(K)$ oriented counterclockwise.

Note that the two circles intersect on the left hand side of the knot but not on the right hand side. The Seifert matrix $V$ is

$$\begin{pmatrix} \operatorname{lk}(a_1, a_1^+) & \operatorname{lk}(a_1, a_2^+) \\ \operatorname{lk}(a_2, a_1^+) & \operatorname{lk}(a_2, a_2^+) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Recall that the eigenvalues of the matrix $V + V^T$ can be found by solving the equation $\det(V + V^T - tI) = 0$ for $t$ ($I$ is the identity matrix). In this example, solving $\det \begin{pmatrix} -2-t & 1 \\ 1 & -2-t \end{pmatrix} = 0$ for $t$ yields $t = -1, -3$. Since there are two negative eigenvalues and no positive eigenvalues, the signature of the right-handed trefoil is $\sigma(K) = 0 - 2 = -2$.

EXAMPLE 2.4.2. The signature of a left-handed trefoil is 2, by Theorem 2.4.1.

EXAMPLE 2.4.3. Example 2.4.1 can be generalized to compute the signature of a right-handed torus knot $T_{2,n}$ ($n > 0$ and $n$ odd). Let $\{a_1, a_2, \cdots, a_{n-1}\}$ be the bases for $H_1(T_{2,n})$. Then the $(n-1) \times (n-1)$ Seifert matrix $V$ is of the form

$$\begin{pmatrix} \operatorname{lk}(a_1, a_1^+) & \operatorname{lk}(a_1, a_2^+) & \operatorname{lk}(a_1, a_3^+) & \cdots & \operatorname{lk}(a_1, a_{n-1}^+) \\ \operatorname{lk}(a_2, a_1^+) & \operatorname{lk}(a_2, a_2^+) & \operatorname{lk}(a_2, a_3^+) & \cdots & \operatorname{lk}(a_2, a_{n-1}^+) \\ \operatorname{lk}(a_3, a_1^+) & \operatorname{lk}(a_3, a_2^+) & \operatorname{lk}(a_3, a_3^+) & \cdots & \operatorname{lk}(a_3, a_{n-1}^+) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \operatorname{lk}(a_{n-1}, a_1^+) & \operatorname{lk}(a_{n-1}, a_2^+) & \operatorname{lk}(a_{n-1}, a_3^+) & \cdots & \operatorname{lk}(a_{n-1}, a_{n-1}^+) \end{pmatrix}.$$

Notice that when the indices $i$ and $j$ differ from each other by at least 2 ignoring the sign, the circles $a_i$ and $a_j$ are already seperate from each other in the Seifert surface so their linking number is zero. Thus $\text{lk}(a_i, a_j^+) = 0$ when $|i-j| \geq 2$. Generalizing the linking numbers obtained in Example 2.4.1 gives that

$$\begin{cases} \text{lk}(a_i, a_j^+) = -1 & j = i, \\ \text{lk}(a_i, a_j^+) = 1 & j = i+1, \\ \text{lk}(a_i, a_j^+) = 0 & j = i-1. \end{cases}$$

Thus, the Seifert matrix becomes

$$V = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}_{(n-1)\times(n-1)}.$$

It is a known fact that if a $n \times n$ matrix is negative definite then the signature of the knot $K$ is $-n$. We will apply this fact on the $(n-1)\times(n-1)$ matrix $V + V^T$ to get that the signature of a right-handed torus knot is $-(n-1) = 1-n$. The matrix $V + V^T = A$ will be shown to be negative definite by an induction argument on its determinant.

$$\det A_1 = \det \begin{pmatrix} -2 \end{pmatrix} = -2 < 0$$

$$\det A_2 = \det \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = 3 > 0$$

$$\det A_3 = \det \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} = -4 < 0$$

The general pattern observed from the calculations above is $\det A_m = (-1)^m(m+1)$ for $1 \leq m \leq n-1$. Since $\det A_m = \sum_{j=1}^{n-1} a_{ij}C_{ij}$ where $C_{ij}$ is the cofactor of $a_{ij}$, $\det A_m = -2 \cdot \det A_{m-1} - 1 \cdot \det M_{12}$ where $M_{12}$ is the matrix obtained by deleting the first row and second column of $A_m$. Again by taking the cofactor approach to calculate the determinant of $M_{12}$ we get that $\det M_{12} = \det A_{m-2}$. Thus,

$$\begin{aligned}\det A_m &= -2 \cdot \det A_{m-1} - 1 \cdot M_{12} \\ &= ((-2)(-1)^{m-1}m) - ((-1)^{m-2}(m-1)) \\ &= -2m(-1)^{m-1} + (-1)^{m-1}(m-1) \\ &= (-1)^{m-1}(-m-1) \\ &= (-1)^m(m+1).\end{aligned}$$

Therefore, $A = V + V^T$ is a negative definite matrix and the signature of a left-handed torus knot $T_{2,n}$ is $1 - n$.

EXAMPLE 2.4.4. The signature of a left-handed torus knot $T_{2,n}$ ($n < 0$ and $n$ odd) is $n - 1$, by Theorem 2.4.1.

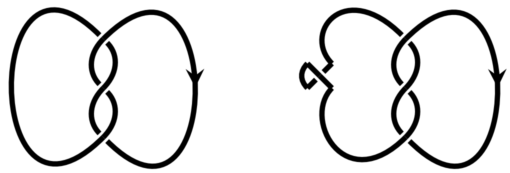
As a result of this theorem, it can be concluded that the left- and right-handed trefoils are not amphicheiral.

**2.4.1. Signature of an Alternating Knot.** This section gives an explicit formula for computing the signature of a specific type of knot called the alternating knot.

DEFINITION 2.4.6 (Alternating diagram). A knot diagram is said to be alternating if the over- and undercrossings of the knot occur in an alternating fashion.

DEFINITION 2.4.7 (Alternating knot). A knot is said to be alternating if it has an alternating diagram without nugatory crossings.

EXAMPLE 2.4.5. The two knots drawn in Figure 2.4.3 are both of a right-handed trefoil. Since the knot diagram on the right has over- and undercrossings in an alternating fashion, it is an alternating knot diagram. However, the knot diagram on the left is not alternating. This shows that an alternating knot can have non-alternating diagrams.



alternating diagram non-alternating diagram

FIGURE 2.4.3. Right-handed trefoil.

Although the method described above can be used to compute the signature of an alternating knot, there is an alternate method available to compute the signatures of these knots.

Consider the knot diagram of an alternating knot $K$ in a plane. The knot diagram divides the plane into a finite number of regions. Note that four regions meet at every crossing. At a crossing, color the regions white that are swept by turning the overcrossing strand counterclockwise until it coincides with the undercrossing strand. Color the remaining two regions black. Note that there must be two such black regions at a given crossing that meet diagonally at the crossing but not along an edge. Proceed to color such regions in black at every crossing. Then the signature of the alternating knot is

$$\sigma(K) = (\text{number of black regions}) - (\text{number of positive crossings}) - 1.$$

EXAMPLE 2.4.6. The results of shading the regions of a figure-8 knot in black and white in an alternating fashion is shown in Figure 2.4.4.

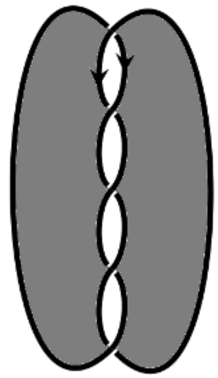


FIGURE 2.4.5. The number of black regions and positive crossings in $T_{2,5}$.

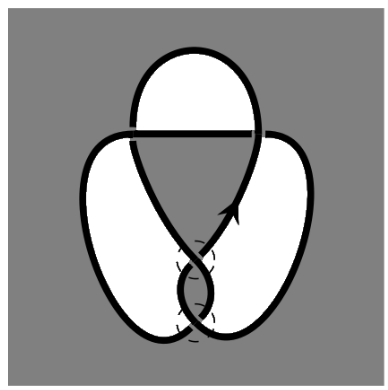


FIGURE 2.4.4. The black regions and positive crossings in a figure-8 knot.

Since there are 3 black regions and 2 positive crossings shown with dashed circles, the signature of a figure-8 knot is

$$\begin{aligned}\sigma(K) &= (\text{number of black regions}) - (\text{number of positive crossings}) - 1 \\ &= 3 - 2 - 1 \\ &= 0.\end{aligned}$$

EXAMPLE 2.4.7. As shown in Figure 2.4.5, $T_{2,5}$ has 2 black regions and each one of its crossing is positive. In general, a right-handed torus knot has $n$ positive crossings and the same 2 black regions. Thus its signature is

$$\begin{aligned}\sigma(T_{2,n}) &= (\text{number of black regions}) - (\text{number of positive crossings}) - 1 \\ &= 2 - n - 1 \\ &= 1 - n\end{aligned}$$

where $n > 0$.

Note that the signature obtained in Example 2.4.7 is consistent with the signature obtained in Example 2.4.1. Also, by Theorem 2.4.1, it can be seen that the left- and right-handed torus knots ($n$ odd) are not amphicheiral.

## 2.5. Conway Polynomial

This section introduces the Conway polynomial which is a polynomial-valued knot/link invariant. It has an axiomatic definition based on relationships between diagrams introduced in Section 2.3.

Definition 2.5.1 (Conway polynomial). The Conway polynomial of an oriented knot/link $K$, denoted $\nabla_K(z)$, is defined by the following three axioms [**1**].

(i) $\nabla_K(z)$ is an invariant under the ambient isotopy of $K$.

(ii) If $K$ is an unknot then $\nabla_K(z) = 1$.

(iii) $\nabla_{K_+}(z) - \nabla_{K_-}(z) = z\nabla_{K_0}(z)$ where $K_+$, $K_-$, $K_0$ as in Figure 2.3.1. This equation is called the Skein relation.

The Conway polynomial is a polynomial in $\mathbb{Z}[z]$.

Theorem 2.5.1. If $L$ is a link, $-L$ is the reverse of $L$ and $\overline{L}$ is the mirror of $L$ then $\nabla_{-L}(z) = \nabla_L(z) = \nabla_{\overline{L}}(z)$ [**1**].

Theorem 2.5.2. If $L$ is a split link then $\nabla_L(z) = 0$ [**1**].

Proof. Let $L_0$ be a disconnected diagram of the split link $L_1 \sqcup L_2$ arranged so that $L_1$ and $L_2$ are disjoint. Then $L_+$ can be obtained from the neighborhood defining $L_0$ such that it contains one arc from both $L_1$ and $L_2$ as shown in Figure 2.5.1 and $L_-$ can be obtained by turning the right half of the diagram through a full twist. Therefore, $L_+$ and $L_-$ are isotopic, and by axiom 1, $\nabla_{L_+}(z) = \nabla_{L_-}(z)$. Thus, the skein relation yields $z\nabla_{L_0}(z) = 0$ proving $\nabla_L(z) = 0$.

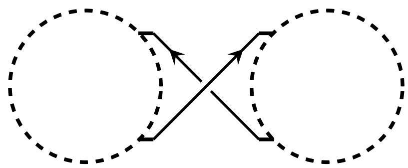


FIGURE 2.5.1

□

EXAMPLE 2.5.1. In this example we compute the Conway polynomial of the Hopf link $\vec{L}$ with the orientation given in Figure 2.5.2.

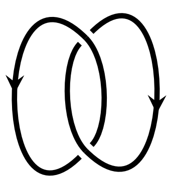
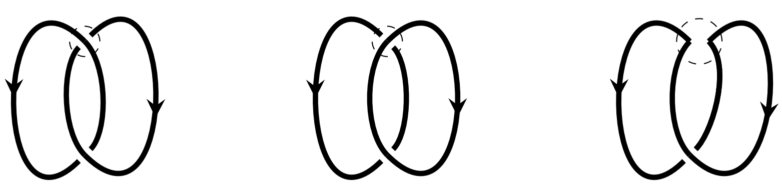


FIGURE 2.5.2. Hopf link with a chosen orientation.

Switching and smoothing a crossing enclosed in the dashed circle of $\vec{L}$ produces $\vec{L}_+$, $\vec{L}_-$ and $\vec{L}_0$ as shown in Figure 2.5.3 where $\vec{L}$=$\vec{L}_-$.

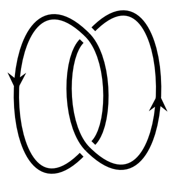


$\vec{L}_-$ $\vec{L}_+$ $\vec{L}_+$

FIGURE 2.5.3. $\vec{L}_+$, $\vec{L}_-$ and $\vec{L}_0$ for the Hopf link with a chosen orientation.

Notice that $\overrightarrow{L}_0$ is the unknot, and so $\nabla_{\overrightarrow{L}_0}(z) = 1$. Since $\overrightarrow{L}_+$ is a split link, $\nabla_{\overrightarrow{L}_+}(z) = 0$ by the previous theorem. Applying the skein relation yields the Conway polynomial of the Hopf link with the given orientation to be $\nabla_{\overrightarrow{L}_-}(z) = -z$.

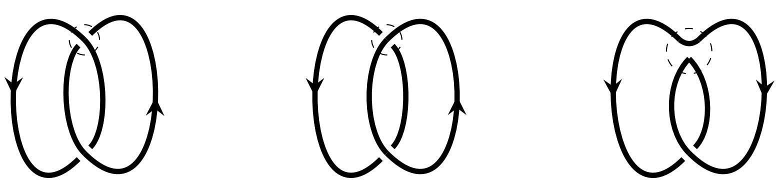
EXAMPLE 2.5.2. The Conway polynomial of the Hopf link $\overleftarrow{L}$ with the orientation given in Figure 2.5.4 is computed as follows.



$L_+$

FIGURE 2.5.4. Hopf link with a chosen orientation.

Consider the oriented link diagrams $\overleftarrow{L}_+$, $\overleftarrow{L}_-$ and $\overleftarrow{L}_0$ in Figure 2.5.5 where $\overleftarrow{L}=\overleftarrow{L}_+$.



$\overleftarrow{L}_+$ $\overleftarrow{L}_-$ $\overleftarrow{L}_0$

FIGURE 2.5.5. $\overleftarrow{L}_+$, $\overleftarrow{L}_-$ and $\overleftarrow{L}_0$ for the Hopf link with the chosen orientation.

Since $\overleftarrow{L}_-$ is a split link, $\nabla_{\overleftarrow{L}_-}(z) = 0$. Since $\overleftarrow{L}_0$ is the unknot, $\nabla_{\overleftarrow{L}_0}(z) = 1$. Thus, applying the skein relation yields the Conway polynomial of the Hopf link with the given orientation to be $\nabla_{\overleftarrow{L}_+}(z) = z$.

EXAMPLE 2.5.3. Let the orientation chosen on the left-handed trefoil $K = K_-$ be as shown in Figure 2.5.6.

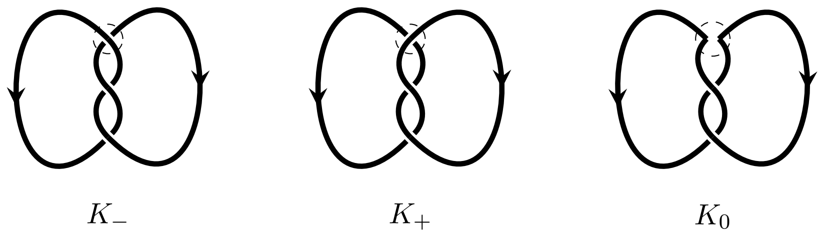



FIGURE 2.5.6. $K_+$, $K_-$ and $K_0$ for a left-handed trefoil with a chosen orientation.

The knot diagrams $K_-$, $K_+$, $K_0$ are obtained after switching and smoothing the crossing inside the dashed circle. Notice that $K_+$ is the unknot and $K_0$ is the Hopf link with the orientation given in Example 2.5.1. Therefore, $\nabla_{K_+}(z) = 1$ and $\nabla_{K_0}(z) = -z$. Thus, the Conway polynomial of the left-handed trefoil is $\nabla_{K_+}(z) = 1 + z^2$.

EXAMPLE 2.5.4. Let the orientation chosen on the right-handed trefoil $K$ be as shown in Figure 2.5.7.

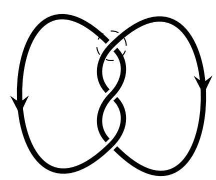
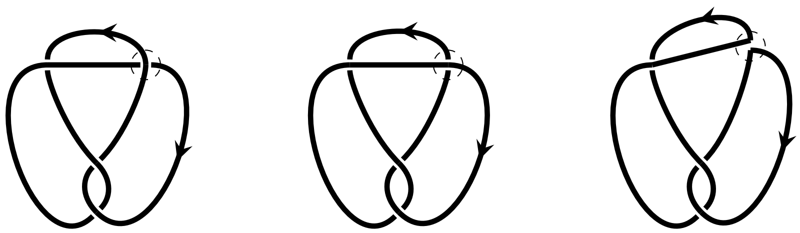


FIGURE 2.5.7. Right-handed trefoil with a chosen orientation.

Then, by Theorem 2.5.1, the Conway polynomial of the right-handed trefoil is $\nabla_K(z) = 1 + z^2$.

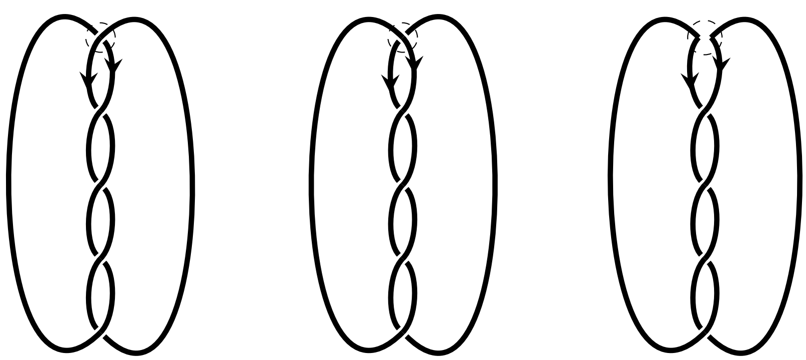
EXAMPLE 2.5.5. Figure 2.5.8 gives the orientation chosen on the figure-8 knot $K = K_-$ and demonstrates the consequent knot diagrams $K_-$, $K_+$, $K_0$ obtained by switching and smoothing a crossing inside the dashed circle. Observe that $K_+$ is the unknot and $K_0$ is isotopic to the Hopf link with the orientation given in Example 2.5.1 so $\nabla_{K_+}(z) = 1$ and $\nabla_{K_0}(z) = -z$. Thus, the Conway polynomial of the figure-8 knot is $\nabla_{K_-}(z) = 1 - z^2$.



$K_-$ $\qquad$ $K_+$ $\qquad$ $K_0$

FIGURE 2.5.8. $K_-$, $K_+$, $K_0$ for a figure-8 knot with the chosen orientation.

EXAMPLE 2.5.6. In this example, we will compute the Conway polynomial of a general right-handed torus knot/link with orientation as given in Figure 2.5.9. However, it will be helpful to first understand the crossings resolutions of a particular odd $n$, say $n = 5$. The knot diagrams $K_+$, $K_-$ and $K_0$ with a chosen orientation that results from switching and smoothing the crossing of $\overrightarrow{T}_{2,5}$ enclosed in dashed circle are shown in Figure 2.5.9.



$K_+$ $\qquad$ $K_-$ $\qquad$ $K_0$

FIGURE 2.5.9. $K_+$, $K_-$, $K_0$ for $\overrightarrow{T}_{2,5}$ with a chosen orientation.

Since $K_+$ is isotopic to $\overrightarrow{T}_{2,3}$, $K_0$ is the link $\overrightarrow{T}_{2,4}$ and $K_-$ is the knot itself, we have $\nabla_{\overrightarrow{T}_{2,5}} - \nabla_{\overrightarrow{T}_{2,3}} = z \cdot \nabla_{\overrightarrow{T}_{2,4}}$. From this calculation, we can generalize that for a given $n$, the

skein relation is $\nabla_{\vec{T}_{2,n}} - \nabla_{\vec{T}_{2,n-2}} = z \cdot \nabla_{\vec{T}_{2,n-1}}$. If $\nabla_{\vec{T}_{2,n}}(z)$ is set to be equal to $X_n$, the skein relation becomes $X_n - X_{n-2} = z \cdot X_{n-1}$. A general formula for $X_n$ can be obtained by solving this recurrence relation with the two initial conditions that $X_0 = \nabla_{\vec{T}_{2,0}}(z) = 0$ since $\vec{T}_{2,0}$ is a split link, and $X_1 = \nabla_{\vec{T}_{2,1}}(z) = 1$ since $\vec{T}_{2,1}$ is an unknot.
The characteristic equation of this recurrence relation becomes $\alpha^2 - z\alpha - 1 = 0$. Upon solving this quadratic equation we get that $\alpha = \dfrac{z \pm \sqrt{z^2+4}}{2}$. Let $\alpha_1 = \dfrac{z + \sqrt{z^2+4}}{2}$ and $\alpha_2 = \dfrac{z - \sqrt{z^2+4}}{2}$, then $X_n = a\alpha_1^n + b\alpha_2^n$ where $a$ and $b$ are constants. These constants can be attained by using the two intial conditions of $X_n$. Thus $1 = a\alpha_1 - a\alpha_2$ yields $a = \dfrac{1}{\alpha_1 - \alpha_2} = \dfrac{1}{\sqrt{z^2+4}}$ and $a\alpha_1^0 + b\alpha_2^0 = 0$ yields $b = -a$. Therefore, the Conway polynomial of $\vec{T}_{2,n}$ is

$$\nabla_{\vec{T}_{2,n}}(z) = \frac{1}{\sqrt{z^2+4}} \left[ \left( \frac{z+\sqrt{z^2+4}}{2} \right)^n - \left( \frac{z-\sqrt{z^2+4}}{2} \right)^n \right].$$

Remark. The torus link $T_{2,n}$ has two possible orientations (up to overall orientation reversal). The orientation as in Figure 2.5.9 will be denoted by $\vec{T}_{2,n}$ and the orientation as in Figure 2.6.1 will be denoted by $\overleftarrow{T}_{2,n}$.

### 2.6. Alexander Polynomial

This section introduces another polynomial-valued invariant called the Alexander polynomial. It is derived from the Seifert matrix introduced in Section2.4.

Definition 2.6.1 (Alexander polynomial). The Alexander polynomial of an oriented knot $K$, denoted $\Delta_K(t)$, is the determinant $\det(t^{1/2}V - t^{-1/2}V^T)$ where $V$ is any Seifert matrix for $K$. It is a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$ [**1**].

Furthermore, applying the change of variable $z \mapsto \sqrt{t} - \sqrt{t^{-1}}$ in the Conway polynomial of an oriented knot $K$ gives rise to the Alexander polynomial of $K$. All the axioms of the Conway polynomial also hold for the Alexander polynomial.

Theorem 2.6.1. If $L$ is a split link then $\Delta_L(t) = 0$ [**1**].

Proof. Let $L_1$ and $L_2$ be disjoint links with the Seifert surface $F_1$ and $F_2$ respectively. Let $V_i$ be the Seifert matrix of $F_i$. Remove a disk $d_1$ from $F_1$ and a disk $d_2$ from $F_2$ and identify the boundary of the disks $\partial d_1$ and $\partial d_2$ with the ends of the tube $S^1 \times [0,1]$. This connects the surface $F_1$ and $F_2$ such that its boundary is ambient isotopic to $L$. Let this connected surface be called $F$. A basis for $H_1(F)$ can be formed by taking the union of the bases for $F_1$ and $F_2$ together with a meridian $m$ of the tube. For any loop $a_j \in H_1(F_i)$, $\text{lk}(a_j, m^+) = \text{lk}(m, a_j^+) = 0$. Thus, a Seifert matrix for $F$ has the form $\begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence, $\det(L) = 0$ and so, $\Delta_L(z) = 0$. □

EXAMPLE 2.6.1. Since the Conway polynomial of both the left-handed and right-handed trefoil $K$ is $\nabla_K(z) = 1 + z^2$, changing the variable $z \mapsto \sqrt{t} - \sqrt{t^{-1}}$ yields the Alexander polynomial of $K$ to be $\Delta_K(t) = 1 + (\sqrt{t} - \sqrt{t^{-1}})^2 = 1 + t - 2 + t^{-1} = t - 1 + t^{-1}$.

EXAMPLE 2.6.2. Applying the change of variable $z \mapsto \sqrt{t} - \sqrt{t^{-1}}$ to the Conway polynomial of the figure-8 knot $\nabla_K(z) = 1 - z^2$ yields its corresponding Alexander polynomial to be $\Delta_K(t) = 1 - (\sqrt{t} - \sqrt{t^{-1}})^2 = 1 - t + 2 - t^{-1} = 3 - t - t^{-1}$.

EXAMPLE 2.6.3. Recall from Example 2.5.6 that the Conway polynomial for $\overrightharpoon{T}_{2,n}$ is

$$\nabla_{\overrightharpoon{T}_{2,n}}(z) = \frac{1}{\sqrt{z^2+4}}\left[\left(\frac{z+\sqrt{z^2+4}}{2}\right)^n - \left(\frac{z-\sqrt{z^2+4}}{2}\right)^n\right].$$

The Alexander polynomial of $\overrightharpoon{T}_{2,n}$ can be obtained by applying the change of variable $z \mapsto \sqrt{t} - \sqrt{t^{-1}}$ to this Conway polynomial. Thus, $\Delta_{\overrightharpoon{T}_{2,n}}(t) = \dfrac{\sqrt{t}^n - (-\sqrt{t^{-1}})^n}{\sqrt{t} + \sqrt{t^{-1}}}$ for all $n \in \mathbb{Z}$. When $n$ is odd, the Alexander polynomial becomes

$$\begin{aligned}
\Delta_{\overrightharpoon{T}_{2,n}}(t) &= \frac{\sqrt{t}^n + \sqrt{t^{-1}}^n}{\sqrt{t} + \sqrt{t^{-1}}} \\
&= \frac{1}{\sqrt{t} + \sqrt{t^{-1}}}\left[\left(\sqrt{t} + \sqrt{t^{-1}}\right)\left(\sqrt{t}^{n-1} - \sqrt{t}^{n-2}\sqrt{t^{-1}} + \sqrt{t}^{n-3}\sqrt{t^{-1}}^2 - \cdots \sqrt{t^{-1}}^{n-1}\right)\right] \\
&= t^{\frac{n-1}{2}} - t^{\frac{n-3}{2}} + t^{\frac{n-5}{2}} - \cdots + t^{-\frac{n-1}{2}} \\
&= \sum_{i=-(n-1)/2}^{(n-1)/2} t^i (-1)^{\frac{(n-1)}{2} - i}, n > 0.
\end{aligned}$$

EXAMPLE 2.6.4. The Conway polynomial of a right-handed torus knot $\overleftharpoon{T}_{2,n}$ with even number of crossings is computed as follows. Let us first understand the case of $\overleftharpoon{T}_{2,4}$ which will then lead to a general case. The knot diagrams $K_+$, $K_-$ and $K_0$ for $\overleftarrow{T}_{2,4}$ with a chosen orientation is given in Figure 2.6.1.

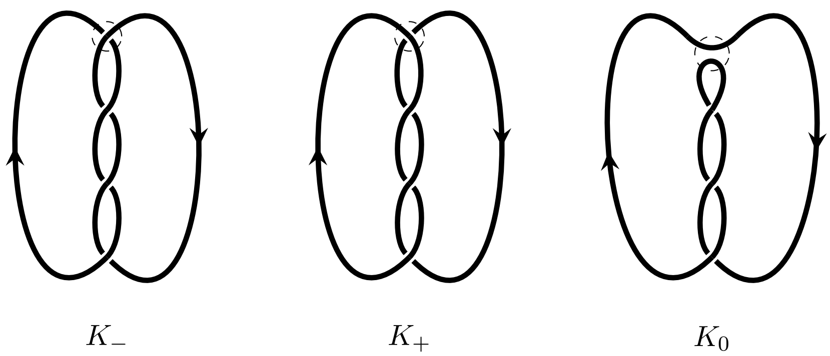



FIGURE 2.6.1. $K_+, K_-, K_0$ for $\overleftarrow{T}_{2,4}$ with a chosen orientation.

Observe that $K_+$ is isotopic to $\overleftarrow{T}_{2,2}$, $K_0$ is isotopic to the unknot and $K_-$ is the torus knot itself. From this we can generalize that for an even $n$, $K_- = \overleftarrow{T}_{2,n}$, $K_+ = \overleftarrow{T}_{2,n-2}$, and $K_0$ is the unknot. Thus, the skein relation becomes $\nabla_{\overleftarrow{T}_{2,n-2}}(z) - \nabla_{\overleftarrow{T}_{2,n}}(z) = z \cdot 1$. On changing the variable $z \mapsto \sqrt{t} - \sqrt{t^{-1}}$, the skein relation become

$$\begin{aligned}\Delta_{\overleftarrow{T}_{2,n}}(t) &= \Delta_{\overleftarrow{T}_{2,n-2}}(t) - \left(\sqrt{t} - \sqrt{t^{-1}}\right) \\ &= \Delta_{\overleftarrow{T}_{2,0}}(t) - \frac{n}{2}\left(\sqrt{t} - \sqrt{t^{-1}}\right) \\ &= -\frac{n}{2}\left(\sqrt{t} - \sqrt{t^{-1}}\right).\end{aligned}$$

Notice that $\Delta_{\overleftarrow{T}_{2,0}}(t) = 0$ since $\overleftarrow{T}_{2,0}$ is the unknot.

EXAMPLE 2.6.5. The knot illustrated in Figure 2.6.2 is called a twist knot where the rectangle represents some odd number of crossings at that location in the knot.

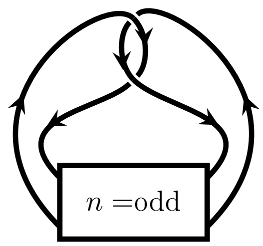



FIGURE 2.6.2. Twist knot.

Consider $K_-$, $K_+$ and $K_0$ of this knot in Figure 2.6.3. Observe that $K_+$ isotopes to an unknot, $K_0$ isotopes $\overleftarrow{T}_{2,n+1}$, and $K_-$ is the twist knot with an odd $n$.

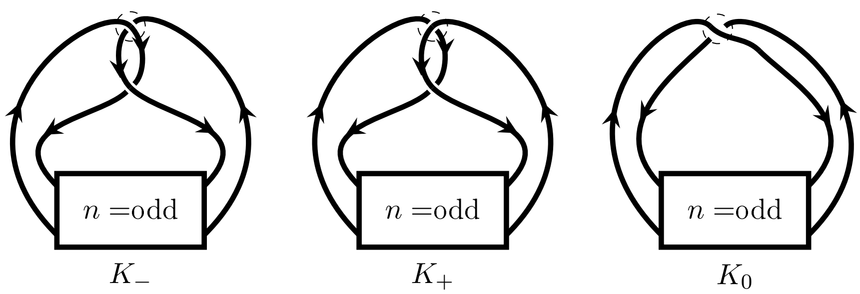



FIGURE 2.6.3. $K_-$, $K_+$, $K_0$ of a twist knot with odd number of twists.

Let us denote $K_- = K_n$, then the skein relation simplies gives us the equation $1 - \nabla_{K_n}(z) = z \cdot \nabla_{\overleftarrow{T}_{2,n+1}}(z)$. After applying the change of variables $z \mapsto \sqrt{t} - \sqrt{t^{-1}}$, the skein relation becomes $1 - \Delta_{K_n}(t) = (\sqrt{t} - \sqrt{t^{-1}})\Delta_{\overleftarrow{T}_{2,n+1}}(t)$. As calculated in the previous example, the Alexander polynomial for a torus knot with an even number of crossings is $\Delta_{\overleftarrow{T}_{2,n+1}} = -\left(\frac{n+1}{2}\right)\left(\sqrt{t} - \sqrt{t^{-1}}\right)$. After putting this result through the skein relation, we get that the Alexander polynomial of a twist knot with odd number of twists is

$$\Delta_{K_n}(t) = \left(\frac{n+1}{2}\right)t - n + \left(\frac{n+1}{2}\right)t^{-1}.$$

EXAMPLE 2.6.6. Figure 2.6.4 illustrates $K_+$, $K_-$, $K_0$ for a twist knot with an even number of crossings $n$.

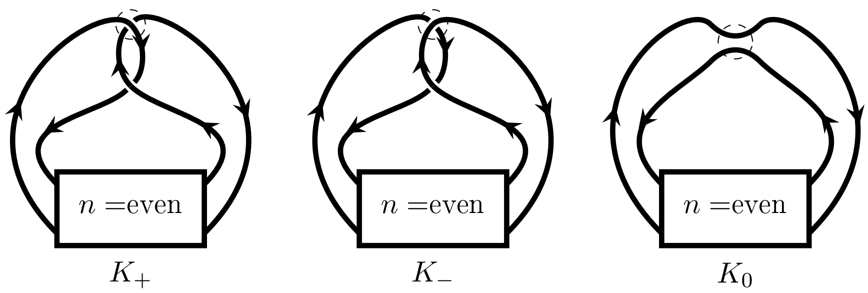



FIGURE 2.6.4. $K_-$, $K_+$, $K_0$ of a twist knot with even number of twists.

Observe that $K_-$ is isotopic to the unknot, $K_0 = T_{2,n}$ with $n$ even, and $K_+ = K_n$ where $K_n$ is the twist knot with an even number of crossings. Thus, the skein relation becomes $\nabla_{K_n}(z) = 1 + z \cdot \nabla_{\overleftarrow{T}_{2,n}}(z)$. Applying the change of variables $z \mapsto \sqrt{t} - \sqrt{t^{-1}}$ and substituting in the Alexander polynomial of $\overleftarrow{T}_{2,n}$ for $n$ even in Example 2.6.4 into the skein relation yields the Alexander polynomial of a twist knot with an even number of crossings to be $\Delta_{K_n}(t) = 1 - \dfrac{nt}{2} + n - \dfrac{nt^{-1}}{2}$.

**2.6.1. Alexander Polynomial of Connected Sum of Knots.** The operation of connected sum forms a knot/link $L$ from two given disjoint knots/links $L_1$, $L_2$. Consequently, there is a relation between the Alexander polynomial of $L$ with the Alexander polynomial of $L_1$, $L_2$.

DEFINITION 2.6.2 (Connected sum). Let $K_1$ and $K_2$ be disjoint knots. Choose a rectangular disk $R$ whose boundary $\partial R$ is composed of four arcs $\{a_1, a_2, a_3, a_4\}$, oriented counterclockwise, such that one pair of arcs go along the edge of each knot with consistent orientation while the other pair is disjoint from the knots, say $K_1 \cap R = a_1$ and $K_2 \cap R = a_3$. Then the two knots can be combined together by the deleting the pair of arcs $a_1$, $a_3$, and adding the arcs $a_2$, $a_4$. Thus the connected sum of $K_1$ and $K_2$, denoted $K_1 \# K_2$, is $(K_1 - a_1) \cup (K_2 - a_3) \cup a_2 \cup a_4$ [**1**].

THEOREM 2.6.2. $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$ [**1**].

PROOF. Let $F_i$ be an orientable surface spanning $K_i$ and let $V_i$ be the Seifert matrix of $F_i$. Add a rectangular disk $R$ such that $R \cap F_i = \partial R \cap \partial F_i = a_i$ is a single arc for each $i$

such that $a_1$ and $a_2$ are opposite sides of the rectangle. This forms a connected surface $F$ that spans $K_1 \# K_2$. To form a Seifert matrix for $F$, $V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$, take the union of the bases for $F_1$ and $F_2$ as the basis for $F$. Since the matrix $t^{1/2}V - t^{-1/2}V^T$ is

$$\begin{pmatrix} t^{1/2}V_1 - t^{-1/2}V_1^T & 0 \\ 0 & t^{1/2}V_2 - t^{-1/2}V_2^T \end{pmatrix},$$

its determinant is

$$\det(t^{1/2}V_1 - t^{-1/2}V_1^T) \cdot \det(t^{1/2}V_2 - t^{-1/2}V_2^T).$$

Thus, $\Delta_{K_1\#K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$. □

EXAMPLE 2.6.7. The knot illustrated in Figure 2.6.5 with $p$, $q$, $r$ number of crossings inside the rectangles respectively is called a pretzel knot, denoted $P(p, q, r)$. If at least two of the integers $p$, $q$, $r$ are even then $P(p, q, r)$ becomes a link. The diagram of a pretzel knot as in Figure 2.6.5 is non-alternating if and only if the integers $p$, $q$, $r$ are not all of the same sign.

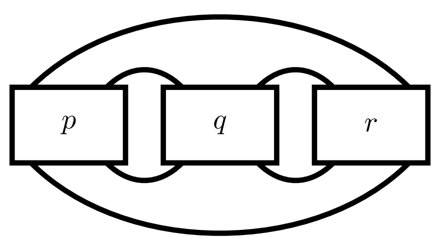



FIGURE 2.6.5. Pretzel knot $P(p, q, r)$.

In this example, we will compute the Alexander polynomial of a pretzel knot $P(p, q, r)$ where $p, q$ are odd integers and $r$ is an even integer. Consider, first, the oriented knot diagrams $K_+$, $K_-$, $K_0$ of a non-alternating pretzel knot $P(5, -3, 4)$ given in Figure 2.6.6.

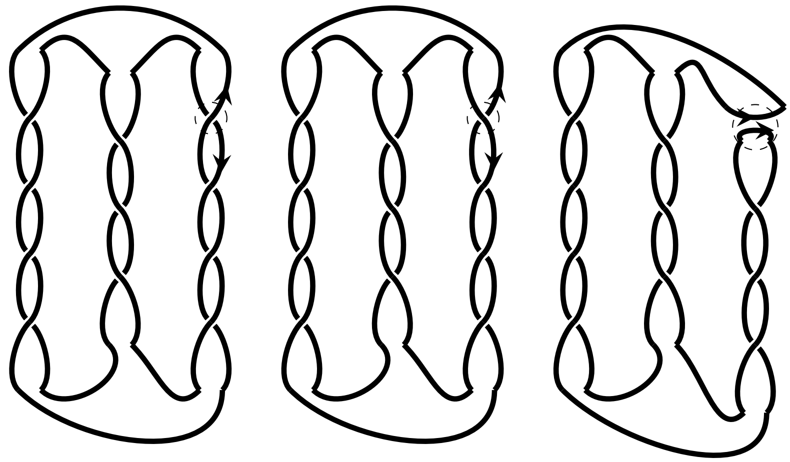



FIGURE 2.6.6. $K_+$, $K_-$, $K_0$ for $P(5,-3,4)$.

Observe that $K_0$ isotopes to $\overrightarrow{T}_{2,5+(-3)}$, $K_+$ resolves into $P(5,-3,2)$, and $K_-$ is $P(5,-3,4)$. In general, $K_0$ will resolve into $\overrightarrow{T}_{2,p+q}$, $K_+$ will resolve into $P(p,q,r-2)$, and $K_-$ will be the non-alternating pretzel knot $P(p,q,r)$ whose Alexander polynomial is to be calculated. Then by the skein relation for Alexander polynomial we have,

$$\Delta_{P(p,q,r-2)}(t) - \Delta_{P(p,q,r)}(t) = (\sqrt{t} - \sqrt{t^{-1}}) \cdot \Delta_{\overrightarrow{T}_{2,p+q}}(t).$$

Notice that upon further taking $P(p,q,r-2)$ as $K_+$, two more of its crossings can be undone and its corresponding skein relation would be

$$\Delta_{P(p,q,r-2)}(t) = \Delta_{P(p,q,r-4)}(t) - (\sqrt{t} - \sqrt{t^{-1}}) \cdot \Delta_{\overrightarrow{T}_{2,p+q}}(t).$$

Substituting this result in the first skein relation gives

$$\Delta_{P(p,q,r)}(t) = \Delta_{P(p,q,r-4)}(t) - 2(\sqrt{t} - \sqrt{t^{-1}}) \cdot \Delta_{\overrightarrow{T}_{2,p+q}}(t).$$

This procedure can be repeated until $P(p,q,0)$ is obtained, that is, all the crossings in the last block are undone. The corresponding skein relation for this procedure is

$$\Delta_{P(p,q,r)}(t) = \Delta_{P(p,q,0)}(t) - \frac{r}{2}(\sqrt{t} - \sqrt{t^{-1}}) \cdot \Delta_{\overrightarrow{T}_{2,p+q}}(t).$$

Observe that $P(p,q,0)$ is a connect sum of $\overrightarrow{T}_{2,p}$ and $\overrightarrow{T}_{2,q}$. So, by Theorem 2.6.2,

$$\Delta_{P(p,q,0)}(t) = \Delta_{\overrightarrow{T}_{2,p}}(t) \cdot \Delta_{\overrightarrow{T}_{2,q}}(t).$$

Since $p$ and $q$ are odd,

$$\Delta_{\overrightarrow{T}_{2,p}}(t) = t^{\frac{p-1}{2}} - t^{\frac{p-3}{2}} + \cdots + t^{-\frac{p-1}{2}}$$

and

$$\Delta_{\overrightarrow{T}_{2,q}}(t) = t^{\frac{q-1}{2}} - t^{\frac{q-3}{2}} + \cdots + t^{-\frac{q-1}{2}}$$

as computed in Example 2.5.6. Since $p+q$ is even,

$$\Delta_{\overrightarrow{T}_{2,p+q}}(t) = \frac{\sqrt{t}^{p+q} - (-\sqrt{t^{-1}})^{p+q}}{\sqrt{t} + \sqrt{t^{-1}}}$$

as shown in the same example. Thus the Alexander polynomial of this pretzel knot is

$$\begin{aligned}
&\Delta_{P(p,q,r)}(t) \\
&= \left(t^{\frac{p-1}{2}} - t^{\frac{p-3}{2}} + \cdots + t^{-\frac{p-1}{2}}\right) \cdot \left(t^{\frac{q-1}{2}} - t^{\frac{q-3}{2}} + \cdots + t^{-\frac{q-1}{2}}\right) - \frac{r}{2}\left(\sqrt{t} - \sqrt{t^{-1}}\right)\left(\frac{\sqrt{t}^{p+q} - (-\sqrt{t^{-1}})^{p+q}}{\sqrt{t} + \sqrt{t^{-1}}}\right) \\
&= \left(\sum_{i=-(p-1)/2}^{(p-1)/2} t^i (-1)^{\frac{(|p|-1)}{2} - i}\right) \cdot \left(\sum_{j=-(q-1)/2}^{(q-1)/2} t^j (-1)^{\frac{(|q|-1)}{2} - j}\right) \\
&= \sum_{i+j=-(\frac{p+q}{2})-1}^{(\frac{p+q}{2})-1} t^{i+j} (-1)^{\frac{|p|+|q|}{2} - 1 - (i+j)}.
\end{aligned}$$

CHAPTER 3

# Grid Diagrams of Knots

The knot/link diagram introduced in the previous chapter can be presented in a grid on a coordinate plane via an algorithm given below. All the proceeding chapters built more upon the grid diagram presentations and introduce all the relevant definitions to discuss the knot Floer homology of a knot. Lastly, Chapter 6.0.3 states an important theorem on the knot Floer homology of a knot. See reference [**4**].

ALGORITHM (Grid diagram construction of a knot). Any knot $K \subset S^3$ can be presented by a diagram placed into a grid in the following way.

(1) Draw a square grid with $n \times n$ cells positioned on a coordinate plane such that the bottom left corner is at the origin and each cell is a square of length one. The number $n$ is called the grid number.

(2) Arrange black and white dots in the center of the respective cells so that
   - every row contains exactly one black dot and one white dot;
   - every column contains exactly one black dot and one white dot;
   - no cell contains more than one dot.

(3) Construct a planar knot projection by drawing horizontal segments from the white to the black dot in each row, and vertical segments from the black to the white dot in each column. At every intersection point, let the horizontal segment be the undercrossing and the vertical segment be the overcrossing. This produces a planar grid diagram of a knot in $S^3$.

These planar grid diagrams can be transfered to a torus to construct a toroidal grid diagram, or simply a grid diagram, denoted $\Gamma$, for a knot $K$ by gluing the topmost edge of the grid to the bottom-most edge, and the leftmost edge to the rightmost edge. Thus the torus inherits its orientation from the plane. Figure 3.0.1 illustrates a toroidal grid diagram of an unknot.

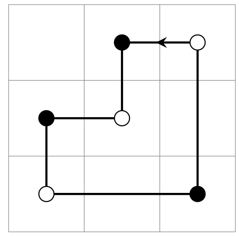


FIGURE 3.0.1. A grid diagram presentation of an unknot.

The intersection points of the horizontal and vertical circles on a grid diagram are called lattice points. Observe that there are $n^2$ lattice points in a $n \times n$ grid diagram since the opposite edges of the grid get glued to each other.

Associated to a given toroidal grid diagram is the chain complex $\{C(\Gamma), \partial\}$ where the generators of $C(\Gamma)$ are indexed by one-to-one correspondences between the $n$ horizontal circles and $n$ vertical circles in a torus. Notice that there are $n!$ such correspondences (and hence generators) in an $n \times n$ toroidal grid.

These correspondences will be enumerated by $n$-sets of points $\{x_{i_1,j_1}, x_{i_2,j_2}, \ldots, x_{i_n,j_n}\}$ where $x_{i,j}$ is the lattice point on the $i$-th horizontal and $j$-th vertical circle, indicating that these two circles correspond to one another.

The boundary operator $\partial$ is more complicated to define, details are given in Chapter 6.

CHAPTER 4

# Graded Groups and Homomorphisms

Since the homology of the chain complex of the grid diagram of a knot introduced in the previous chapter is defined in terms of bigraded groups, we find it relevant to define a graded group.

Definition 4.0.3 ($G$-graded group). An Abelian group $A$ is called $G$-graded if $A$ has a decomposition $A = \bigoplus_{g \in G} A_{(g)}$. Then an element $a \in A_{(g)}$ is said to have grading $g$ and is written as $\mathrm{gr}(a) = g$.

Definition 4.0.4 (Homomorphism of graded groups). Let $A = \bigoplus_{g \in G} A_{(g)}$ and $B = \bigoplus_{g \in G} B_{(g)}$ be two $G$-graded groups. A group homomorphism $f : A \to B$ is called a homomorphism of $G$-graded groups if $f(A_{(g)}) \subseteq B_{(g+h)}$ for some $h \in G$ and for all $g \in G$. This element $h$ is called the degree *of* $f$. A $G$-graded isomorphism is a $G$-graded isomorphism of degree 0.

If $G$ is taken to be $\mathbb{Z} \oplus \mathbb{Z}$, the degree of the graded homomorphism $(n, m) \in \mathbb{Z}^2$ is called a bigrading of the graded homomorphism. A tensor product of two bigraded groups $A = \bigoplus_{(m,n) \in \mathbb{Z}^2} A_{(m,n)}$ and $B = \bigoplus_{(m,n) \in \mathbb{Z}^2} B_{(m,n)}$ is also bigraded with

$$(A \otimes B)_{(m,n)} = \bigoplus_{(m,n)=(m_1+m_2, n_1+n_2)} (A_{(m_1,n_1)} \otimes B_{(m_2,n_2)}).$$

CHAPTER 5

# Alexander and Maslov Gradings

This chapter defines the Alexander and Maslov gradings. These form the bigradings for the homology of the chain complex of the grid diagram of a knot. Chapter 6 describes how these gradings can be used to compute the homology of the chain complex of the grid diagram of a knot. See reference [**4**].

Let $X$ be the generators of $C(\Gamma)$. The Alexander grading on $X$ is defined as follows. Define a function $a$ on lattice points $p$ to be minus one times the winding number of the knot projection around $p$. Let $\{c_{ij}\}$ with $i \in \{1, \ldots, 2n\}$ and $j \in \{1, \ldots, 4\}$ be the set of corners of the $1 \times 1$ squares within the grid diagram that contain either a black or white dot. Then the function $A : X \to \mathbb{Z}$ defined by

$$A(\mathbf{x}) = \sum_{p \in \mathbf{x}} a(p) - \frac{1}{8}(\sum_{i,j} a(c_{ij})) - \frac{n-1}{2}$$

is called the Alexander grading.

Example 5.0.8. The Alexander grading $A(\mathbf{x})$ of the $3 \times 3$ grid presentation of an unknot given in Figure 5.0.1 is computed as follows. The generators of $C(\Gamma)$ are

$$\alpha_1 = \{\mathbf{x}_{11}, \mathbf{x}_{23}, \mathbf{x}_{32}\}, \qquad \beta_1 = \{\mathbf{x}_{13}, \mathbf{x}_{22}, \mathbf{x}_{31}\},$$

$$\alpha_2 = \{\mathbf{x}_{11}, \mathbf{x}_{22}, \mathbf{x}_{33}\}, \qquad \beta_2 = \{\mathbf{x}_{13}, \mathbf{x}_{12}, \mathbf{x}_{32}\},$$

$$\alpha_3 = \{\mathbf{x}_{12}, \mathbf{x}_{21}, \mathbf{x}_{33}\}, \qquad \beta_3 = \{\mathbf{x}_{12}, \mathbf{x}_{23}, \mathbf{x}_{31}\},$$

where each $\mathbf{x}_{ij}$ is the one-to-one correspondence between the $i^{th}$ horizontal circle and $j^{th}$ vertical circle in the grid diagram. For example, consider the generator $\alpha_1$ indicated in Figure 5.0.1.

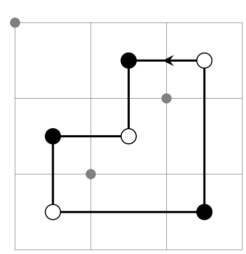


Figure 5.0.1. A grid diagram presentation of an unknot with generator $\alpha_1$ labeled.

Let $\{c_{11}, c_{12}, c_{13}, c_{14}\}$, $\{c_{31}, c_{32}, c_{33}, c_{34}\}$, $\{c_{61}, c_{62}, c_{63}, c_{64}\}$ denote the corners of the square that contain a black dot in the first row, second row and third row respectively. Of these, the only corners with a nonzero winding number are $\{c_{13}, c_{33}, c_{61}\}$ with the value being 1. Thus, $a(c_{13}) = a(c_{33}) = a(c_{61}) = -1$.

Let $\{c_{21}, c_{22}, c_{23}, c_{24}\}$, $\{c_{51}, c_{52}, c_{53}, c_{54}\}$, $\{c_{41}, c_{42}, c_{43}, c_{44}\}$ denote the corners of the square that contain a white dot in the first row, second row and third row respectively. Observe that $\{c_{52}, c_{53}, c_{54}, c_{24}, c_{42}\}$ are the only corners with a nonzero winding number with the value being 1. Thus $a(c_{52}) = a(c_{53}) = a(c_{54}) = a(c_{24}) = a(c_{42}) = -1$. Therefore, $\frac{1}{8}(\sum_{i,j} a(c_{ij})) = \frac{-8}{8} = -1$.

Observe also that $\{\mathbf{x}_{23}, \mathbf{x}_{32}, \mathbf{x}_{33}\}$ are the only $\mathbf{x}_{ij}$'s with a nonzero winding number of 1. Thus,

$$\sum_{p\in\alpha_1} a(p) = a(\mathbf{x}_{11}) + a(\mathbf{x}_{23}) + a(\mathbf{x}_{32}) = 0 - 1 - 1 = -2,$$

$$\sum_{p\in\alpha_2} a(p) = a(\mathbf{x}_{11}) + a(\mathbf{x}_{22}) + a(\mathbf{x}_{33}) = 0 + 0 - 1 = -1,$$

$$\sum_{p\in\alpha_3} a(p) = a(\mathbf{x}_{12}) + a(\mathbf{x}_{21}) + a(\mathbf{x}_{33}) = 0 + 0 - 1 = -1,$$

$$\sum_{p\in\beta_1} a(p) = a(\mathbf{x}_{13}) + a(\mathbf{x}_{22}) + a(\mathbf{x}_{31}) = 0 + 0 + 0 = 0,$$

$$\sum_{p\in\beta_2} a(p) = a(\mathbf{x}_{13}) + a(\mathbf{x}_{12}) + a(\mathbf{x}_{32}) = 0 + 0 - 1 = -1,$$

$$\sum_{p\in\beta_3} a(p) = a(\mathbf{x}_{12}) + a(\mathbf{x}_{23}) + a(\mathbf{x}_{31}) = 0 - 1 + 0 = -1.$$

Therefore, the Alexander grading for each generator is

$$A(\alpha_1) = \sum_{p\in\alpha_1} a(p) - \frac{1}{8}(\sum_{i,j} a(c_{ij})) - \frac{n-1}{2} = -2+1-1 = -2,$$

$$A(\alpha_2) = \sum_{p\in\alpha_2} a(p) - \frac{1}{8}(\sum_{i,j} a(c_{ij})) - \frac{n-1}{2} = -1+1-1 = -1,$$

$$A(\alpha_3) = \sum_{p\in\alpha_3} a(p) - \frac{1}{8}(\sum_{i,j} a(c_{ij})) - \frac{n-1}{2} = -1+1-1 = -1,$$

$$A(\beta_1) = \sum_{p\in\beta_1} a(p) - \frac{1}{8}(\sum_{i,j} a(c_{ij})) - \frac{n-1}{2} = 0+1-1 = 0,$$

$$A(\beta_2) = \sum_{p\in\beta_2} a(p) - \frac{1}{8}(\sum_{i,j} a(c_{ij})) - \frac{n-1}{2} = -1+1-1 = -1,$$

$$A(\beta_3) = \sum_{p\in\beta_3} a(p) - \frac{1}{8}(\sum_{i,j} a(c_{ij})) - \frac{n-1}{2} = -1+1-1 = -1.$$

The Maslov grading $M : X \to \mathbb{Z}$ is defined as follows. Pick a pair of generators $\mathbf{x}$, $\mathbf{y} \in X$ that differ along two horizontal circles. Let $r$ be a rectangle embedded in the torus such that its horizontal edges are formed by connecting $\mathbf{x}$ to $\mathbf{y}$ and its vertical edges are formed by connecting $\mathbf{y}$ to $\mathbf{x}$. Hence, the generators $\mathbf{x}$, $\mathbf{y}$ dictate an orientation on $r$. Let $R_{\mathbf{x},\mathbf{y}}$ be the collection of rectangles connecting $\mathbf{x}$ to $\mathbf{y}$. Notice that $R_{\mathbf{x},\mathbf{y}} \neq R_{\mathbf{y},\mathbf{x}}$.

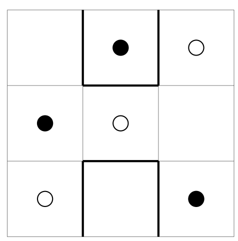
Given $\mathbf{x}$, $\mathbf{y} \in X$, find an oriented null-homologous curve $\gamma_{\mathbf{x},\mathbf{y}}$ such that its each horizontal arc goes from a point in $\mathbf{x}$ to a point in $\mathbf{y}$ (and hence its each vertical arc goes from a point in $\mathbf{y}$ to a point in $\mathbf{x}$). Let $D$ be a two-chain whose boundary is $\gamma_{\mathbf{x},\mathbf{y}}$, and let $W(D)$ and $B(D)$ denote the number of white and black dots in $D$ respectively. Furthermore, $D$ has four local multiplities at each intersection point $x$ of the horizontal and vertical circles. Define the local multiplicity of $D$ at $x$, denoted $p_x(D)$, to be the average of these four local multiplicities. Moreover, given $\mathbf{x} \in X$, let $P_{\mathbf{x}}(D) = \sum_{x\in\mathbf{x}} p_x(D)$. Now, $M$ is uniquely characterized up to an additive constant by the property that for each $\mathbf{x}$, $\mathbf{y} \in X$,

$$M(\mathbf{x}) - M(\mathbf{y}) = P_{\mathbf{x}}(D) + P_{\mathbf{y}}(D) - 2W(D).$$

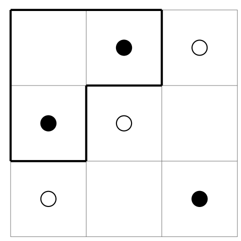
Consider the generator $\mathbf{x}_0$ which occupies the lower left-hand corner of each square which contains a white dot. We declare that $M(\mathbf{x}_0) = 1 - n$.

Example 5.0.9. We continue with Example 5.0.8 to calculate the Maslov grading of the $3 \times 3$ grid presentation of an unknot given in Figure 5.0.1. Observe that the generator $\alpha_1$ occupies the lower left-hand corner of each square with a white dot, so $M(\alpha_1) = 1 - n = -2$. The Maslov grading of the remaining generators is calculated as follows.

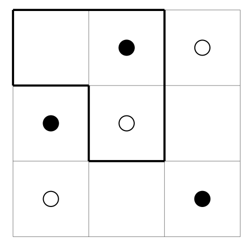
Figure 5.0.2 describes a 2-chain $D$ whose boundary is the curve $\gamma_{\mathbf{x},\mathbf{y}}$ that horizontally connects from $\mathbf{x}$ to $\mathbf{y}$ and vertically connects from $\mathbf{y}$ to $\mathbf{x}$ for indicated pairs of generators $\mathbf{x}$ and $\mathbf{y}$.



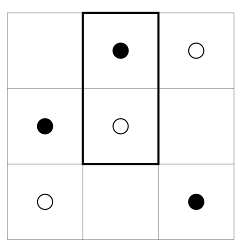
$\mathbf{x} = \alpha_1, \mathbf{y} = \alpha_1.$



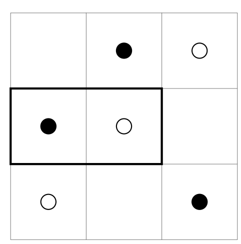
$\mathbf{x} = \beta_1, \mathbf{y} = \alpha_1.$



$\mathbf{x} = \beta_2, \mathbf{y} = \alpha_2.$



$\mathbf{x} = \beta_2, \mathbf{y} = \alpha_3.$



$\mathbf{x} = \beta_3, \mathbf{y} = \alpha_3.$

Figure 5.0.2. A 2-chain $D$ for the indicated pairs of generators $\mathbf{x}$,$\mathbf{y}$.

For $\mathbf{x} = \alpha_2$ and $\mathbf{y} = \alpha_1$, the local multiplicities are $P_{\alpha_2}(D) = \frac{1+1}{4} = \frac{1}{2} = P_{\alpha_1}(D)$. Since the rectangle does not enclose any square with a white dot, $W(D) = 0$. Thus,

$$M(\alpha_2) - M(\alpha_1) = P_{\alpha_2}(D) + P_{\alpha_1}(D) - 2W(D),$$

$$M(\alpha_2) - (-2) = \frac{1}{2} + \frac{1}{2} - 0$$

gives us that $M(\alpha_2) = -1$.

When $\mathbf{x} = \beta_1$ and $\mathbf{y} = \alpha_1$, $W(D) = 0$ and the local multiplities are

$$P_{\beta_1}(D) = \frac{1+3+1}{4} = \frac{5}{4},$$
$$P_{\alpha_1}(D) = \frac{1+1+1}{4} = \frac{3}{4}.$$

Thus, from $M(\beta_1) - M(\alpha_1) = P_{\beta_1}(D) + P_{\alpha_1}(D) - 2W(D)$, we get that $M(\beta_1) = 0$.

When $\mathbf{x} = \beta_2$ and $\mathbf{y} = \alpha_2$, the curve $\gamma_{\mathbf{x},\mathbf{y}}$ contains one white dot, so $W(D) = 1$. The local multiplicities are $P_{\beta_2}(D) = \frac{1+1+1}{4} = \frac{3}{4}$ and $P_{\alpha_2}(D) = \frac{1+3+1}{4} = \frac{5}{4}$. Therefore, $M(\beta_2) = -1$.

When $\mathbf{x} = \beta_2$ to $\mathbf{y} = \alpha_3$,

$$W(D) = 1,$$
$$P_{\beta_2}(D) = \frac{1+1}{4} = \frac{1}{2},$$
$$P_{\alpha_3}(D) = \frac{1+1}{4} = \frac{1}{2}.$$

and so $M(\alpha_3) = 0$.

When $\mathbf{x} = \beta_3$ and $\mathbf{y} = \alpha_3$,

$$W(D) = 1,$$
$$P_{\beta_3}(D) = \frac{1+1}{4} = \frac{1}{2},$$
$$P_{\alpha_3}(D) = \frac{1+1}{4} = \frac{1}{2}.$$

and so $M(\beta_3) = -1$.

In summary, the Maslov grading of all the generators are

$$M(\alpha_1) = -2, \qquad M(\beta_1) = 0,$$
$$M(\alpha_2) = -1, \qquad M(\beta_2) = -1,$$
$$M(\alpha_3) = 0, \qquad M(\beta_3) = -1.$$

CHAPTER 6

# Homology of the Chain Complex

For a chain complex $C(\Gamma)$ and the $\mathbb{F}$-vector space generated by the elements of $X$ ($\mathbb{F}$ is a field), the differential $\partial : C(\Gamma) \to C(\Gamma)$ is defined by the formula

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in X} \sum_{r \in R_{\mathbf{x},\mathbf{y}}} \begin{Bmatrix} 1; & \text{if } P_{\mathbf{x}}(r) + P_{\mathbf{y}}(r) = 1 \text{ and } W(r) = B(r) = 0 \\ 0; & \text{otherwise} \end{Bmatrix} \cdot \mathbf{y}.$$

The differential $\partial$ drops Maslov grading by one and preserves Alexander grading. It is also true that $\partial^2 = 0$. Thus, the homology of this complex can be taken to obtain a bigraded vector space over $\mathbb{F}$.

Theorem 6.0.3. Let $V$ be the two-dimensional bigraded vector space spanned by one generator in bigrading $(-1, -1)$ and another in bigrading $(0, 0)$. Then for a fixed grid presentation $\Gamma$ of a knot $K$ with grid number $n$, the homology of the above chain complex $H_\star(C(\Gamma), \partial)$ is isomorphic to the bigraded group $\widehat{HFK}(K) \otimes V^{\otimes(n-1)}$ [**4**].

Example 6.0.10. The Alexander and Maslov gradings computed in Examples 5.0.8 and 5.0.9 are summarized below.

$$\begin{aligned} A(\alpha_1) &= -2, & M(\alpha_1) &= -2, \\ A(\alpha_2) &= -1, & M(\alpha_2) &= -1, \\ A(\alpha_3) &= -1, & M(\alpha_3) &= 0, \\ A(\beta_1) &= 0, & M(\beta_1) &= 2, \\ A(\beta_2) &= -1, & M(\beta_2) &= -1, \\ A(\beta_3) &= -1, & M(\beta_3) &= -1. \end{aligned}$$

We want to understand how the differential $\partial$ maps the chain complex $C(\Gamma) = \mathbb{F}\alpha_1 \oplus (\mathbb{F}\alpha_2 \oplus \mathbb{F}\beta_2 \oplus \mathbb{F}\alpha_3 \oplus \mathbb{F}\beta_3) \oplus \mathbb{F}\beta_1$ to itself. Since $\partial$ drops Maslov grading by one and preserves Alexander grading, we can create a list of Alexander and Maslov gradings for each $\partial\alpha_i$ and

$\partial\beta_i$.

$$A(\partial\alpha_1) = -2, \qquad M(\partial\alpha_1) = -3,$$

$$A(\partial\alpha_2) = -1, \qquad M(\partial\alpha_2) = -2,$$

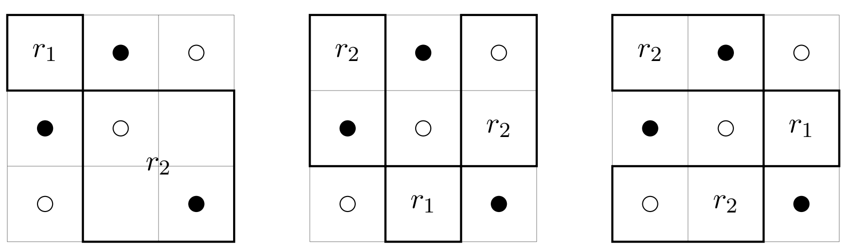
$$A(\partial\alpha_3) = -1, \qquad M(\partial\alpha_3) = -1,$$

$$A(\partial\beta_1) = 0, \qquad M(\partial\beta_1) = 1,$$

$$A(\partial\beta_2) = -1, \qquad M(\partial\beta_2) = -2,$$

$$A(\partial\beta_3) = -1, \qquad M(\partial\beta_3) = -2.$$

Since $M(\partial\alpha_1) = -3$ and $A(\partial\alpha_1) = -2$, and there are no corresponding $\alpha_i$ and $\beta_i$ in the first list above that has these gradings, it must be that $\partial\alpha_1 = 0$ which means $\alpha_1$ is a cycle. For the same reasons, each $\beta_i$ and $\alpha_2$ is a cycle. In particular, $\beta_1$ is a cycle. Thus, under the differential $\partial$, $\mathbb{F}\alpha_1$ and $\mathbb{F}\beta_1$ map to zero. Notice that $\alpha_2$, $\beta_2$, $\beta_3$ have the same Alexander and Maslov gradings as $\partial\alpha_3$, so $\partial\alpha_3 = \lambda_1\alpha_2 + \lambda_2\beta_2 + \lambda_3\beta_3$. In order to find the values of $\lambda_1$, $\lambda_2$, $\lambda_3$, we will use the formula for $\partial\mathbf{x}$.




$\mathbf{x} = \alpha_3, \mathbf{y} = \alpha_2.$ $\qquad$ $\mathbf{x} = \alpha_3, \mathbf{y} = \beta_2.$ $\qquad$ $\mathbf{x} = \alpha_3, \mathbf{y} = \beta_3.$

FIGURE 6.0.1. Rectangles $r_1$, $r_2$ for a pair of generators $\mathbf{x}$, $\mathbf{y}$ of an unknot.

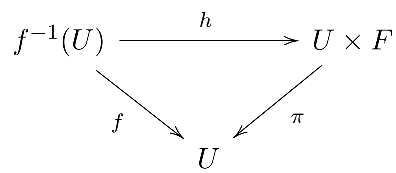
Consider the rectangles $r_1$, $r_2$ in each of the three grids in Figure 6.0.1. Since $W(r_2) \neq 0$, $B(r_2) \neq 0$, and $W(r_1) = B(r_1) = 0$ in all three grids, we will only consider $r_1$ in our further calculations. Since $P_{\mathbf{x}}(r_1) = \frac{1}{2} = P_{\mathbf{y}}(r_1)$, we have $P_{\mathbf{x}}(r_1) + P_{\mathbf{y}}(r_1) = 1$ for $r_1$ in all three grids, and so $\partial\alpha_3 = \sum\limits_{\mathbf{y}\in\alpha_3} 1 \cdot \mathbf{y} = 1 \cdot \alpha_2 + 1 \cdot \beta_2 + 1 \cdot \beta_3$. Thus, the image of $\alpha_3$ under $\partial$ is $\mathbb{F}(\alpha_2 + \beta_2 + \beta_3)$. Since $\partial\alpha_2 = \partial\beta_2 = \partial\beta_3 = 0$, the kernel of $\partial$ is $\mathbb{F}\alpha_2 \oplus \beta_2 \oplus \mathbb{F}\beta_3$. Thus, $\partial$ maps $(\mathbb{F}\alpha_2 \oplus \mathbb{F}\beta_2 \oplus \mathbb{F}\alpha_3 \oplus \mathbb{F}\beta_3)$ to $\dfrac{\mathbb{F}\alpha_2 \oplus \mathbb{F}\beta_2 \oplus \mathbb{F}\beta_3}{\mathbb{F}(\alpha_2 + \beta_2 + \beta_3)}$ which is isomorphic to $\mathbb{F}^2$. Therefore, the homology of the unknot with the grid presentation given in Figure 3.0.1 is $H_\star(C(\Gamma),\partial) \cong \mathbb{F}_{(-2,-2)} \oplus \mathbb{F}^2_{(-1,-1)} \oplus \mathbb{F}_{(0,0)}$. On the other hand, by Theorem 6.0.3, $H_\star(C(\Gamma),\partial) \cong \widehat{HFK}(K) \otimes (\mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(-1,-1)})^{\otimes(3-1)}$. Equating the two relations yields that $\widehat{HFK}(\text{unknot}) \cong \mathbb{F}_{(0,0)}$.

CHAPTER 7

# Applications

The knot Floer homology of a knot has various useful applications which are provided in this chapter.

Definition 7.0.5 (Fibration map). A map $f : E \to B$ is said to be a fibration with fiber $F$ if each point of $B$ has a neighborhood $U$ and a trivializing homeomorphism $h : f^{-1}(U) \to U \times F$ for which the following diagram commutes. ($\pi$ is the projection map.)




$E$ and $B$ are known as total and base spaces, respectively. Each set $f^{-1}(b)$ is called a fiber and is homeomorphic to $F$ [**11**].

Definition 7.0.6 (Fibered knot). A knot or link $L$ in $S^3$ is fibered if there exists a fibration map $f : S^3 - L \to S^1$ that is well-behaved near $L$. That is, each component $L_i$ is to have a neighborhood framed as $S^1 \times D^2$, with $L_i \cong S^1 \times 0$, in such a way that the restriction of $f$ to $S^1 \times (D^2 - 0)$ is the map into $S^1$ given by $(x, y) \to \frac{y}{|y|}$ [**11**].

Theorem 7.0.4. A knot $K$ of genus $g$ is fibered if and only if $\widehat{HFK}(K, g)$ is isomorphic to $\mathbb{Z}$ [**2**][**6**].

Theorem 7.0.5. The genus $g(K)$ of a knot $K$ is $g(K) = \max\left\{j \in \mathbb{Z} | \widehat{HFK}(K, j) \neq 0\right\}$ where $j$ is the Alexander grading of the knot [**7**].

Remark 7.0.1. In view of Theorem 7.0.5, the largest Alexander grading of any generator forms an upper bound of the genus. Moreover, if there is a unique generator in the top Alexander grading $j$ then the corresponding $\widehat{HFK}(K, j)$ is isomorphic to $\mathbb{Z}$. Hence, the knot is fibered by Theorem 7.0.4 and is of genus $j$ by Theorem 7.0.5.

Example 7.0.11. Consider the grid diagram presentation of $T_{2,-5}$ in Figure 7.0.1. We will compute its largest Alexander grading. The negative one times the winding number for all lattice points $p$ and corners $\{c_{ij}\}$ that contains a dot of either kind is summarized in the same figure. Since there can only be one generator along each row and each column, the gray dots in the same figure is the unique generator $\mathbf{x}'$ that has a nonnegative Alexander grading and $\sum_{p\in\mathbf{x}'} a(p) = 1+1+1+1+1 = 5$. Therefore, the Alexander grading of this unique generator is

$$\begin{aligned} A(x) &= \sum_{p\in\mathbf{x}'} a(p) - \frac{1}{8}(\sum_{i,j} a(c_{ij})) - \frac{n-1}{2} \\ &= 5-0-3 \\ &= 2. \end{aligned}$$

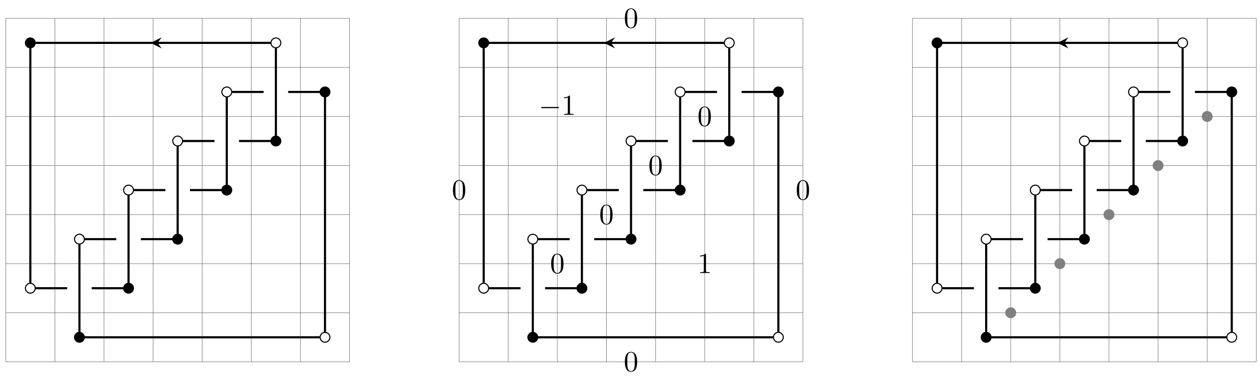



FIGURE 7.0.1. A grid diagram presentation of $T_{2,-5}$.

In general, for any torus knot $T_{2,m}$ ($m \in \mathbb{Z}$, $m$ odd), the lattice points that are off-diagonal from the crossings in the bottom half of the grid form the unique generator with the largest Alexander grading of $A(x) = m - 0 - \left(\frac{m+2-1}{2}\right) = \frac{m-1}{2}$. Thus, a torus knot is fibered. Since this generator is unique, its genus is $\frac{m-1}{2}$ by Remark 7.0.1.

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