University of Nevada, Reno

# Geometric, Algebraic and Topological Connections in the Historical Sphere of the Platonic Solids

A thesis submitted in partial fulfillment of the
requirements for the degree of
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by

James Adam Smith

Dr. Alexander Kumjian/Thesis Advisor

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# THE GRADUATE SCHOOL

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Alexander Kumjian, Ph.D., Advisor

Christopher Herald, Ph.D., Committee Member

Bruce Moran, Ph.D., Committee Member

Tom Nickles, Ph.D., Graduate School Representative

Marsha H. Read, Ph.D., Associate Dean, Graduate School

December 2012

# Abstract

The Platonic solids have made prominent appearances in the history of pure mathematics: in Euclid's *Elements*, which contains early geometric constructions of the solids and demonstrates the special manner in which each is comprehended by a sphere; in William Hamilton's geometric interpretation of the icosians, a non-abelian group that describes certain "passages" between faces of the Platonic solids; and in Henri Poincaré's *Analysis situs*, in which two of the solids are used to obtain 3-manifolds for which 0- and 1-dimensional homology are determined. Incidentally, Poincaré's non-polyhedral construction of a homology 3-sphere with non-trivial fundamental group revealed the same icosahedral relations that Hamilton had invented. Topologists later realized that this important construction could be obtained as a polyhedral manifold using the dodecahedron. On the basis of these historical observations, a salient pattern emerges: The recurring uses of the Platonic solids in key episodes of mathematical development that have connected Euclidean geometry, non-commutative algebra and topology in important ways.

# Acknowledgements

I thank Professor Kumjian for originally suggesting to me the possibility of writing a history of the Platonic solids. I believe that my work on the topic has proved to be valuable and interesting; I hope the reader will agree. I also thank my committee members — above all Professor Kumjian — for their constructive input and encouragement at various stages of the project. Finally, I thank my family and friends back home for their encouraging words and interest in my work.

# Contents

# Chapter 1

# Introduction

We do mathematics with concepts — and with some concepts more than with others.

Ludwig Wittgenstein, *Remarks on the Foundations of Mathematics*

In this thesis I argue that the Platonic solids have been at the center of key stages of development in the history of pure mathematics. These stages are where major connections have been made between Euclidean geometry, modern algebra and topology. In writing a history of these connections, I follow two interweaving paths: one historical and the other mathematical. The two need to be distinguished due to the contingent aspects of mathematics in practice, and the logically necessary aspects of mathematics in theory. This distinction is important, for there is nothing intrinsically necessary about the historical direction or order of mathematical development. It could have been the case that the Platonic solids were never used in the ways to be considered or even never mathematically conceptualized. However, the contrary is true, and that they have been so conceptualized and used in various ways constitute the basic facts of the history with which this thesis is concerned.

## 1.1 The Platonic Solids: An overview

What are the Platonic solids? And why are they imbued with theoretical and historical significance? The Platonic solids are the five regular solids known as the tetrahedron, octahedron, cube (or hexahedron), icosahedron, and dodecahedron. There are only five such solids due to the regular features of their geometry.

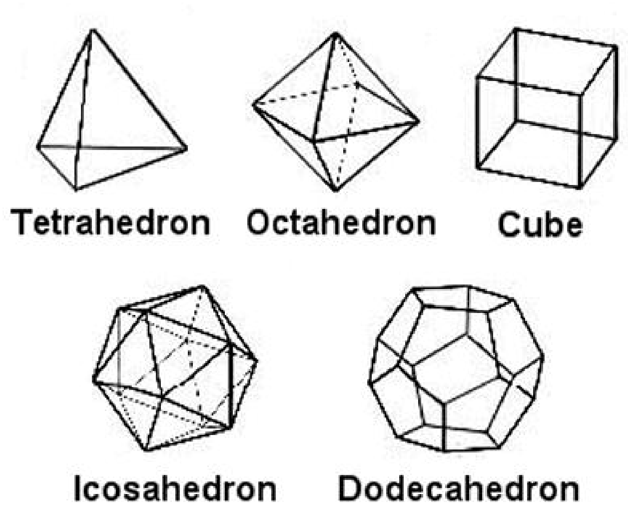


Figure 1.1: The Platonic solids, or the five regular solids.

The story of the Platonic solids in the history of pure mathematics is a long one, going back to the Classic Greek thought of Euclid (c. 300 B.C.), Plato (427 - 347B.C.) and Pythagoras (*c.* 550 B.C.). Plato observed the basic symmetric properties of the solids in relation to the sphere; while Euclid, in the last book of his *Elements*, provided constructions of the solids so as to circumscribe each one by a sphere. They make a prominent appearance fifteen hundred years after Euclid in the work of

Renaissance mathematician Luca Pacioli (1445-1509). Pacioli's *De divine proportion* was illustrated by Leonardo da Vinci (A.D.1452-1519), who provided majestic drawings of the solids. Pacioli also provided a fascinating construction of the icosahedron using the Golden rectangle. Three hundred years later, in the work of nineteenth century mathematician William Hamilton, we find original algebraic views of the solids and therefore an important theoretical connection between Euclidean geometry and modern (non-commutative) algebra. Shortly after the publication of Hamilton's results, Felix Klein fully explicated the connection in terms of the rotation groups of the solids, each of which is a subgroup of the rotation group of the sphere. Finally, in the work of early twentieth century topology, beginning with Henri Poincaré (1854 - 1912), the Platonic solids are found to be at the center of mathematical discovery that builds on the geometry and algebra of the solids. In particular, Poincaré used two of the solids (the cube and the octahedron) to introduce and explain the basic concepts of algebraic topology, a branch of mathematics that is largely concerned to say when one kind of geometric figure is topologically equivalent to another. Later topologists considered a geometric construction, based on the dodecahedron, that was shown to be equivalent to Poincaré's counterexample to a historically and mathematically significant conjecture concerning 3-manifolds.

While the Platonic solids may have been the subject of thought between these episodes of historical development, they are of special significance in three places: namely, those configured by Euclid's solid geometry, Hamilton's modern non-commutative algebraic interpretation of the solids, and Poincaré's use of the solids to demonstrate the new ideas of topology. The works of the other individuals mentioned play important roles as well, but they do so rather in terms of subsequent development based on, or related to, the work of Euclid, Hamilton and Poincaré.

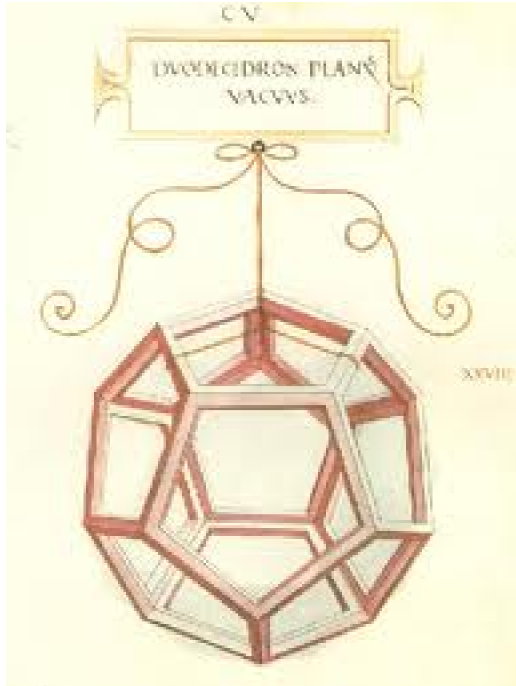


Figure 1.2: Leonardo da Vinci's representation of the dodecahedron.

Aside from these connections that bridge Euclidean geometry, modern algebra and topology, there is another feature of the history that singles out the solids for historical investigations: the philosophical contexts of their appearance. As Plato is the namesake of the Platonic solids due to the central role he assigned to them in his cosmology it is not surprising that the solids should be linked to the philosophical contexts that accompanied, or shaped to some degree, the mathematical theory. But the historical links may only be indirect, for there is no explicit philosophy expressed anywhere in Euclid's *Elements*. However, as Euclid was likely trained as a second-generation student at Plato's Academy, the Platonist theme of relating the solids to the sphere could explain why Euclid constructed the solids in relation to the sphere. However, the remaining links between philosophical and mathematical contexts appear to be direct: before "discovering" the quaternions by introducing two new imaginary units, Hamilton was initially concerned to justify the use of such units by appealing to the philosophy of Immanuel Kant (1724 -1804), who argued that the cognitive faculties of the imagination constituted the basis for the possibility of the mind's ability to represent mathematical objects. Poincaré similarly justified the study of topology, claiming that the use of figures in geometry served the purpose of aiding the kind of mathematical understanding that goes beyond what the senses are capable of representing. However, Poincaré also relied on modern algebraic techniques in order to distinguish between the kinds of geometric figures that the senses cannot directly perceive.

Exploring these immediate philosophical contexts of the mathematical appearances of the solids may allow us to better understand the history of the Platonic solids.

## 1.2 Methodological approach to the history

My approach to the history of the Platonic solids is to focus on the historical connections by discerning and relating the mathematical and philosophical contexts. This is to be contrasted with any other approach that arbitrarily selects static theoretical connections over historical ones. A clear example of this approach is found in John Stillwell's "The story of the 120-cell" (2001), which begins with an aesthetic valuation of the mathematical object of historical study. He finds the 120-cell (a higher dimensional analogue of the dodecahedron) to be "one of the most beautiful objects in mathematics" for the following reason:

> This 120-cell is a rarity among rarities because it lives in three special worlds. Its home is among the regular polytopes in $\mathbb{R}^4$, but it also lives in the remarkable sphere $\mathbb{S}^3$ and in the quaternions $\mathbb{H}$ [Stillwell 2001, p.17].

He goes on to specify additional features of the 120-cell that supposedly further justify his claim regarding the significance of this polytope, including its modern theoretical characterization and the possibility of its illustration using computer graphing techniques. However, according to Stillwell, this modern capacity has potentially detrimental but permissible consequences for our *understanding* of the history of this mathematical idea and, perhaps, more generally for the history of mathematics as a whole:

> Telling the story in contemporary language has the danger that certain connections become "obvious", and it is hard to understand how our ancestors could have overlooked them. However, there is no turning back; we cannot stop seeing the connections we see now, so the best thing to do is describe them as clearly as possible and recognize that our ancestors lacked our advantages [ibid.].

Presumably, what Stillwell intends is the description of mathematical connections from the point of view of the present. I will refer to this way of understanding history from a temporaneous external position as presentism. The presentist prescription is problematic and unjustified for an historical understanding of mathematical connections, since it essentially discounts potentially relevant or significant historical connections in its aim of historical understanding. In fact, the advantage of which Stillwell speaks might presuppose *any* relevant historical connections, so that his kind of "story" would have the effect of making relevant historical facts appear as strictly mathematical facts. The present state of mathematics is an historical moment that is merely a single slice in a much longer development of the subject. Granted, Stillwell pays attention to the chronology of mathematical discoveries, but the implicit implication appears to be that such discoveries were simply waiting to be made. Should

Euclid's work be disparaged for not having included the geometry of four-dimensional space, or for not having considered the possibility of a geometry without a parallel postulate? Should Hamilton's invention of the icosian system be downplayed since he didn't recognize the close link between the icosians and the quaternions? Presumably not. Stillwell may very well appreciate and agree to this point. But the further point to make is that the results of Euclid, Hamilton, and other mathematicians of the past can certainly be appreciated within the mathematical contexts of their own times by reasoning about their subject matter in terms of the categories specific to their theoretical frameworks. I take it to be a worthwhile exercise in history, as well as in mathematics, to historically reconstruct mathematical objects in their original contexts and to consider both their development over time and the way in which they have served to bridge mathematical connections between different areas of mathematics. The mathematical connections of the present are indeed obvious from our perspective, and it might only be difficult for us to understand how our ancestors lacked certain insight if we were to unjustifiably exclude important historical connections in the first place. On the contrary, there is no stopping us from turning back with an eye to the interconnected history of mathematics.

## 1.3 The Platonic Solids' namesake

The argument against the presentist view can be illustrated by a consideration of Plato's cosmology, which takes the sphere as the archetypal image and fundamental organizing principle of the cosmos. Plato, in his dialogue *Timaeus*, has the Pythagorean figure Timaeus of Locri describe and aesthetically evaluate the sphere to reveal its cosmological significance. When Socrates prompts Timaeus for an account of the cosmos as it can be known in its state of being and process of becoming, Timaeus replies:

> For god, wanting to make the world as similar as possible to the most beautiful and most complete of intelligible things, composed it as a single living being, which contains in itself all living beings of the same natural order [30d-31a].

In addition:

> A suitable shape for a living being that was to contain within itself all living beings would be a figure that contains all possible figures within itself. Therefore he turned it into a round spherical shape, with the extremes equidistant in all directions from the centre, the figure that of all is the most complete and like itself [32b].

Insofar as Plato was advancing a mathematical cosmology, he would have borrowed from the Pythagoreans. In fact, Plato is believed to have studied with Archytas, a member of the Pythagorean school, and it may have been during the course of his studies that he learned about the specific cosmological ideas involving the regular solids. The early Pythagoreans are likely the first to have drawn the mystico-religious correspondence between three of the five regular solids and the supposed four basic elements of the cosmos. The tetrahedron, octahedron and icosahedron corresponded to the solid shapes of fire, air and water, respectively. The Pythagorean element of earth might have rather been given the polygonal shape of the hexagon, perhaps due to a special preference for equilateral triangles in the construction of the geometric shapes (see Smith, 1958). But the early Pythagoreans also took the dodecahedron to represent the form of the cosmos, presumably since the twelve faces of figure corresponded to the twelve signs of the living forms belonging to the zodiac.

The Pythagorean correspondences between the five figures and the elements of the world ws therefore a kind of *model* of the universe.

Plato's cosmology (insofar as we can ascribe one to him) can rightfully be seen as a development of the Pythagoreans', since it evidently builds on the known parts of their cosmology. In Plato's cosmology, as articulated by Timaeus, we find a *complete* use of the solids, where, in place of the hexagon, the cube is taken to correspond to the element of earth. Timaeus's construction of the cube is, furthermore, based on the configuration of twelve equal isosceles triangles, any two of which can be used to form the square face of a cube. Finally, Plato preserved the Pythagorean representation of the cosmos by the dodecahedron. Timaeus claims it was the "fifth construction, which the god used for embroidering the constellations on the whole heaven" (*Timaeus*, 55c). And in Plato's dialogue *Phaedo*, Socrates, having had the occasion to ponder the afterlife and the immortality of the soul shortly before his execution, imagined "the earth when looked at from above . . . as streaked like one of those balls which have leather coverings in twelve pieces" (*Phaedo*, 110). Socrates, to use modern terminology, was imagining a tessellation of the spherical earth by the faces of a dodecahedron. The idea of tessellation is also present, and more clear, in Plato's observation of the relation between the tetrahedron and the sphere: "when four such [solid] angles have been formed the result is the simplest figure, which divides the surface [of the sphere] circumscribing it into equal and similar parts" (55a). Plato could have easily recognized (and perhaps did) the other possible tessellations of the sphere based on the remaining solids. It is interesting to note that Euclid didn't consider this aspect of the circumscribed solids in the solid geometry of his *Elements*. The reason for this may be that he intended to distinguish his work strictly as a treatise in mathematics, in contradistinction to cosmology or philosophy where tessellations were thought about. Even Heath, in the first volume of his comprehensive

*History of Greek Mathematics* (1981 [1921]), doesn't appear to recognize the mathematical character of Plato's observations of the relation between the solids and the sphere.

We may recognize two additional prominent points of use of the Platonic solids in the Western cosmological tradition. First, Proclus Diadochus (*c.*412-485), the Greek pagan philosopher and mathematician, imagined that "the figures of the five elements delivered in geometrical proportions in the *Timaeus*, represent in images the peculiarities of the Gods who ride on the parts of the Universe" (Proclus, p.11). Commenting on Euclid's definition of figure, Proclus found the heavenly bodies to exhibit shapes like the intelligible forms studied in the *Elements* (Proclus 1970, pp. 136-137). But it was Johannes Kepler (1571-1630) who, in his *Mystery of the Universe*, wrote about the intelligible figures that exist *between* the heavenly bodies, having modeled the relative proportions of the distances between the planets' circular orbits by a nested sequence of the Platonic solids and an associated set of circumscribing spheres. Even more fantastically, in the preface of the first edition of his work, he offers an account of the solids at the moment of Creation:

> [T]he most great and good Creator, in the creation of this moving universe, and the arrangement of the heavens, looked to those five regular solids, which have been so celebrated from the time of Pythagoras and Plato down to our own, and that he fitted to the nature of those solids, the number of the heavens, their proportions, and the law of their motions [1981, p. 63].

The religious profundity of Kepler's vision indicates something profound about the recurring use and imaginings of the Platonic solids. Indeed, the Platonic tradition

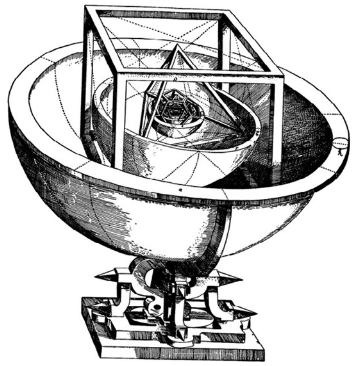

Figure 1.3: Kepler's geometric model of the universe.

in cosmology appears to run right up to a very recent cosmological investigation which posits that the shape of the universe can be described by a shape constructed from a dodecahedron (see Luminet et. al., 2003; Luminet, 2005) — the dodecahedral space. However, the appearance of the Platonic solids is just as striking in the Western tradition of pure mathematics, where we also find evaluative judgments being made of certain mathematical objects.

A brief comparison of Stillwell's description of the 120-cell with Plato's description of the sphere bears on this point. First, there is the obvious difference between the two: Plato's characterization of the sphere is made within the context of an ancient cosmology, whereas Stillwell's technical characterization of the 120-cell is made within the context of modern mathematical theory. But then there is an interesting similarity: both authors make use of metaphor to characterize the containment of one living being, or "rarity," by that of another. In other words, the relation by which one object can be said to "live" in whatever "special world" that it does, as Stillwell puts it; or, the relation by which the shape of one living being can be said to contain another living being, as we find in Plato. The comparison indicates that the aesthetic and metaphorical language, as well as the logic, of Stillwell's view is not entirely contemporary but quite ancient indeed. As Jeremy Gray remarks in his *Plato's Ghost*, an historical account of the modern transformation of mathematics: Plato is "the presiding ancestor of mathematicians, and Platonism commonly supposed to be their default philosophy" (2008, p.14). Plato most certainly had no conception of the 120-cell, but he was quite aware of other mathematical ideas (e.g. tessellations of the sphere by the dodecahedron) which warrant just as much notice today by mathematicians as they may have by those who lived over two thousand years ago.

A history of the Platonic solids in pure mathematics would be deficient without an adequate account of why Plato's name became attached to the five regular solids. A consideration of his philosophy helps to explain why, and I have tried to recognize the historical significance of his philosophy while avoiding any simplistic or contextually insensitive notion that regards it as, say, a "fantasy . . . due to the Pythagoreans," as it has been put by Boyer and Merzbach in *A History of Mathematics* (1988); or simply as "mysterious," as Gabor Toth puts it in his *Glimpses of Algebra and Topology* (1998), when in fact Plato took the models to be perfectly intelligible under the auspices of his theory of knowledge and ontology; or even as childlike, as we find in Coxeter's account of Plato's understanding of the solids (1973). The historical contexts of the Platonic solids in theory are far richer than what these remarks might otherwise suggest.

## 1.4 Chapter Outline

The mathematics done by the individuals mentioned above in connection with the Platonic solids, as well as the philosophical contexts of the theory involved, are the historical subjects of the chapters to follow. In Chapter 2, I take up Euclid's constructions of the solids, highlighting aspects of the *Elements* that illuminate the theoretical context of the constructions. From Euclid's constructions arises the theme of relating the Platonic solids to the sphere as well as the idea of a complete enumeration of the regular solids that essentially constitutes a classification theorem. In the final sections of each chapter I further discuss additional mathematical aspects of the history or theory of the solids, beginning with Pacioli's Euclidean construction of the icosahedron in chapter 2. There is an intrinsic interest in this construction, but it is also one that we consider in order to understand how Euclidean geometry gets connected to algebra and topology.

In chapter 3, I turn to Hamilton's related algebraic systems, the quaternions and icosians, both of which would lead to a higher-dimensional geometric view of the solids which would eventually carry over to topology. While Hamilton did not investigate the rotations of the solids in relation to the sphere, his results concerning quaternions permit not only an algebraic description of the three-dimensional sphere but also a correspondence between such a description and the rotations of the two-dimensional sphere and, hence, the Platonic solids. Klein's later specification of the rotation groups of the solids will in turn motivate a general linear interpretation of the relation between Hamilton's quaternions and icosians.

In chapter 4, I conclude with an account of Poincaré's topological use of the solids for obtaining 3-dimensional manifolds (a topological space that is locally homeomorphic to 3-dimensional Euclidean space), or polyhedral manifolds, and of how such manifolds relate algebraically to $S^3$, the 3-dimensional sphere, in terms of homology or the fundamental group. In the case of the Platonic solids, each one is topolically identical to the 2-sphere, as can be seen by considering a radial projection of, say, the dodecahedron onto the sphere which circumscribes it. However, in the case of the polyhedral manifold obtained by identifying the opposite faces of a dodecahedron by a certain rotation, we obtain a manifold which exhibits trivial homology but non-trivial fundamental group; the latter fact shows that the constructed 3-manifold (known as the Poincaré homology sphere or dodecahedral space) is not topologically equivalent to $S^3$. I go on to provide a brief survey of some the important work of a few later topologists who established new results concerning the Poincaré homology sphere along the theoretical lines laid down by Poincaré. Finally, I discuss a characterization of the dodecahedral space based on the results of the final sections of the previous chapters.

This history is necessarily selective with respect to the historical facts with which it deals, for it does not purport to be anything near a complete history of the Platonic solids. However, this history offers a compelling case for the central role that the solids have played in the history of pure mathematics that connects these fields. Taking the use of the Platonic solids as a recurring historical fact, the shape of the history is suggested by the very subject matter itself. Figuratively speaking, Plato, Euclid, Hamilton and Poincaré can be seen as chronologically ordered vertices that are located on the boundaries of the relevant historical planes of theory — the boundaries between Plato's cosmology, the classic geometry of Euclid, the non-commutative algebra of Hamilton, and the topology of Poincaré. The choice of these four individuals may appear arbitrary, but this is not the case. For their work, or the ideas that their names now symbolize in theory, are the prominent points in the history that centers these three areas of mathematics on the Platonic solids. They are nevertheless only four in number and are embedded in a much larger manifold of historical relations than I can possibly account for — a manifold that we might picture, on first approximation, as the historical sphere of the Platonic solids.

# Chapter 2

# Euclidean Constructions of the Platonic Solids

In this chapter I provide an account of Euclid's construction of the Platonic solids, as found in Heath's translation of the *Elements* (1956). In section 2.1, I offer a few biographical and historiographical remarks about Euclid, and I further consider the problem of presentism in relation to the *Elements*. In section 2.2, I develop the parts of the *Elements* that are relevant to the solid geometry of the Platonic solids. In section 2.3, I examine in detail Euclid's propositions concerning the cube and dodecahedron and the proofs regarding their constructions and circumscriptions by a sphere. In section 2.4 I discuss Euclid's complete enumeration of the Platonic solids. Finally, in section 2.5, I characterize Euclidean solid geometry in terms of Cartesian coordinates and illustrate Pacioli's construction of the icosahedron thereby.

## 2.1 Biographical and Historiographical remarks

Euclid flourished during the time of the first Ptolemy's rule of Alexandria, where he is thought to have founded a school after his time studying at Plato's Academy in Athens. As Plato died in the year 348 B.C., Euclid would have likely learned from the generation of Plato's pupils. To what extent he adhered to a Platonist philosophy or cosmology cannot be ascertained on the basis of his *Elements* alone, for there is nothing in it essentially philosophical or cosmological in it. Heath laments the fact that Euclid wrote no preface, by which such items regarding his possible Platonist views, for example, might have been gathered; as Heath notes, Euclid "plunges at once into his subject". Euclid's great work is a comprehensive mathematical treatise devoted exclusively to the systematic development of the geometry and arithmetic with which it deals. We may therefore regard the text as one of the first, original works in pure mathematics.

Heath's analysis and summary of the *Elements*, as found in the first volume of his *History of Greek Mathematics*, is helpful for the purpose of gaining an appreciation for the content and patterns of reasoning found in the *Elements*. For instance, Heath's analysis includes a discussion of Euclid's postulates and considers the apparent Platonist reasoning implicit therein. Moreover, his summaries of the constructions of the five regular solids closely follow Euclid's arguments, with use of modern algebraic notation only in places where the relations between magnitudes may be more easily understood in symbolic form.

Benno Artmann, in his *Euclid: The Origin of Mathematics* (1999), finds Heath's discussion of Euclid limited since he provides few mathematical details of the text and "refrains from any judgement of the relative importance of various propositions or theories" found there. (p. xvi). Artmann's first criticism of Heath seems unwarranted since we do in fact find in Heath's analysis an overview of each book of the *Elements*. And Artmann's second criticism is unjustified, if we want to understand Euclid with respect to the context of his theory and not from a predominantly modernist perspective that focuses on limitations as viewed from the present state of mathematics. The latter criticism that Artmann voices nevertheless motivates him to take a "clear position" in his interpretation of Euclid: "The *Elements* are read, interpreted, and commented upon from the point of view of modern mathematics" (ibid., vii). As discussed in the previous chapter, the presentist perspective leads to the danger of devaluing mathematical or historical significance of texts. Even Felix Klein, who was a serious modern critic of the axiomatic ideal exemplified by Euclid's work, wrote that "the *Elements* were, and are, by no means a school textbook.... The *Elements* presuppose, rather, a mature reader capable of scientific thinking" (Klein 1939, p. 193).

Artmann's interpretation is nevertheless helpful in gaining another mathematical overview of the *Elements* that complements Heath's. However, I aim to treat the relevant parts of Euclid's solid geometry from a *Euclidean* perspective, and to reason about the solid geometry of the Platonic solids as Euclid or his contemporaries might have. My position in this respect is therefore sympathetic to Heath's view of the value of the *Elements*:

> Euclid is far from being defunct or even dormant, and that, so long as mathematics is studied, mathematicians will find it necessary and worth while to come back again and again, for one purpose or another, to the twenty-two-centuries-old book which, notwithstanding its imperfections, remains the greatest elementary textbook in mathematics that the world is privileged to possess [1925, p. ix].

If we recognize the mathematical and historical value of the *Elements*, as with Klein and Heath, then my approach to Euclid's text may not be without merit.

## 2.2 The elements of Euclid's solid geometry

Euclid first defines the Platonic solids, or "the five figures" as he calls them, in Book XI, which marks the beginning of his treatment of solid geometry. The first definition of Book XI, accordingly, is made for the term *solid*, or "that which has length, breadth and depth." In this section I collect the elements required for an interpretation of the solid geometry of the Platonic solids before stating Euclid's propositions concerning their individual constructions. As the present topic now concerns the Platonic solids, we may take a first take a look at the statements of the definitions of each solid, as listed in the order given in Book XI:

> 12. A **pyramid** is a solid figure, contained by planes, which is constructed from a plane to a point.
>
> 25. A **cube** is a solid figure contained by six equal squares.
>
> 26. An **octahedron** is a solid figure contained by eight equal and equilateral triangles.
>
> 27. An **icosahedron** is a solid figure contained by twenty equal and equilateral triangles.
>
> 28. A **dodecahedron** is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

Now a tetrahedron is a type of pyramid, so Definition XI.12 is not exactly the definition of the tetrahedron since a pyramid isn't necessarily contained by four equal or equilateral triangles. Presumably, Euclid would have defined the tetrahedron as the solid figure contained by four of equal and equilateral triangles. Definitions XI.25-28 go without comment by Heath, and perhaps this is due to the logically satisfactory and intuitive aspects of the definitions. However, Heath remarks about Definition XI.12 that the "definition is by no means clear" and he discusses attempts at a satisfactory definition, beginning with Heron of Alexandrian and running up through Legendre and early twentieth century commentators. For the purposes of my discussion, I recognize the appropriate relation between pyramid and tetrahedron while taking the tetrahedron to be the solid figure contained by four equal and equilateral triangles.

One of Euclid's goals involving the five figures is to "comprehend" them by a sphere, i.e. to circumscribe each figure by a sphere in such a way that the sphere meets the figure at each of its vertices but no other points. In constructing the figures and proving that they can be comprehended by a sphere, Euclid draws on the postulates and common notions stated in Book I. In particular, he postulates that

- between any two points, a straight line can be drawn
- a finite straight line can be produced continuously in a straight line
- a circle with any center and any distance can be described
- all right angles are equal to one another

Each of these postulates can be seen to enter into the constructions of the Platonic solids. Among two of the five common notions that Euclid observes are the following, which are used frequently in each construction:

- things which are equal to the same thing are also equal to one another
- things which coincide with one another are equal to one another

We can take 'equal' in both notions to mean that things (e.g. squares) are equal when they are congruent and of the same size. On this understanding, we can quite clearly see how the second common notion is relevant to constructing the Platonic solids since the triangles of an octahedron, for instance, are found to coincide with each other. Euclid's postulates and common notions are the fundamental geometric and logical principles that underpin his arguments. In the remainder of this section, I summarize the elements of Books I, VI, XI and XIII that figure in Euclid's solid geometry of the Platonic solids.

The first definition of Book I is made for the term *point*, or "that which has no part." The second definition of the same book is made for *line*, that which "is breadthless width". When length is added to the breadth in a *figure*, or that which is contained by any boundary (Def. I.14), we obtain a *surface.* A special kind of surface is a *plane surface*, the straight lines on which "lie evenly" in a plane (Def. I.7). And a *plane angle* is the inclination to one another of two straight lines on such a surface (Def. I.8). A *rectilinear angle* is then an angle contained by straight lines (Def. I.9), so that a rectilinear figure may be *trilateral*, *quadrilateral*, or *multilateral*, according to the number of sides it contains (Def. I.19). Hence trilateral figures may have all three sides equal, only two sids equal, or no two sides equal — that is, such a figure may either be an equilateral, an isosceles, or a scalene triangle, respectively (Def. I.19). A square, for example, is a quadrilateral figure defined not only in terms of equal sides but also in terms of *right angles*, a term that Euclid takes more than the average count of words to define. It is worthwhile to quote the definition in full, given the importance of the term.

> *Definition* I.10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.

A square is therefore "both equilateral and right-angled" (Def. I.22), whereby we may observe that the straight lines on the opposite sides of a square are parallel since they can be produced (by the first postulate) on parallel straight lines (Def. I.23). Angles less than a right angle are *acute* (Def. I.12), while angles greater than a right angle are *obtuse* (Def. I.11).

Also contained in the first set of definitions are those made for the parts of a *circle*, which is also an example of a plane figure that is:

> contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another [Def. I.15]; and the point is called the center of the circle [Def. I.16].

Euclid evidently takes the line that describes a circle to be the circle's circumference, whereas the *diameter* of a circle is any straight line that terminates in both directions on the circumference (Def. I.17). Of special importance to Euclid's goal of comprehending the five figures by a sphere is the idea of semicircle, "the figure contained by the diameter and the circumference cut off by it" (Def. I.18). This will be seen below in the case of the cube and dodecahedron.

In constructing the five solids, Euclid cites four propositions of Book I, and one in particular stands out:

> Proposition I.47. *In right-angled triangles the square on the side subtending the right angle is equal to the [sum of the] squares on the sides containing the right angle.*

As I.47 has become known to posterity, the *Pythagorean Theorem* is used four times in three of Euclid's five constructions of the Platonic solids, including those for the cube and dodecahedron.

In Book VI we find a definition that is frequently used in the solid geometry of Book XIII and which has played a significant role outside of pure mathematics. A (finite) straight line "is said to have been cut in extreme and mean ratio," according to Euclid, "when, as the whole line is to the greater segment, so is the greater to the less" (Def. VI.3). This ratio has come to be known as the Golden ratio and will be further discussed when we consider Pacioli's construction of the icosahedron. Within the context of the *Elements*, the importance of the extreme and mean ratio becomes especially visible in Euclid's construction of the dodecahedron. In fact, the ratio is used in constructions of both the icosahedron and the dodecahedron, since it is directly linked to the property of the sides of these two figures being instances of *incommensurable magnitudes* which, according to Euclid's definitions, cannot be measured by any common measure, in contrast to commensurable magnitudes, which can be so measured (X.1).

Returning to the solid geometry of Book XI, in which the five figures are first defined, we find that Euclid defines a *surface* of a solid to be the "extremity" or boundary of the solid (Def. XI.2), and it is at this stage that we may understand the difference between a solid and its boundary. On the one hand, there is the solid figure of the tetrahedron, contained by four equal and equilateral triangles; on the other hand, there is just that figure consisting of the configuration of triangles alone. How did Euclid define the solid angles of the five regular figures, which consist of straight line segments that coincide at the vertices of the figure?

First, a straight line or plane may be *at right angles to a plane* when the former straight line, or any straight line contained in the plane, are at right angles to the latter plane (Defs. XI. 3-4). Accordingly, a *solid angle* "is that which is contained by more than two plane angles which are not in the same plane and are constructed to a point" (Def. XI.11).

Between the definitions of the pyramid and the cube, Euclid defines what may be the most important solid of all, since it is the one to which all of the five figures are ultimately related — and the one on which Plato laid great emphasis — namely, the sphere:

> *Definition* XI.14. When, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a **sphere**.

Heath remarks that Euclid's definition of the sphere is a bit clumsy, as compared with Plato's seemingly more precise definition of the sphere as that figure with "extremes equidistant in all directions from the center." In fact, Plato's definition of the sphere is analogous to Euclid's definition of a circle (see above). However, it may be that Euclid formulated the definition of the sphere as he did in order to illustrate more clearly the manner in which the five figures can be comprehended by a sphere.

It may also be worth noting that Euclid supplies no definitions in Book XIII, as it only contains ten propositions prior to Proposition 13 that he draws on in order to construct the five figures. Five of these propositions are found in the case of the dodecahedron.

Finally, Euclid states the five propositions of Book XIII regarding the possibility of constructing the Platonic solids in relation to the sphere:

> Proposition XIII.13. *To construct a pyramid [viz. a tetrahedron], to comprehend it in a given sphere, and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.*
>
> Proposition XIII.14. *To construct an octahedron and comprehend it in a sphere, as in the preceding case; and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron.*
>
> Proposition XIII.15. *To construct a cube and comprehend it in a sphere, like the pyramid; and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.*
>
> Proposition XIII.16. *To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the side of the icosahedron is the irrational straight line called minor.*
>
> Proposition XIII. 17. *To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures, and to prove that the side of the dodecahedron is the irrational straight line called apotome.*

We find that each proposition consists of three parts: (1) A claim regarding the possibility of constructing any one of the given figures; (2) a claim regarding the possibility of comprehending each of the five figures by a sphere; and (3) a claim regarding (i) the proportion of the diameter of the sphere to the sides of each of the first three figures, or (ii) the irrationality of the sides of the latter two figures. Euclid's aim in the third part of each proposition is related to the additional goal of Proposition XIII.18, in which each solid is comprehended by the same sphere for the purpose of comparing the lengths of each of the figure's side to that of every other.

The lengths of the individual proofs of propositions 13 - 17 vary from short to long. While the tetrahedron may be the simplest of the five figures (as Plato suggests), its construction is in fact more involved than that of the cube. The construction of the cube and the proof of its comprehension by a sphere appears to be the least complex of the five constructions, which evidently requires no proposition of the *Elements* other than several applications of the Pythagorean theorem. By contrast, the corresponding construction of the dodecahedron involves the use of twelve propositions taken from Books I, V, VI, XI and XIII.

## 2.3 Euclid's construction of the cube and the dodecahedron

In order to illuminate the range of complexity involved in the construction of the Platonic solids, I focus on the cube and the dodecahedron. An interesting feature of these two constructions is the fact that the result of the former is used in the construction of the latter. Euclid's construction of the dodecahedron is the only one of the five figures that draws on the results of a previous construction.

### 2.3.1 Euclid's construction of the cube

Recall the objective of Proposition XIII.15

> *To construct a cube and comprehend it in a sphere ... and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.*

Euclid begins the construction by setting out the diameter of sphere so as to ascribe appropriate lengths to the sides of the cube in order to prove the second and third parts of the proposition. The sphere is therefore central to the construction at the outset. Denoting the diameter by $AB$, he cuts the given straight line at a point $C$ so that the segment $AC$ is double that of $CB$. Euclid then draws a perpendicular line from the point $C$ on the diameter to the point of the semicircle labeled $D$. Since $AC$ is double $CB$, the idea of the third part of the proof is clear: to construct a cube with sides that have the same magnitude of $CB$ so that the sphere which comprehends the cube will be the one constructed with diameter $AB$, the square of which will be found triple the square of the side of the cube. I take up each part in turn.

(i) To construct the cube with sides equal to $DB$, Euclid simply assumes, or "sets out", the square $EFGH$, the sides of which are equal to $DB$. He then draws at right angles to the plane of the square four finite lines beginning at $E$, $F$, $G$, $H$ and terminating in $K$, $L$, $M$, $N$, respectively. Each straight line produced is then "cut off" at a magnitude equal to $DB$, so that the eight lines to this point will all have the same length. Joining by straight lines the remaining points of the terminated lines produces the square $KLMN$ on a plane parallel to the plane in which $EFGH$ is contained. Euclid has now constructed a solid with six squares; hence he has

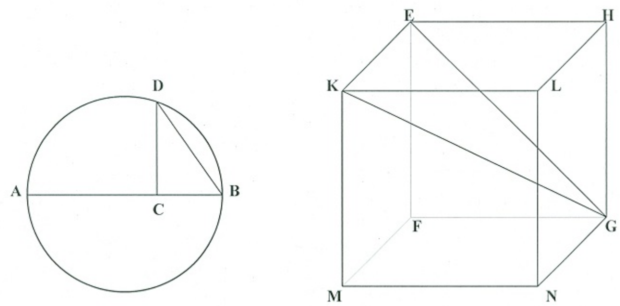


Figure 2.1: Construction of cube.

constructed a cube.

(ii) "It is then required to comprehend it in the given sphere." Drawing straight lines from $K$ and $E$ to the point $G$, Euclid produces a right angle in $KEG$. The semicircle described on $KG$ then passes through the point $E$. By observing that a semicircle passes through the remaining points of the cube, the sphere that is described by the movement of the semicircle that carries it around to its original position "comprehends" the cube; i.e. the sphere passes through the vertices of the cube and no other points of the cube.

(iii) Finally, the square of the diameter of the sphere that comprehends the cube is triple that of the side of the cube. This follows from 1.47, or the Pythagorean theorem, applied to the square on $EG$, which is double the square on $EF$, since $EF$ is equal to $GF$. As $EF$ is also equal to $EK$, the square on $EG$ is double the square on $EK$. The remainder of the proof consists of showing that the side of the cube $KE$ was assumed to equal $DB$, while the diagonal $KG$ is the diameter of $AB$.

### 2.3.2 Euclid's Construction of the dodecahedron

We turn now to an account of Euclid's objectives in Proposition XIII.17:

> *To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures, and to prove that the side of the dodecahedron is the irrational straight line called apotome.*

In proving this proposition, Euclid draws on, in addition to the postulates and common notions, twelve previous propositions of the *Elements*: I.47, V.15, VI.32, XI.1, 6 and 38, and XIII.4-7, and 15. The claim is analyzed according to its three parts. In order to elucidate Euclid's proof of Proposition 17, I identify each of the propositions listed and indicate how they are used in each part of the proof.

(i) *To construct a dodecahedron.* At the outset of the proof Euclid labels the points of two squares that are to be two faces of a cube that are at right angles: $ABCD$ and $CBEF$. Each square is bisected along each side. In the case of the first square, the bisections result in two lines: $GK$, parallel to $AD$; and $BC$ and $HL$, parallel to $AB$ and $CD$. In the case of the second square, the bisections result in the two lines $MH$, parallel to $BE$ and $CF$, and $NO$, parallel to $BC$ and $EF$. The point at which $MH$ and $NO$ meet is labeled $P$, and the point at which $GK$ and $HL$ meet in the square $ABCD$ is labeled $Q$ (see figure).

Next, the bisecting lines and their common points consequently determine four additional straight lines, three of which Euclid labels $NP$, $PO$ and $HQ$. Each of these lines are, in turn, cut in extreme and mean ratio, respectively, at $R$, $S$, and $T$. Drawing straight lines from $R$, $S$, and $T$ at magnitudes equal to $RP$, $PS$, $TQ$,

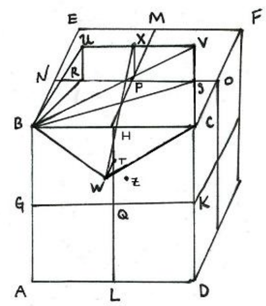


Figure 2.2: Construction of one face of the dodecahedron using the cube.

and at right angles to the planes in which $P$ and $Q$ lie, Euclid produces three new points, $U$, $V$, and $W$; i.e. those points that are the extremities "outside of the cube" on the lines $RU$, $SV$, and $TW$. By joining the straight lines $UB$, $BW$, $WC$, $CV$, and $VU$, Euclid shows that the pentagon $UBWCV$ lies on the side $BC$ of the cube. At this stage, the process can be repeated eleven times to produce eleven additional pentagons similarly situated on the cube, as with $UBWCV$.

Euclid now proves that the pentagon is equilateral, in one plane, and equiangular, which he does in that order and according to the twelve propositions that are cited in his proof. A summary of the argument follows.

(a) *The pentagon is equilateral.* First, Euclid joins $RB$, $SB$, and $VB$. He then shows that the constructed pentagon is equilateral by using proposition XIII.4, which appears in subsequent places in the proof:

> Proposition XIII. 4. *If a straight line be cut in extreme and mean ratio, the square on the whole and the square on the lesser segment together are triple of the square on the greater segment.*

$NP$ has been cut in extreme and mean ratio at $R$ with greater segment $RP$. Therefore, the square on $NP$ and the square on $NR$ together are triple of the square on $RP$. As Euclid observes that $PN$ is equal and at right angles to $NB$, and $PR$ equal and at right angles to $RU$, he is also able to conclude that the square on $BN$ and the square on $NR$ is triple the square on $RU$.

By further observing that the square on $BR$ is equal to the squares $BN$ and $NR$ by I.47, Euclid's argument now relies on the common notions. In particular, he concludes that the square on $BR$ is triple that on $RU$, so that the squares on $BR$ and $RU$ is quadruple the square on $RU$. Hence $BU$ is twice $RU$. And since $SR$ is twice $PR$ and $SR$ twice $RU$, Euclid finds that $BU$ is equal to $UV$. Observing that the same pattern of reasoning can be applied to obtain equality for the remaining pairs of the five sides of the pentagon, he concludes that the pentagon is equilateral.

(b) "*I say next that it is also in one plane*". Employing the first postulate, Euclid draws the straight line $PX$ from $P$ parallel to each of the straight lines $RU$, $SV$ and "towards the outside of the cube." He next joins $XH$ and $HW$ and proceeds to show that $XHW$ is a straight line, i.e. is in one plane As the same argument can be made for the remaining four lines that can be produced, it follows that the pentagon $UBWCV$ is in one plane.

(c) *The pentagon is equiangular.* Finally, to conclude the first part of the proof, Euclid first applies Propositions XIII.4-5 to the line $NS$ cut in extreme and mean ratio. After a chain of reasoning that shows that the diagonals $BV$ and $BC$ of the pentagon are equal and that side $BU$ equals $BW$ and side $UV$ equals $WC$, Euclid finds that the angle $BUV$ equals $UVC$ and that $UVC$ equals $BWC$. Hence $BUV$ equals $BWC$. The following proposition is used to show that the pentagon is in fact equiangular.

Proposition XIII. 7. *If three angles of an equilateral pentagon, taken either in order or not in order, be equal, the pentagon will be equiangular.*

Euclid notes that this argument can be applied to the remaining eleven pentagons that can similarly be constructed on the remaining eleven sides of the cube.

Now that Euclid has shown that the constructed pentagon is equilateral, in one plane and equiangular, he is able to conclude that the "solid figure will have been constructed which is contained by twelve equilateral and equiangular pentagons, and which is called a dodecahedron."

(ii) *The dodecahedron can be comprehended by a sphere.* The strategy of Euclid's proof is to show that the radius of the sphere drawn from the center of the sphere to a vertex of the cube that coincides with a vertex of the dodecahedron is equal to the length of the straight line drawn from the center of the sphere to a vertex of the dodecahedron that is not on the cube. In this way, he can show that the semi-circle with the corresponding diameter passes through all of the vertices of the dodecahedron, as it does for the remaining four positions that the cube alternately assumes in the dodecahedron. (Note in Fig. 2.2 the four circles on the surface of the sphere traced out by the the vertices belonging to the inscribed cube.)

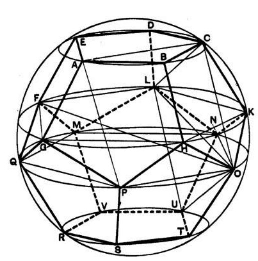


Figure 2.3: Comprehending sphere of the dodecahedron and cube.

(iii) "*I say next that the side of the dodecahedron is the irrational straight line called apotome.*" Finally, Euclid proves the last part of the claim in rather short order based on the facts that $NP$ and $NO$ have been cut in extreme and mean ratio. The diameter of the sphere is rational, and the square on the diameter is triple of the square on the side of the cube, as we learned in Euclid's proof of XIII.15. This amounts to a recognition that $NO$ is rational. By Proposition XIII.6, if a rational straight line be cut in extreme and mean ratio, each of the segments is an irrational magnitudes. These magnitudes are called the apotomes of the line so cut. As $UV$ is the side of the dodecahedron and is equal to the greater segment of a line cut in extreme and mean ratio, it is then an apotome.

I have focused on Euclid's constructions of the cube and the dodecahedron. The next two propositions that Euclid states in Book XIII go on to relate the five solids to a single comprehending sphere. In particular, in Proposition XIII. 18 Euclid sets out the sides of the five solids and compares them to one another. The basis for the comparison is supplied by the third parts of each of propositions XIII.13-17.

## 2.4 Euclid's complete enumeration of the Platonic solids

Euclid concludes the *Elements* with the following unnumbered proposition of Book XIII:

> I say next that *no other figures, besides the said five figures, can be constructed which is contained by equilateral and equiangular figures equal to one another* [p. 507].

This statement offers a complete enumeration of the solids that can be constructed in the manner indicated. There are only five such solids — the Platonic solids. The proof is quite short and appeals to a single proposition from Book XI:

> Proposition XI.21. *Any solid angle is contained by plane angles less than four right angles.*

Now by definition XI.11, we have that a solid angle "is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point." Therefore, no solid angle can be constructed by anything less than three plane angles. But when three, four, or five equilateral triangles are brought to together to form a solid angle, a tetrahedron, octahedron or icosahedron, respectively, is obtained. In all of three cases, the solid angle is less than four right angles. The interior angles of a square are all right angles; however, when only three squares are brought together, the solid angle of the cube is obtained. When six equilateral triangles are brought together, we obtain an angle equal to four right angles; and when more than six are attempted to construct a solid angle, the resulting angle exceeds four right angles.

Finally, in the case of the dodecahedron, Euclid proves in a lemma that the angle of an equilateral and equiangular pentagon is equal to a right angle added to a fifth of a right angle (that is, 108 degrees). Then three such pentagons when brought together constitute the solid angle of a dodecahedron, but to bring together any more than three contradicts Proposition XI. As Euclid notes, similar arguments show that no other polygonal figures can be brought together so as to form a solid angle without contradicting the given proposition.

The final proposition of the *Elements* is an important fact of Euclidean solid geometry, and it would eventually find a parallel in the theory pertaining rotations of the sphere and the five solids contained thereby. Before moving on to that part of the history, we consider a Renaissance construction of the icosahedron from the theoretical point of view of Cartesian coordinate geometry.

## 2.5 Pacioli's construction and Cartesian coordinates

Pacioli's construction of the icosahedron is based on "the Golden rectangle," the sides of which form the Golden ratio which is precisely the extreme and mean ratio of Euclid's Book VI of the *Elements*. Euclid defined the ratio in terms of a straight line cut so that "as the whole line is to the greater segment, so is the greater to the less." In other words, if $\alpha$ is a straight line segment divided in extreme and mean ratio with greater segment $\beta$ and lesser segment $\gamma$, then we will have $\frac{\alpha}{\beta} = \frac{\beta}{\gamma}$. Thus if the magnitude $\alpha$ of a straight line segment is $\alpha = \frac{\sqrt{5}+3}{2}$, then the ratio of the Golden rectangle satisfies this property, with width $\gamma = 1$ and height $\beta = \frac{\sqrt{5}+1}{2}$. It should be noted that Pacioli likely learned about the construction from his teacher, Piero Della Francesca (*c.* 1412 - 1492), who may have actually been the first to provide the details of the construction. Pacioli has been suspected of plagiarism for this and other items(see Boyer and Merzbach 1989, p.313). However, the supposed problem of plagiarism here may be an anachronism since such "borrowing" may have been "part of the practical tradition" (Field 2005, p. 6). Such a tradition appears to have persisted to the present day: e.g. Montisenos (1985) borrows freely of the same construction of the icosahedron for use in a certain construction of the 3-manifold known as the dodecahedral space (discussed in Chapter 4).

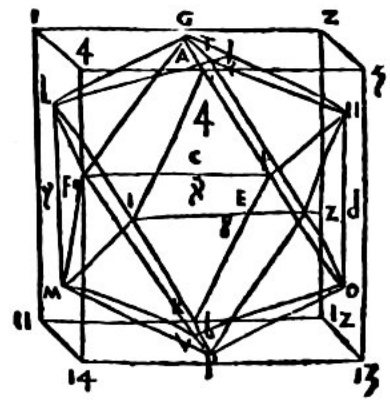

Figure 2.4: Francesca's sketch of the icosahedron using the Golden ratio.

Before proceeding to Pacioli's construction, I provide a definition that is intended to highlight the Euclidean context of Pacioli's construction as well as additional historical results to follow. I assume the standard definition of the field of real numbers $\mathbb{R}$ under the two algebraic operations of addition $(+)$ and multiplication $(\times)$ and the order relation $<$. Just as is with Hamilton's quaternions, which are often denoted by $\mathbb{H}$, I then define $\mathbb{E}$ (for Euclid) as the set $\mathbb{R}$ along with the given operations and ordering (as do Montesinos [1985] and Thurston [1997]).

The definition of $\mathbb{E}$ thus corresponds to the real number line as it is theoretically understood in real analysis, and we can just as well picture $\mathbb{E}$ as a straight line extending indefinitely in both directions. Consequently, we can picture an additional real number line perpendicular to the first, so as to obtain the Euclidean plane, $\mathbb{E}^2$, the two-fold Cartesian product of $\mathbb{E}$ with itself. A third real line perpendicular to the previous two then yields Euclidean space, $\mathbb{E}^3$, the three-fold Cartesian product of $\mathbb{E}$ with itself. The points of the Euclidean plane are then located by ordered pairs $(x, y)$, where $x$ is the first coordinate and $y$ the second coordinate. The points of Euclidean space $\mathbb{E}^3$ are similarly located by triples of real numbers with corresponding first, second and third coordinates. We could go on to consider $n$- dimensional Euclidean space, taking $\mathbb{E}^n$ to be an $n$-fold product of $\mathbb{E}$ with itself.

Now, Pacioli's construction of the icosahedron consists of taking three copies of the Golden rectangle so that each lies in one of the three coordinate planes of Euclidean space. The single common point of intersection of all three is taken to be the midpoint of each copy and is located at the origin. The twelve corners of the figure then determine twenty equilateral and congruent (or, as in Euclid, equal) triangles that constitute the faces of the constructed figure, which in fact is the icosahedron. Pacioli's construction proceeds under the auspices of Euclidean geometry, but the reconstruction I offer here is achieved using Cartesian coordinates of Euclidean space. The theory of Cartesian coordinate geometry grew out of Rene Descartes' (1596 -1650) creation of analytic geometry. It was Descartes' goal to show how "reduce" problems of geometry to the quantities and operations of arithmetic (Descartes [1637] 1954, p.2). The Cartesian context of the reconstruction to be undertaken therefore signals an historical development by which the classic geometry of Euclid is interpreted by an early modern mathematical approach: By specifying the Cartesian coordinates of the icosahedron centered at the origin of Euclidean space and by applying the Pythagorean Theorem to the appropriate points of space, the distances between any two consecutive vertices of the figure are found to be equal.

Let $\tau = \frac{\sqrt{5}+1}{2}$, the greater part of a line cut in extreme and mean ratio. Then the three copies of the Golden rectangle, positioned in $\mathbb{E}^3$ as described, determine three sets of four coordinates according to the Euclidean planes in which they are

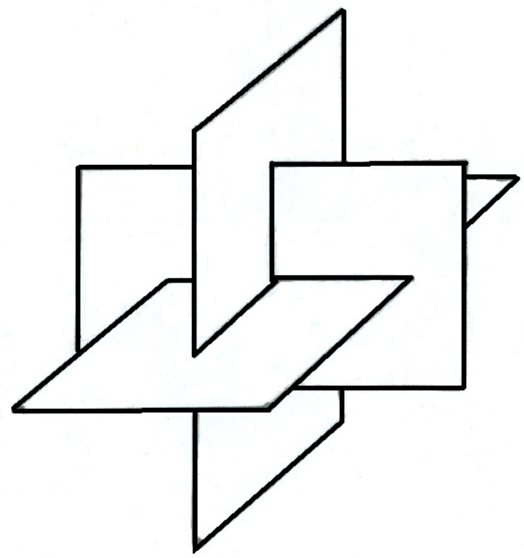

Figure 2.5: Three copies of the Golden rectangle.

positioned:

$$\begin{aligned}
xy \text{ plane} &: \left(\pm\frac{\tau}{2}, \frac{1}{2}, 0\right); \left(\pm\frac{\tau}{2}, -\frac{1}{2}, 0\right) \\
xz \text{ plane} &: \left(-\frac{1}{2}, 0, \pm\frac{\tau}{2}\right); \left(\frac{1}{2}, 0, \pm\frac{\tau}{2}\right) \\
yz \text{ plane} &: \left(0, \pm\frac{\tau}{2}, \frac{1}{2}\right); \left(0, \pm\frac{\tau}{2}, -\frac{1}{2}\right).
\end{aligned}$$

Let $C_0$ denote the point of the figure located at the top corner of the Golden rectangle positioned in the $xz$ plane: That is, $C_0 = \left(\frac{1}{2}, 0, \frac{\tau}{2}\right)$. We then find five points of the figure immediately adjacent to $C_0$:

$$\begin{aligned} C_1 &= \left(\frac{\tau}{2}, -\frac{1}{2}, 0\right), \\ C_2 &= \left(\frac{\tau}{2}, \frac{1}{2}, 0\right), \\ C_3 &= \left(0, \frac{\tau}{2}, \frac{1}{2}\right), \\ C_4 &= \left(-\frac{1}{2}, 0, \frac{\tau}{2}\right), \\ C_5 &= \left(0, -\frac{\tau}{2}, \frac{1}{2}\right). \end{aligned}$$

In order to show that the triangles are equilateral, we need to verify that the lengths of any the three sides determined by any of the three adjacent points are equal. In particular, the lengths $C_0C_1, C_0C_2, C_1C_2$ are equal. By the Pythagorean Theorem (Proposition I.47 of the *Elements*) we find that

$$C_0C_1 = C_0C_2 = \left[\left(\frac{1}{2} - \frac{\tau}{2}\right)^2 + \left(0 \pm \frac{1}{2}\right)^2 + \left(\frac{\tau}{2} - 0\right)^2\right]^{1/2} = 1,$$

$$C_0C_3 = C_0C_5 = \left[\left(\frac{1}{2} - 0\right)^2 + \left(0 \pm \frac{\tau}{2}\right)^2 + \left(\frac{\tau}{2} - \frac{1}{2}\right)^2\right]^{1/2} = 1,$$

and

$$C_0C_4 = \left[\left(\frac{1}{2} + \frac{1}{2}\right)^2 + (0 - 0)^2 + \left(\frac{\tau}{2} - \frac{\tau}{2}\right)^2\right]^{1/2} = 1.$$

The distance between each of the points of the five pairs of adjacent points are therefore equal to 1. The distances between any of the pairs

$$C_1C_2, C_2C_3, C_3C_4, C_4C_5, C_5C_1$$

may similarly found to be equal to 1. Hence the five triangles that constitute the solid angle at $C_0$ are equilateral and equal. As the result can be reproduced for the remaining solid angles located at $C_1$, $C_2$ through $C_{11}$, we find that the vertices of the three configured copies of the Golden rectangle correspond to the vertices of an icosahedron.

Moreover, we find that the distance $r$ from the origin $O$ to every corner $C$ of the icosahedron is

$$r = \left[\left(0 \pm \frac{\tau}{2}\right)^2 + \left(0 \pm \frac{1}{2}\right)^2 + (0 \pm 0)^2\right]^{1/2} = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}.$$

The vertices of the icosahedron are therefore points on the sphere of radius $r$. The icosahedron is thereby “comprehended” by a sphere, as in Euclid’s proposition XIII.16.

# Chapter 3

# Hamilton's non-commutative algebra and the Platonic solids

In chapter 3, I provide an account of William Hamilton's original work in algebra which are relevant to a history of the Platonic solids in pure mathematics. In section 3.1 I offer a biographical sketch of Hamilton and discern his philosophical view concerning his theory of the quaternions. In section 3.2 I take a closer look at Hamilton's discovery of the quaternions and single out the results that he established that are of importance to the rotation groups of the solids. In sections 3.3-3.5, I turn to Hamilton's icosians, a system that he found to have a fascinating connection to the Platonic solids. In the final section, 3.6, I discuss two related algebraic aspects of the solids which Hamilton may not have noticed: (i) the rotation groups of the solids, as contained by the rotation group of the sphere; and (ii) the two-to- one homomorphism from the group of unit quaternions to the rotation group of the sphere and, hence, to such groups of the Platonic solids.

## 3.1 Biographical and historiographical remarks

William Rowan Hamilton (1805-1865), born in Ireland, may best be remembered by the mathematical community for his 1843 discovery and development of the quaternions, a non-commutative algebra which has become specially symbolized by the first letter of his last name, $\mathbb{H}$. The record of Hamilton's achievements leading up to 1843 is impressive, which included his working knowledge of twelve languages by the age of fourteen (Smith 1958, p. 461). Hamilton's work and professional achievements were not strictly confined to pure mathematics. While he was still an undergraduate at Trinity College, Dublin, we was appointed professor of astronomy, and his work in this field continued along side his purely mathematical researches: his applications of the quaternions included modeling the motion of the moon and the calculation of distances of certain comets and planets.

Hamilton's philosophical concerns about his mathematical work are evident in his 1833 paper, "On conjugate functions", which discusses the role of an "imaginary" part of an ordered pair of numbers. Examples of conjugate functions are complex numbers of the form $a + ib$, where $a$ and $b$ are any real numbers and $i$ the number defined to be $\sqrt{-1}$. Hence the status of $i$ as imaginary, since no *real* number satisfies the equation $x^2 = -1$. In his paper, Hamilton goes on to quote Euclid in the Greek and Newton in Latin while expanding on a claim he makes concerning the difference between geometry and algebra. "It is the genius of Algebra," Hamilton remarks, "to consider what it reasons on as *flowing*, as it was the genius of Geometry to consider what it reasoned on as *fixed*" (Hamilton 1967, p.5; emphasis in the original). For instance, the (commutative) algebra of the real numbers permits the possibility of modeling continuous phenomena, as Newton had done with his creation of the calculus. And while Euclid had actually set in motion the diameter of a semi-circle to define a sphere, we can confirm the fixed nature of his geometric reasoning: e.g. those parts of the figures given in the construction of the solids.

How does Hamilton's distinction between algebra and geometry bear on his philosophical view of his work? As the editors to his collected papers on algebra note, it is quite evident in this paper that Hamilton's overall philosophical view was very much indebted to the Prussian philosopher Immanuel Kant (1724 - 1804). Kant, in his monumental *Critique of Pure Reason* (first published in 1781) argued that the concepts of space and time were *a priori* and served as the necessary conditions for experience and understanding. In Kant's view, an outer sense of space allowed for the possibility of representing objects as existing outside the self, while an inner sense of time allowed for the representation of objects in space as temporally ordered, or as *flowing* in time. While Kant was concerned with the general phenomenon of human cognition, one of the major questions that he addressed was how pure mathematics was possible. He found mathematics to be essentially *a priori*, independent of any kind of sense experience. However, he also argued that mathematical concepts were synthetic, and not strictly analytic. The answer to his question, in brief, was that the concepts of geometry and arithmetic derive from the *a priori* forms of space and time, respectively, and that the possible syntheses between the two kinds of mathematical concepts lay in the mind's active capacity to imagine *a priori*, synthetic concepts. For Kant, this was one of the primary functions of the imagination, "the faculty for representing an object even without its presence in intuition" (Kant 1998, p. 258).

The Euclidean and Newtonian theoretical contexts of Kant's time thus underpinned his philosophical theories of cognition and mathematics, and Hamilton essentially adopted this feature of his philosophy. It can rightfully be seen as a philosophical basis for his justification of a four-dimensional geometric interpretation of the quaternions, a system that included three so-called imaginary units. Indeed, to whatever extent Kant might have been able to relate his three- dimensional Euclidean assumptions about the mind, he might have judged Hamilton's significant connection between imaginary numbers and a fourth spatial dimension to be an exemplary synthetic *a priori* concept. In any case, Hamilton took the connection between geometry and algebra to be a mathematical phenomenon, as can be seen in his mathematical treatment of the quaternion and icosian systems. The possible uses of imaginary numbers in geometry may have first been considered two full centuries before Hamilton (see Smith 1958, p. 423), but Hamilton appears to be the first to have made a significant theoretical connection between Euclidean geometry and an original, non-commutative algebra.

## 3.2 Hamilton's "discovery" of the quaternions

Hamilton had been working to develop, as other mathematicians had been, a "theory of triples" analogous to the theory of complex numbers but which would have used a second imaginary part in such a triple. His goal was to discern such properties of couples that might be found to hold in the case of triples: e.g. how the complex number property that states that $(a + ib)(a - ib) = a^2 + b^2$ might be extended to triples so that $(a + ib + jc)(a - ib - jc) = a^2 + b^2 + c^2$ where $a$, $b$, and $c$ are real numbers and $i$ and $j$ imaginary units (i.e. $i^2 = j^2 = -1$). In fact, this was exactly the problem that Hamilton was working on when he conceived the possibility of introducing a total of three imaginary numbers, so that the desired result for triples would rather be found to hold true for quadruples. He characterized this possibility as a "discovery" in his notebook on 16 October, 1843:

> I, this morning, was led to what seems to me a theory of *quaternions*, which may have interesting developments. *Couples* being supposed known, and known to be representable by points in a plane, so that $\sqrt{-1}$ is perpendicular to 1, it is natural to conceive that there may be another sort of $\sqrt{-1}$, perpendicular to the plane itself. Let this new imaginary be $j$; so that $j^2 = -1$, as well as $i^2 = -1$ [Hamilton 1967, p. 103; emphasis in the original].

This was Hamilton's first step toward a theory of quaternions, which clearly included a geometric interpretation in terms perpendicular quantities.

From the definition of the modulus, or magnitude, of the complex number $(a + ib)$, that is,

$$|a + ib| = \sqrt{a^2 + b^2},$$

Hamilton knew that

$$(a + ib)\,(a - ib) = a^2 + b^2.$$

What he was now investigating was the reason why an analogous result for triples of numbers with two imaginary parts would not hold in general. He considered the equation

$$(a + ib + jc)\,(a - ib - jc) = a^2 + b^2 + c^2,$$

in which no imaginary parts remain after multiplication. But then he considered the product

$$(a + ib + jc)\,(a - ib + jc) = a^2 - b^2 - c^2 + i2ab + j2ac + 2bcij + 2bcji,$$

and upon so doing "was thus led to conceive that the product $ij$ might be equated to a *new imaginary*, say $k$, while $ji = -k$" (ibid., pp. 103- 104; emphasis in the original).

He was then able to cancel the last two terms of the product, which, in his words, "destroy" each other. But more importantly, Hamilton had conceived the system of quaternions, the formula for multiplying any two quaternions given by:

$$
\begin{aligned}
&(a + ib + jc + kd)\,(\alpha + i\beta + j\gamma + k\delta) = \\
&a\alpha - b\beta - c\gamma - d\delta + i\,(a\beta + b\alpha + c\delta - d\gamma) + \\
&j\,(a\gamma - b\delta + c\alpha + d\beta) + k\,(a\delta + b\gamma - c\beta + d\alpha)
\end{aligned}
$$

Hamilton's formulation of the rule for quaternion multiplication is the one used today. He was also able to show that the modulus of a product of a quaternion with its conjugate is in fact equal to the square of the modulus of the quaternion — precisely what could not be found to hold true for only two imaginaries. In other words, given a quaternion $q = a + ib + jc + kd$

$$|q\bar{q}| = |\,(a + ib + jc + kd)\,(a - ib - jc - kd)\,| = \left(\sqrt{a^2 + b^2 + c^2 + d^2}\right)^2 = a^2 + b^2 + c^2 + d^2.$$

He established this result and others on the basis of the rules for multiplying unit quaternions, which he also states in his letter (p. 104):

$$i^2 = j^2 = k^2 = -1;\; ij = k,\; jk = i,\; ki = j;\; ji = -k,\; kj = -i,\; ik = -j.$$

Hamilton concluded his notebook entry with several formulae that he found to follow “on the plan of [the] theory of couples:”

$$(a,b,c,d)^2 = \left(a^2-b^2-c^2-d^2, 2ab, 2ac, 2ad\right);$$
$$(0,x,y,z)^2 = -\left(x^2+y^2+z^2\right);$$
$$(0,x,y,z)^3 = -\left(x^2+y^2+z^2\right)(0,x,y,z);$$
$$(0,x,y,z)^4 = \left(x^2+y^2+z^2\right)^2.$$

Of special interest is Hamilton's derivation of the formula that describes the angular directions of a quaternion. Letting $\rho = \sqrt{x^2+y^2+z^2}$, $x=\rho\cos\phi$, $y=\rho\sin\phi\;\cos\psi$, and $z=\rho\sin\phi\sin\psi$, he found that

$$e^{\rho(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)}$$

$$= \cos\rho + \sin\rho\,(i\cos\phi + j\sin\phi\cos\psi + k\sin\phi\sin\psi)\,.$$

As he further notes, when $\phi = 0$, the equation reduces to $e^{i\rho} = \cos\rho + i\sin\rho$.

On 17 October, 1843, the day after he recorded the above results concerning quaternions, he wrote to John Graves, his friend and colleague (as well as the discoverer of the octonians, the eight dimensional analogue of the quaternions), about the “very curious train of mathematical speculation” that had occurred to him. After recounting the problem of trying to find an analogous theory of couples for three dimensions, he explained to Graves that “here there dawned on me the notion that we must admit in some sense, *a fourth dimension* of space for the purpose of calculating with triplets; or transferring the paradox to algebra, must admit a *third* distinct imaginary symbol $k$, not to be confounded with either $i$ or $j$ ... and therefore was led to introduce *quaternions*" (ibid., p. 108; emphasis in the original). The natural geometric interpretation takes a quaternion $a+ib+jc+kd$ to correspond to the vector $(a,b,c,d)$ in a four- dimensional space consisting of four mutually perpendicular axes, just as we find in four dimensional Euclidean space, $\mathbb{E}^4$ (ibid. p. 105). Hamilton's appeal to a fourth dimension for the purpose of interpreting the quaternions will be found to underscore in large part the theoretical context for understanding the rotation groups of the Platonic solids, although Hamilton wasn't aware of this theory. And this is an interesting fact of the history, since Hamilton was the first to state the pertinent algebraic relations in his later work on another non-commutative system, the icosians.

## 3.3 Hamilton's "invention" of the icosians

On 17 October, 1856 — nearly thirteen years to the day after his letter on quaternions to John Graves — Hamilton wrote again to Graves to announce his "invention" of a new non-commutative system of symbols, "the icosians." The letter includes the algebraic definition of the system and a geometric interpretation using the Platonic solids, and it comprises more than one-hundred-and-twenty equations. However, before writing down any of these equations or even a definition of the system, Hamilton was quick to acknowledge Graves' influence as regards the role of the Platonic solids in his invention:

> I feel sure that if you had not lately pressed on my attention the geometrical interest of the polyhedra, although the feeling of such an interest is among my very earliest mathematical recollections, I should not have been conducted to

that novel system of symbols [Hamilton, p. 612].

Hamilton was indeed inspired by, to use his words, the “ancient geometry” of the solids, as clearly indicated by the fact that he developed the purely algebraic system in conjunction with a detailed geometric interpretation focused exclusively on the Platonic solids. This is in contrast to his extensive purely mathematical work on the quaternions, which can be separated from his applications of the system in geometry and mechanics.

Hamilton’s letter on the icosians can be divided into four parts: (1) the abstract theory of the icosian system in sections [1] through [4]; (2) a substantial geometric interpretation of the icosians in terms of the “passages” from one face of the solid to another and the cycles that can be traversed across the faces of the icosahedron in sections [5] through [14]; (3) a similar but brief account of the geometric interpretation of the dodecahedron, which is essentially a corollary to the previous interpretation; and finally (4), additional but brief interpretations of icosian calculi based on the cycles that can be described on the octahedron, cube, and tetrahedron. In order to lay due emphasis on Hamilton’s focus on the icosahedron, I summarize the first part of the letter concerning the definition of the system before turning to his detailed interpretation of the system using the icosahedron.

## 3.4 The icosian system

In laying out another non-commutative system, Hamilton introduced three symbols, $\iota$, $\kappa$, and $\lambda$ and defined each in terms of the relations they satisfy as square, cube and fifth roots of unity, respectively, along with an additional relatoin:

$$\iota^2 = 1,\ \kappa^3 = 1,\ \lambda^5 = 1;\ \lambda = \iota\kappa.$$

As Stillwell points out, the symbol $\lambda$ is redundant since $\lambda$ is defined in terms of $\iota$ and $\kappa$. The fifth root of unity is nevertheless important within the context of Hamilton's exposition, since he used it to define additional fifth roots of unity in connection to his geometric interpretation of the system. The defining feature of the icosian system is that it is non-commutative: in particular, $\iota\lambda \neq \lambda\iota$. The system therefore consists of "non-ordinary" roots of unity, as Hamilton put it.

He goes on to deduce relations that he finds to be of some interest and illustrative of the system. For example, he finds that

$$(\iota\kappa)^5 = (\iota\lambda)^3 = 1,$$

which easily follows from the defining relations: because $\iota\kappa = \lambda$, we find, after multiplying both sides of the equation on the left by $\iota$, that $\kappa = \iota\lambda$. Taking the third power of each side and using the second relation of the definition, it follows that $\kappa^3 = (\iota\lambda)^3 = 1$.

He now introduces a new symbol, $\mu$, by setting $\mu = \iota\kappa^2$, thus providing a subsequent fifth root of unity: $\mu^5 = 1$. The symbol $\mu$ henceforth appears repeatedly over the course of Hamilton's letter, and it appears quite specifically along with $\lambda$ in a relation that belongs to a collection of reduction formulae. He lists four of these along with the introduction of yet another symbol, $\omega$, which he takes to be an abbreviation for a product of symbols that is another square root of unity. That is:

$$\lambda\mu^2\lambda = \mu\lambda\mu, \text{ or } \mu\lambda^2\mu = \lambda\mu\lambda \ ;$$

$$\lambda\mu^3\lambda = \mu^2, \text{ or } \mu\lambda^3\mu = \lambda^2$$

and

$$\omega = \lambda\mu\lambda\mu\lambda = \mu\lambda\mu\lambda\mu.$$

Hence

$$\omega^2 = 1.$$

More complex relations are deduced, such as

$$\left(\lambda^3\mu^3\left(\lambda\mu\right)^2\right)^2 = 1,$$

which doesn't readily appear to have any significance within the abstract setting of the icosians, but is in fact quite significant in the geometric interpretation that he provides in the next part of his letter. The relation describes certain kinds of cycles of faces on the icosahedron, and it also resonates with his sense of achievement as regards a solution to the algebraically characterized geometric problem that he posed to himself: namely, the complete enumeration of the distinct and complete cycles that are possible on the faces of an icosahedron. This problem properly belongs to the interpretation of the icosians that is focused on the icosahedron.

## 3.5 The icosians and the icosahedron

Hamilton's geometric interpretation of the icosians is based on the Euclidean definitions of the Platonic solids, and his goal in using the solids is to interpret the cycles described on the faces (or vertices) of the solids according to the properties of the icosian calculus:

> every symbolic result of which ... admits of easy and often interesting interpretations, with reference to the passage from face to face, or from corner to corner, of one or the other of the solids considered in the ancient geometry [ibid.].

Hamilton's interpretation is focused on the icosahedron, but he also considered similar systems (with slight modification to the relations) and indicates the similar manner in which they are interpreted in the four remaining cases of the solids.

As a first step of the interpretation, Hamilton labels the twelve vertices of an icosahedron by

$$A, B, C, D, E, F, A', B', C', D', E' \text{ and } F'.$$

Vertex $A'$ is at the opposite position of $A$, $B'$ at the opposite of $B$, etc. The faces of the icosahedron are then labeled by the following schema, moving in order from the top vertex of the icosahedron at $F$, to the bottom vertex, $F'$: (*i*) $a, b, c, d$ and $e$ represent the five successive faces of the icosahedron about the corner $F$; (*ii*) $\alpha, \beta, \gamma, \delta$ and $\epsilon$ represent the five respective faces adjacent to the first set of faces; (*iii*) $\alpha', \beta', \gamma', \delta'$ and $\epsilon'$ represent the five respective faces adjacent to the faces of the second set of faces; and (*iv*) $a', b', c', d'$ and $e'$ represent the five successive faces about the corner $F'$, all of which are opposite to the faces of the first set (see figure).

Next, Hamilton describes the twelve "quines" corresponding to the twelve corners of the icosahedron. He takes a quine to be a configuration of adjacent faces centered about a vertex and ordered by a counter-clockwise rotation. (Note that this characterization of the order of the faces as a rotation is describing something about a quine, not the solid.) Listed according to opposite pairs of vertices, he specifies the twelve quines of the icosahedron:

| | |
|---|---|
| $F:\ abcde$ | $F':\ a'e'd'c'b'$ |
| $A:\ \alpha'\delta dc\gamma$ | $A':\alpha\gamma'c'd'\delta'$ |
| $B:\ \beta'\epsilon' ed\delta$ | $B':\beta\delta'd'e'\epsilon'$ |
| $C:\gamma'\alpha ae\epsilon$ | $C':\gamma\epsilon'e'a'\alpha'$ |
| $D:\delta'\beta ba\alpha$ | $D':\delta\alpha'a'b'\beta'$ |
| $E:\epsilon'\gamma cb\beta$ | $E':\epsilon\beta'b'c'\gamma'$ |

Hamilton's interpretions of the individual icosian symbols are based on the geometric meanings of cycles, transpositions, or reflections. With respect to quine $(F : abcde)$, $\lambda$ is the symbol that represents the operation of passing from one pair of adjacent faces to a subsequent pair.

$$\lambda\,(ab) = bc, \lambda\,(bc) = cd, \lambda\,(cd) = de, \lambda\,(de) = ea, \lambda\,(ea) = ab.$$

Hence $\lambda^5\,(ab) = ab$, "the character of a fifth root of positive unity being evidently thus presented" (p.614). Next, $\iota$ is the symbol that operates on a pair of adjacent faces so as to transpose the order of the faces; that is, $\iota\,(ab) = ba$, whereby $\iota^2\,(ab) = ab$. The interpretation of $\kappa$, in contrast to $\lambda$ and $\iota$, doesn't have the intuitive kind of meaning for Hamilton as he found to be the case with the previous two symbols, so it is "less interesting," in his words. Of course, we could interpret the relation as a $2\pi/3$ rotation about an axis through the centroid of one of the faces, but Hamilton was not interpreting the icosians as rotations of the solid. Nevertheless, he verifies that the relation is satisfied: $\kappa^3\left(ab\right) = ab$.

As noted, Hamilton introduced two additional symbols, $\mu$ and $\omega$, in order to abbreviate relations that constitute fifth and square roots of unity, respectively. They therefore inherit their separate geometric meanings as consequences of the relations that define the icosians. First, $\mu = \iota\kappa^2$ represents the operation of passing in the order by which a complete cycle on the icosahedron is completed with respect to quine $(D : ab\beta\delta'\alpha')$ (see Figure 3.2):

$$\mu^5\left(ab\right) = \mu^4\left(b\beta\right) = \mu^3\left(\beta\delta'\right) = \mu^2\left(\delta'\alpha\right) = \mu\left(\alpha a\right) = ab.$$

Hence $\mu^5(ab) = 1$ and therefore displays another "character of a fifth root."

Second, $\omega$ "receives an extremely simple geometrical interpretation", inasmuch as it "implies the passage from one pair of adjacent faces of the icosahedron to the pair which is opposite thereto" (p. 615.):

$$\omega\left(ab\right) = \lambda\mu\lambda\mu\lambda\left(ab\right) = \lambda\mu\lambda\mu\left(bc\right) = \lambda\mu\lambda\left(c\gamma\right) = \lambda\mu\left(\gamma\alpha'\right) = \lambda\left(\alpha'a'\right) = a'b'.$$

Hamilton explains the geometric meaning of several icosian formulae in the case of the icosahedron, and he includes descriptions of several subcycles of various lengths. But what stands out in the interpretation at this point, however, is his claim to have enumerated the "complete cyclical successions" of the twenty faces that constitute an icosahedron. The possibility of enumerating these cycles in fact depends on the complex relation $\left(\lambda^3\mu^3\left(\lambda\mu\right)^2\right)^2 = 1$ that he deduced as a useful relation within the abstract setting of the icosians. He claims that this relation represents "the only method of cyclic succession, whereby all the faces of that body can be passed over, one after another, so as to end in immediate proximity to the face with which we had begun" (p. 615). The corresponding succession of twenty symbols that he enumerates, as determined by the pertinent relation, is then

$$\lambda\,\lambda\,\lambda\,\mu\,\mu\,\mu\,\lambda\,\mu\,\lambda\,\mu\,\lambda\,\lambda\,\lambda\,\mu\,\mu\,\mu\,\lambda\,\mu\,\lambda\,\mu.$$

Thus if the operation $\left(\lambda^3\mu^3\,(\lambda\mu)^2\right)^2$ is applied to $ab$, we find, as with Hamilton, a complete succession of the twenty faces of the icosahedron, specified in the given order:

$$a\ b\ c\ d\ e\ \epsilon\ \beta'\ \delta\ \alpha'\ \gamma\ \epsilon'\ \beta\ \delta'\ d'\ e'\ a'\ b'\ c'\ \gamma'\ \alpha.$$

This complete cycle can be traced out over the faces of the icosahedron. I conclude this section by noting that at this stage of Hamilton's letter, he enumerates a total of ten complete cycles. This appears to have given him sufficient cause to reflect on the apparent significance of the result. The relevant passage is worth quoting at length, for it expresses both modesty and excitement that he seems to have experienced in deriving these particular results of the icosian calculus:

> On the same plan I have found nine other complete and cyclical successions, commencing with the three cases here called $a, b, c$, and do not believe any others, under the same conditions, remain to be discovered: but speak, of course, with a due consciousness of the difficulty of being sure that any subject, of even moderate complexity, has been exhausted . . . Yet the intellect of man recognizes an irrepressible instinct to seek for such a completion: and as deeply enjoyed, when almost a child, the proof that (in the ancient sense of the words) only five regular solids are possible, I shall hazard here what I suppose to be an exhaustive statement . . . of the [ten] ways in which . . . the twenty faces of the icosahedron can be cyclically traversed, if these successive faces $a, b, c$ be given, or assumed, as the initial ones [p. 616].

## 3.6 On the relation between the quaternions and the icosians

Stillwell notes in his "Story of the 120-cell" that Hamilton may not have seen that the icosians can be interpreted as rotations of the Platonic solids. This may be true. However, Hamilton did apply quaternions to the mechanics of rotating solid bodies in 1848 (pp.381-391). What's more, in the same year that he invented the icosians he wrote a memorandum (p.610) in which he notes that "additional remarks on this subject may soon be offered ... under the title ... of 'Extensions of the Quaternions'." But he doesn't appear to say in what way the quaternions are supposed to be extended, and in particular whether they might correspond to rotations of the Platonic solids.

Hamilton's geometric interpretation of the icosians is nevertheless an early connection between group theory and Euclidean geometry. Further theoretical aspects of the connection followed soon thereafter, as we find in the work of Felix Klein (1849-1925). Klein's 1884 *Lectures on the Icosahedron* (1956) amounted to, in his words, a "theory of the icosahedron." Klein had originally been interested in classifying finite groups of fractional linear transformations before viewing the rotations of the solids as rotations of the sphere. But in his lecture, he specified the different rotation groups of the solids by enumerating the possible rotations of the sphere tessellated by the radial projections of the boundaries of the circumscribed solids onto the sphere. As shown in Figure 3.3, Klein enumerated the possible rotations of the solids by identifying the rotational axes of symmetry. In the case of the tetrahedron, three axes of rotations intersect both a vertex and the centroid of an opposite face, while an additional three axes bisect the midpoints of opposite edges. In the case of the cube, axes of symmetry intersect opposite vertices, the centroids of opposite faces, and midpoints of opposite edges. The axes of symmetry for the remaining solids may be similarly specified.

In fact, Camille Jordan (1838-1922), under whom Klein had studied, had earlier classified the finite groups of rotations of Euclidean space, but Klein's approach makes clear that the rotation group of each solid is *contained* by that of the sphere (see van der Waerden, 1985). Having determined each group of rotations, Klein classified their isomorphism types accordingly: The rotation group of the tetrahedron is isomorphic to $A_4$, the alternating group on four elements; the rotation groups of the octahedron and cube are both isomorphic to $S_4$, the full symmetry group on four elements; and the symmetry groups of the icosahedron and dodecahedron are both isomorphic to $A_5$, the alternating group on five elements. This is as far as I consider Klein's theory of the icosahedron, which included the figure's connection to fifth-degree equations and solvability of algebraic equations. In this section, I am only concerned to provide an account of the relation between Hamilton's quaternions and icosians in light of Klein's statements while using the terminology and notation of matrices and homomorpisms.

### 3.6.1 Rotation matrices of the Platonic solids

The general linear group, $GL_3(\mathbb{E})$, is the needed theoretical context for the interpretation of rotations in three-dimensional Euclidean space, since it is the group of linear transformations of which rotations are examples. A group is a set closed under its group operation '$\cdot$' that maps pairs of elements in $G$ to $G$, and such that the

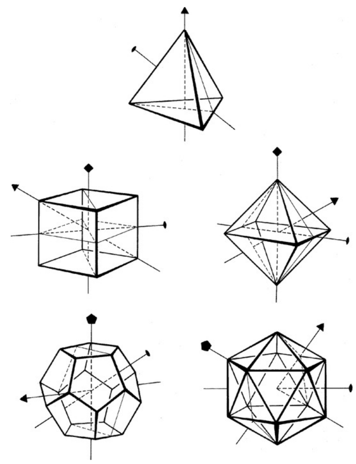

Figure 3.1: Rotational axes of Platonic solids.

group satisfies associativity and contains both an identity element and inverses of each element in $G$. By definition,

$$GL_3(\mathbb{E}) = \{A \in M_{3\times 3} \mid \det(A) \neq 0\}.$$

In particular, $GL_3(\mathbb{E})$ is a group under matrix multiplication. Interpreting $3 \times 1$ matrices as points $x = (x_1, x_2, x_3)$ of 3-space, a map $L : \mathbb{E}^3 \longrightarrow \mathbb{E}^3$ is a linear transformation if: (i) $L(cx) = cL(x)$, where $c$ is a constant; and (ii) $L(x + y) = L(x) + L(y)$. Let $L_A : \mathbb{E}^3 \longrightarrow \mathbb{E}^3$ be the linear transformation that results by left multiplication of matrix $A$ to vector $x^T$; the transpose of this product is then another $3 \times 1$ matrix, or a point in 3-space.

Under matrix multiplication, the subset

$$O(3) = \left\{A \in GL_3(\mathbb{E}) \mid AA^T = I\right\},$$

of $GL_3(\mathbb{E})$ is a subgroup of the general linear group. Restricting this conditition to matrices with determinate equal to 1, we have the special orthogonal group, $SO(3)$, a subgroup of $O(3)$ that consists of rotations of 3-space. An important property of these two groups is that the rows (or columns) of the matrices contained thereby are of unit length and are mutually orthogonal. Hence they constitute orthonormal basis for 3-space. In particular, $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{E}^3$, where the standard rotation matrix for rotations about $e_1$ is

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

The angle $\theta$ is a rotation in the $yz$ plane in a counterclockwise direction when viewed from the point at $e_1$.

In order to determine the matrices that represent the rotational symmetries of a Platonic solid, an axis of symmetry must be specified for each order of symmetry belonging to the solid. Take the icosahedron, the vertices of which were found in section 2.5 using Pacioli's construction. There are three rotational symmetries about three axes, as Klein had observed: a five-fold axis of symmetry intersecting antipodal vertices; a three-fold axis of symmetry intersecting the centroids of opposite faces; and a two-fold symmetry about the $x$-axis.

Consider the vertex located at $v = (1/2, 0, \tau/2)$. In order to find the matrix representing the five-fold rotational symmetry about the axis through this vertex and the origin, we must find the change of basis matrix $V = [v_1\ v_2\ v_3]$, where $v_1$ is a unit vector in the same direction as $v$. (Goodman [1989] describes the procedure for finding the rotation matrices of all five solids in this manner). Then the matrix $VR_{2k\pi/5}V^T$ is a rotation $2k\pi/5$, $1 \leq k \leq 5$, about the vector $(1/2, 0, \tau/2)$. Moreover, the matrix $V$ is orthogonal since the product of $V$ with its transpose is the identity, i.e. $VV^T = I$.

Similarly, the vector originating at the origin with end-point located at the centroid of a face of the icosahedron can be used to determine the change of basis matrix $W$ necessary to represent threefold symmetry. As above, we find that the matrix representing a $2k\pi/3$ rotation, $1 \leq k \leq 3$, about $w_1$ is $WR_{2k\pi/3}W^T$.

Finally, the matrix $R_\theta$ evaluated at $\theta = \pi$ is the matrix that rotates the icosahedron $\pi$ radians with respect to the $x$ axis. In particular, the linear map associated with this matrix sends $(1/2, 0, \tau/2)$ to $(1/2, 0, -\tau/2)$. As $V$, $W$, and $U$ are all orthogonal matrices with determinant equal to one, each matrix belongs to $SO(3)$. Moreover, the order of the rotation matrices are found to be as follows:

$$\left(VR_{2k\pi/5}V^T\right)^5 = VR^5_{2k\pi/5}V^T = \left(VR_{2\pi}V^T\right) = I;$$

$$\left(WR_{2k\pi/3}W^T\right)^3 = WR^3_{2k\pi/3}W^T = \left(WR_{2\pi}W^T\right) = I;$$

$$U^2 = I.$$

From an abstract algebraic point of view, these relations are precisely those given by Hamilton for the icosian system. The subgroup of $SO(3)$ generated by these matrices is, therefore, isomorphic to the rotation group of the icosahedron, or to the alternating group $A_5$. Let $\Lambda = VR_{2k\pi/5}V^T$ and $K = WR_{2k\pi/3}V^T$. Since $\lambda = \iota\kappa$ in Hamilton's system, we can describe this subgroup of $SO(3)$ generated by $\Lambda$ and $K$ with given relations:

$$R_I = \left\langle K, \Lambda \mid \Lambda^5 = K^3 = \Lambda K^{-1} \right\rangle.$$

Our next task is to consider the quaternions that correspond to $\Lambda$ and $K$, or, equivalently, to $\lambda$ and $\kappa$.

### 3.6.2 The two-to-one homomorphism $\varphi : S^3 \longrightarrow SO(3)$

The set of quaternions $\mathbb{H} - \{0\} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{E}\} - \{0\}$, under the rule of multiplication given by Hamilton, is also a (non-commutative) group. The unit quaternions $\{q \in \mathbb{H} \mid |q| = 1\}$ is also a group, since $|q_1 q_2| = |q_1||q_2| = 1$. Given the Euclidean geometric interpretation of the underlying set with the 3-dimensional sphere, denote this group by $S^3$. In accord with Hamilton's observation concerning the angular directions of a quaternion, a unit quaternion $q$ can be written as $q = \cos\theta + u \sin\theta$, where $u$ is a unit imaginary quaternion $xi + yj + zk$. In what follows, I develop the relevant mathematics along the lines found in Montesinos (1985) and Tapp (2005).

We want to understand the relationship between quaternions and rotations of both the sphere and the Platonic solids. Let $q$ be as given, and let $v$ be a purely imaginary unit quaternion. Both $u$ and $v$ belong to the subspace $\{i, j, k\}$ of $\mathbb{H}$. Now consider the action of the unit quaternion $q$ on this subspace under the map $\varphi_q$ defined by

$$w \longmapsto qw\overline{q},$$

where $\overline{q}$ is the conjugate of $q$. (Note: $\overline{q} = q^{-1}$, since $q$ is a unit quaternion.)

There are several observations to make about the map $\varphi_q$. First, $\{i, j, k\}$ is linearly isomorphic to $\mathbb{E}^3$ under the map between vector spaces that sends $i$, $j$, $k$ to $e_1$, $e_2$, $e_3$, respectively. Second, identifying purely imaginary quaternions as vectors in space under the isomorphism, the map $\varphi_q$ is a linear map from $\mathbb{E}^3$ to $\mathbb{E}^3$. In other words, for any real number $c$ and vectors $u$ and $v$ in $\mathbb{E}^3$:

- $cv \longmapsto q\,(cv)\,\overline{q} = (qc)\,v\overline{q} = (cq)\,v\overline{q} = c\,(qvq)$
- $v + w \longmapsto q\,(v + w)\,\overline{q} = (qv + qw)\,\overline{q} = qv\overline{q} + qw\overline{q}$

We therefore know that there is a matrix $A \in GL_3\,(\mathbb{E})$ representing the linear map $\varphi_q$, so that $\varphi_q\,(v) = A_q x$.

Three additional properties of the map are observed in the following cases. First, when $w = \lambda u$, a scalar multiple of the purely imaginary part of $q$. We have that $u \longmapsto qu\overline{q}$. Then

$$q \cdot (\lambda u) \cdot \overline{q} = \lambda u,$$

as can be shown by the properties of quaternion multiplication, including the fact that $q\overline{q} = 1$. This implies that the quaternion $u$ is a fixed point of the map, that is, $u \longmapsto u$. The same result holds for $-u$. Both vectors represent the axis of rotation.

Second, when $w$ is perpendicular to the axis of rotation $u$, we have $w \longmapsto qw\overline{q}$. Then

$$qw\overline{q} = (\cos\theta + u\sin\theta)\,w\,(\cos\theta - u\sin\theta) = \cos 2\theta w + \sin 2\theta\,(u \times w)\,,$$

as can be shown by the properties of quaternion multiplication, including the double-angle formulas for sine and cosine.

Finally, any vector $v$ that is a linear combination of $u$ and $u \times w$, say $v = \lambda_1 u + \lambda_2\,(u \times v)$, can be conjugated by $q$ by the map $\varphi_q$. That is,

$$qv\overline{q} = q\,[\lambda_1 u + \lambda_2\,(u \times w)]\,\overline{q} = \lambda_1 q\,(u)\,\overline{q} + \lambda_2 q\,(u \times w)\,\overline{q}.$$

The quaternion $q = \cos\frac{\theta}{2} + u\sin\frac{\theta}{2}$ therefore corresponds to a rotation of the sphere by $\theta$ radians about the axis $u = (x, y, z)$, the imaginary part of $q$. Because the map is linear, there is a transition matrix $V$ for which $q$ is mapped to $VR_\theta V^T$. Hence $\varphi_q$ can be interpreted as a rotation of 3-space about the origin. Moreover, $\varphi_q$ is implemented by a matrix $A_q$, so that $\varphi_q(v) = A_q(v)$. The matrix $A_q$ is invertible since $v = \overline{q}(qv\overline{q})q = v$; i.e. $A_q^{-1} = A_{\overline{q}}$.

Now both the set of unit quaternions and $SO(3)$ are groups, whereby the foregoing observations suggest a map from the set of unit quaternions, $S^3$, to the rotation group of the sphere. Define the map

$$\varphi : Sp(1) \longrightarrow SO(3),$$

by

$$q \longmapsto A_q.$$

The map $\varphi$ is a homomorphism. For if $q, r \in S^3$, then $qr \longmapsto A_{qr} = A_q A_r$. And using the fact that the kernel of $\varphi$ is $\{\pm 1\}$, it can be shown that $\varphi$ is a two-to-one epimorphism by which $\pm q$ is identified with a rotation of the sphere (see Tapp 2005, p. 128).

In particular, if

$$q = \cos\frac{\pi}{5} + \sin\frac{\pi}{5}(ai + a\tau k)$$

and

$$r = \cos\frac{\pi}{3} + \sin\frac{\pi}{3}(b(3\tau + 1)i + b\tau k),$$

then the unit quaternions $\pm q$ and $\pm r$ correspond to a $2\pi/5$ and $2\pi/3$ rotations, respectively, about the two axes of symmetry of the icosahedron. That is,

$$q v \overline{q} = V R_{2\pi/5} V^{T} \left( v \right)$$

and

$$r v \overline{r} = W R_{2\pi/3} W^{T} \left( v \right).$$

# Chapter 4

# The Platonic solids and Polyhedral manifolds

In the final chapter, I follow the historical path of the Platonic solids into the theoretical context of topology at the turn of the twentieth century. The roots of topology can actually be traced back to Leonhard Euler (1707-1783) and Descartes (Stillwell 1989, pp. 293-294). Descartes had studied the combinatorics of the Platonic solids in connection with the formula that now bears Euler's name: The Euler characteristic, or $V - E + F = 2$, which relates the total number of vertices, edges, and faces of a convex polyhedra — of which the Platonic solids are indeed examples — and which is a numerical invariant that can be used to distinguish different types of surfaces. Although Descartes and Euler's work inform the history in an important way, I begin this chapter with Poincaré's foundational work in topology. It is with Poincaré that the *theory* of topology comes into being, and it is here that two of the Platonic solids were used to represent and construct geometric objects of higher dimension.

In section 4.1, I discuss Poincaré's methodology and philosophical view, both of which furnish a context for the historically relevant topological aspects of the Platonic solids. In section 4.2, I consider Poincaré's example of the 2-dimensional torus that he uses to motivate homology and the fundamental group. I then take a closer look at Poincaré's computation of 0- and 1- dimensional homology of two polyhedral manifolds obtained from the cube and the octahedron. Finally, in section 4.4, I review the history of topology following Poincaré's discovery of a homology sphere with non-trivial fundamental group, including the fact that the dodecahedron was found to yield another polyhedral manifold — the dodecahedral space — that is homeomorphic to Poincaré's homology sphere.

## 4.1 Poincaré's scientific method

Topologist Soloman Lefschetz (1884-1972) claims that the historical roots of algebraic topology are "more-or-less obscure," but he nevertheless identifies Henri Poincaré as the founder of this field of mathematics (James 1999, p 531). Poincaré's founding document, *Analysis situs* (1895), is concerned with the key concepts of manifold and homeomorphism and takes up the central task of stating when two manifolds are not homeomorphic by examining differences in their homology or fundamental group structures; when there are no differences between these structures, the manifolds may or may not be homeomorphic. The five subsequently published "Complements" (1899 - 1904) to the text further elaborated on earlier points and clarified issues in the *Analysis*. Poincaré is credited with having "opened a vast new area of mathematics" (Stillwell 2009, p.9), and his explicit methodological approach to scientific practice may help to explain why: It emphasizes the practice of distinguishing objects of study or, in the case of topology, geometric objects (or classes of such objects) up to the topological notion of equivalence, namely, homeomorphism. (Poincaré's methodological view may thus be contrasted to Descartes' view of the reduction of geometry to algebra.)

In his *Science and Method* (1914), Poincaré had asked the general question, What approach should be taken to the practice of science and history? It is interesting that he considers history in addition to science, and he in fact addresses the former at greater length than the latter as regards the question of method. Claiming that history generally doesn't repeat itself, he took the problem of methodology to be one of knowing which kinds of facts to select. For Poincaré, it is the "interesting fact" of the subject, or the fact that is repeated, that is first selected and made the focus of historical investigation. However, in time "regular facts" lose their interest once they are understood, so that additional features of the subject's history must be considered.

> Then it is the exception which becomes important. We cease to look for resemblances, and apply ourselves before all else to differences, and of these differences we select first those that are most accentuated, not only because they are the most striking, but because they are the most instructive [p. 20].

However illuminating his method may be for history, it undoubtedly sheds light on the nature of his work in topology; it is illustrated quite clearly in the case of two different (but related) kinds of algebraic structures that he defined for topological spaces: namely, homology and the fundamental group.

Now Poincaré did not focus on the Platonic solids in the way that Hamilton did in his letter to Graves. However, his apparently original uses of the cube and octahedron in order to illustrate homology theory show that (at least two) of the Platonic solids are serving to bridge algebra and topology. However, just as Hamilton was concerned to provide a philosophical justification for the mathematical theory of imaginary units and a fourth dimension, Poincaré was taking a philosophical position regarding both the construction of higher dimensional geometric figures and the mathematical science aimed at the study of those features of manifolds that remain unaltered up to continuous transformation. This is seen quite clearly in the introduction to the *Analysis*, in which Poincaré motivates the study of topology — or what he called *analysis situs*, or analysis of position — from a philosophical point of view. We find here an allusion to the Kantian distinction between the pure and sensible aspects of mathematics with respect to the use of figures in topological reasoning.

> Nobody doubts nowadays that the geometry of $N$ dimensions is a real object. Figures in hyperspace are as susceptible to precise definition as those in ordinary space, and even if we cannot represent them, we can still conceive of them and study them.... Geometry, in fact, has a unique *raison d'etre* as the immediate description of the structures which underlie our senses [Poincaré 2009, pp. 18-19].

For Poincaré, part of that immediate description was based on using geometric figures, which he supposed to be especially useful in the case of imaginary functions. But there is a more general point that he was attempting to make about the use of figures in topology:

> The use of figures is, above all, then, for the purpose of making known certain relations between the objects that we study, and these relations are those which occupy the branch of geometry that we have called *Analysis situs*, and which describes the relative situation of points and lines on surfaces, without consideration of their magnitude [ibid.].

Two figures that Poincaré made use of in the study of higher dimensional objects were the cube and the octahedron. The dodecahedron also makes a prominent appearance in the early history of topology following the results of Poincaré.

## 4.2 Poincaré's definitions of manifold

Poincaré begins the formal exposition of the *Analysis* with a definition of a term that also happens to be the first term defined in Euclid's *Elements*: "Any sequence of $n$ variables will be called a *point*" (Poincaré, p. 21; emphasis in the original). In other words, he takes a point to be a sequence of $n$ variables $x_1, x_2, ..., x_n$ that specify the coordinates of that point in an $n$-dimensional space. Certain collections of points may then constitute *manifolds*, the unifying concept of *Analysis situs*. The primary problem for topology is then to determine when two manifolds are homeomorphic (when one can be continuously deformed into the other) or not. Homology and the fundamental group, both algebraic concepts, become the tools that Poincaré uses to determine when two manifolds are not homeomorphic.

Poincaré offers two definitions of *manifold*, the concept of which (as Poincaré was fully aware) had already been advanced by Riemann in 1854 (James 1999, pp.26-27). First, an $m$-dimensional manifold $V$ is a collection of $n$-dimensional points that satisfy a system of $p$ continuous functions with continuous first derivatives (which never simultaneously vanish) and $q$ inequalities, where $m = n - p$. The condition of continuous first derivatives indicates that he is assuming the notion of a differentiable manifold. A submanifold of an $m$-dimensional manifold is then a part of the manifold that has dimension less than $m$.

His second definition of an $m$-manifold consists of a system of $n$ holomorphic functions:

$$x_1 = \theta_1 \left(y_1, y_2, ..., y_m\right)$$

$$x_2 = \theta_2 \left(y_1, y_2, ..., y_m\right)$$

$$\vdots$$

$$x_n = \theta_n \left(y_1, y_2, ..., y_m\right).$$

The condition that the functions be holomorphic presumably means that they are differentiable functions of a complex variable, but he doesn't specify the meaning of the term (nor does the translator in his commentary).

Poincaré's definitions of manifold are related, as seen by considering Poincaré's first example of a manifold, namely, the torus. He defines the torus by the equation

$$\left(x_1^2 + x_2^2 + x_3^2 + R^2 - r^2\right)^2 - 4R^2\left(x_1^2 + x_2^2\right) = 0.$$

where

$$x_1 = (R + r\cos\ y_1)\cos\ y_2,$$

$$x_2 = (R + r\cos\ y_1)\sin\ y_2,$$

$$x_3 = r\sin\ y_1.$$

By the second definition of manifold, the torus is a two-manifold, since it is a system of three (differentiable) functions of two variables. And by the first definition of manifold, we find that the dimension of the manifold is $m = n - p = 3 - 1 = 2$. However, for every point of the torus, as Poincaré notes, there "corresponds an infinite number of systems of values of the $y$" due to the periodicity of the sine and cosine functions. Hence limiting the variables by taking $0 \leq y_1, y_2 < 2\pi$ yields one system that describes the given 2-manifold.

The torus can also be described by what Poincaré calls "the method of geometric representation," which consists in taking the square $[0, 2\pi] \times [0, 2\pi]$ in the Euclidean plane and identifying, or "conjugating," opposite edges. Each edge has an *orientation*, determined by its boundary: if the edge $E$ has boundary consisting of two ordered points $e_i, e_j, i < j$, then the edge is oriented in the direction that takes $e_i$ is initial point and $e_j$ as final point. The path from $e_i$ to $e_j$ along the edge will be denoted by $E$, since the path then defines the edge. Traversing the same edge but in the opposite direction by the path with initial point $e_j$ and final point $e_i$ will be denoted by $E^{-1}$ (for fundamental group) or $-E$ (for homology). (Note that an edge $E$ isn't necessarily a loop in the space, but that it may map to a loop following identifications of edges. The convention of sign simply denotes the direction by which an oriented edge is traversed.)

The identifications of the square required to obtain the torus thus respect orientations according to the following labels. Identifying edge $E_0 = [0, 2\pi] \times \{0\}$ with edge $E_3 = [0, 2\pi] \times \{2\pi\}$ gives a latitudinal circle $a$ of the torus, while identifying $E_1 = \{2\pi\} \times [0, 2\pi]$ with edge $E_2 = \{0\} \times [0, 2\pi]$ gives a longitudinal circle $b$ of the torus. By letting $e_0 = (0, 0), e_1 = (2\pi, 0), e_2 = (0, 2\pi)$ and $e_3 = (2\pi, 2\pi)$, the intended orientations of the edges are determined. Both circles constitute 1-dimensional submanifolds of the torus, and neither can be brought to coincide with the other in the resulting manifold, but they intersect at the point $p$ of the manifold to which the four vertices of the rectangle are mapped. Poincaré leaves the example at this stage. However, in order to motivate his concepts of homology and fundamental group for polyhedral manifolds, I briefly explicate how Poincaré would show that these identificaions figure into the homology and fundamental group of the torus, which happen to be isomorphic.

The single point of intersection, $p$, of the circles, generates the 0-dimensional homology of the torus. The two independent 1-dimensional cycles $\gamma_1$ and $\gamma_2$ generate the 1-dimensional homology of the torus. And the 2-manifold itself, $T$, generates the 2-dimensional homology of the torus. These are the non-trivial homologies of the torus. (The higher dimensional homology is trivial, since $T$ is a 2-manifold.) Now Poincaré didn't further explain these relations as constituting the homology *groups* of the torus, which was done later by Emily Noether (Poincaré 2009, translater's introduction). But this is not to say that Poincaré wouldn't have understood these homology relations as constituting groups isomorphic to the abelian groups $\mathbb{Z}$ $\mathbb{Z} \oplus \mathbb{Z}$, and $\mathbb{Z}$ respectively, where $\mathbb{Z}$ is the abelian group of integers.

The fundamental group of the torus is then determined by considering the relation between all possible loops, or "closed contours," based at $p$. The multiplicative operation of the group is path concatenation, defined for equivalence classes of loops $[\alpha]$ by adjoining terminal points of a representative path to an initial point of another. Two loops $\alpha_1$ and $\alpha_2$ are equivalent when one can be continuously deformed to the other; in this case both loops will belong to the same equivalence class, which we denote by $[\alpha]$.

The elements of the fundamental group $G$ of the torus are the equivalence classes of the two representative loops $[a]$ and $[b]$ and combinations thereof. The group $G$ has identity element $[1]$, determined by the constant path 1. Both homology and fundamental groups of the torus consequently have identical group structures, which can be explained by observing the key relation that holds in the square $[0, 2\pi] \times [0, 2\pi]$ prior to identification of edges.

In the case of the fundamental group, by starting at initial point $e_0 = (0, 0)$ we find that the concatenated path $E_0E_1E_3^{-1}E_2^{-1}$ returns to the initial point $e_0$. Because the square is convex, the loop is equivalent to the constant path at $e_0$ (i.e. the path can be continuously transformed to the constant path at $e_0$). This is expressed by the equation $E_0E_1E_3^{-1}E_2^{-1} = 1$. In the case of first-dimensional homology, the relation is expressed by $E_0 + E_1 - E_3 - E_2 = 0$.

The homology group of a manifold is always abelian, but the fundamental group of a manifold is not always abelian, as illustrated by Poincaré's famous example of a homology sphere. However, the first-dimensional homology and the fundamental groups are related by the fact that the quotient of the latter by the commutator subgroup is isomorphic to the former. Poincare was not aware of this general fact, but he did recognize several instances of the proposition (pp. 59, 61).

The torus is a comparatively simple example to consider, despite its non-trivial homology and fundamental group. An even simpler example is the sphere $S^2$, the 2-manifold which has been a major part of the story of the Platonic solids. Because any closed contour on the sphere based at any one of its points, say $p$, can be continuously deformed to the constant path at $p$, the sphere has trivial fundamental group and, therefore, trivial first-dimensional homology. Poincaré knew this fact, as well as the analogous fact pertaining to the 3-sphere. The main task that he undertook was then to determine when a given 3-manifold was homeomorphic to the 3-sphere. If two manifolds are homeomorphic, then they have precisely the same homology and fundamental groups. So finding that one or the other of the latter conditions fails for two manifolds would suffice to show that they are not homeomorphic. Poincaré drew on this fact when he used the fundamental group to distinguish the 3-sphere from a certain 3-manifold with trivial homology but non-trivial fundamental group (considered below).

## 4.3 Polyhedral manifolds

In this section I provide an account of Poincaré's use of the cube and octahedron in obtaining 3-manifolds. Poincaré was aware of Jordan and Klein's work on finite linear groups, taking the latter's to be "a geometric method of rare elegance" (Poincaré 2011, p.19). He may therefore have had occasion to think about the use of Platonic solids, especially the cube, in setting out to construct 3-manifolds out of 2-dimensional ones by his particular methods. In any case, the cube may simply have been a natural place to start, as may become clear.

Poincaré first introduced the concept of homology in section 5 of the *Analysis*. While his account of the concept is quite short, its application runs throughout the subsequent sections of his work and is further developed in the third and fourth complements to the *Analysis*. His overall approach is to take a $p$-dimensional manifold $V$ and analyzes its submanifolds, the $q$ manifolds $W$, $q \leq p$. The submanifold $W$ has a boundary consisting of $\lambda$ $(q-1)$-dimensional manifolds, denoted by $v_1, v_2, ..., v_\lambda$, and from these the $(q-1)$-dimensional homology of the manifold can be determined. His account is thus completely general with respect to dimension. However, to remain specific to the important ideas of the history of the Platonic solids, I restrict his account of homology to two examples of *polyhedral manifolds*, in which he restricts his computations of homology to dimensions 0 and 1.

Poincaré draws on two kinds of representation of polyhedral manifolds. First, he uses the geometric method of representation for the "purpose of making known certain relations" of a polyhedral manifold based on a polyhedron in 3-space. "The simplest example," as Poincaré writes, "is that where we have a single polyhedron which is a cube $ABCDA'B'C'D'$" (Poincare 2009, p. 50). This description takes the cube to be a solid, so that the resulting identifications result in a 3-manifold. Using Cartesian coordinates, he then locates the eight vertices of the unit cube in the first octant of Euclidean space:

$$A = (0,0,0) \qquad A' = (0,0,1)$$
$$B = (0,1,0) \qquad B' = (0,1,1)$$
$$C = (1,0,0) \qquad C' = (1,0,1)$$
$$D = (1,1,0) \qquad D' = (1,1,1)$$

Viewing the cube from above, Poincaré assumes that the vertices of the face $ABCD$ are traversed in a counter-clockwise direction (in the order $ACDB$) and that the vertices of the face $A'B'C'D'$ are traversed in a clockwise direction (in the order $A'B'D'C'$). These specifications determine a set of orientations of the four 1-dimensional manifolds that constitute the individual boundaries of these two faces. In particular, the one-dimensional manifolds, or the edges, that constitute the boundary $ABCD$ are represented by vectors

$$v_0 = AB,\ v_1 = BD,\ v_2 = AC, \text{ and } v_3 = CD,$$

while the one-dimensional manifolds that constitute the boundary of $A'B'C'D'$ are represented by vectors

$$v_0' = A'B',\ v_1' = B'D',\ v_2' = A'C', \text{ and } v_3' = C'D'.$$

He then provides a schema for the identification of the faces of the cube that yield the 3-dimensional polyhedral manifold that he constructs. Describing the identifications of the opposite faces of the cube as "conjugation" of the paired faces, he uses the symbol '$\equiv$' to denote each identification:

$$ABDC \equiv A'B'D'C'$$

$$ACC'A' \equiv BDD'B'$$

$$CDD'C' \equiv ABB'A'$$

So in the first conjugation, face $ABDC$ is identified with $A'B'D'C'$ so that the vertices of the first face occur in the order listed, and so that these vertices correspond in order to the vertices $A'$, $B'$, $C'$ and $D'$ of the second face; and so on for the remaining two pairs of faces. As Poincaré observes, this identification preserves orientation in the resulting polyhedral manifold, which is the 3-dimensional analogue of the 2-dimensional torus.

A loop around the boundary of the first face can be described by taking the orientation of the one-manifolds to be determined by the direction of the vectors, according to which there are two sets of pairs of one-dimensional manifolds that are identified by having the same direction. In the notation that Poincaré used, we can then write that:

$$v_0 + v_1 \sim v_2 + v_3,$$

or, alternatively,

$$v_0 + v_1 - v_2 - v_3 \sim 0.$$

The notation here signifies the fact that the boundaries of the 2-dimensional submanifolds of the cube are trivial. However, in the polyhedral manifold that results by conjugating the faces of the cube in this manner, Poincaré finds that: (i) the 0-dimensional homology of the manifold is generated by the single point at which the four vertices are identified, since all vertices of the cube are identified; and (ii) the 1-dimensional homology is generated by the three homology classes of edges, namely, those represented by the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. In terms of groups, these homology groups are isomorphic to $\mathbb{Z}$ and $\mathbb{Z}^3$, respectively.

Using the same geometric method of conjugation but a different schema of identification, Poincaré considers the 0- and 1-dimensional homology of a polyhedral manifold using the octahedron.

Poincaré's second method of representing polyhedral manifolds is by way of a discontinuous group. Fixing a point $(x, y, z)$ in $\mathbb{E}^3$, e.g. in the unit cube, he considers a *system of substitutions* that consists in translating the coordinates of the point by one unit along the $x$-, $y$- or $z$ axes. He denoted the substitutions by

$$(x, y, z; x + 1, y, z; x, y + 1, z; x, y, z + 1)\,,$$

$$0 \leq x < 1,\ 0 \leq y < 1,\ 0 \leq z < 1.$$

The notation here is intended to show that the three substitutions

$$(x + 1, y, z)\,, (x, y + 1, z)\,, \text{ and } (x, y, z + 1)$$

generate the system. (In other words, $(x, y, z; x + 1, y, z)$, $(x, y, z; x, y + 1, z)$, and $(x, y, z; x, y, z + 1)$ are three particular substitutions corresponding to a translation of $(x, y, z)$ by one unit along the $x$, $y$ and $z$ axes, respectively.)

The system of substitutions also applies to sufficiently small neighborhoods of points, so Poincaré is able to show that the group action is *properly discontinuous*, the definition of which he presupposes of the reader. A properly discontinuous group action on $\mathbb{E}^3$ is a group of homeomorphisms $g$ of $\mathbb{E}^3$ such that for every point $p$ in space, there is an open ball $B$ centered at $p$ but disjoint from $g(B)$, where $g$ is taken to be other than the identity map (Munkres 2000, p. 490). Of the infinitely many cubes that tessellate $\mathbb{E}^3$ by the action of this group, he takes the unit cube to be the *fundamental domain* of the group. Applying the substitutions of this group to the faces of the cube that coincide with the $xy$, $xz$ and $yz$ planes, the group of substitutions recovers the mode of conjugating faces of the cube, as in the example of the first polyhedral manifold; the polyhedral manifold obtained by the method of a discontinuous group is again the 3-torus.

## 4.4 The dodecahedral space

In section 2.2 of the *Analysis*, Poincaré defines the central concept of *homeomorphism.* As Poincaré described it, a homeomorphism is a continuous map that achieves a particular correspondence between two manifolds $V$ and $V'$. He specifies two conditions of the correspondence: (i) any point of either manifold corresponds to one and only one point of the other, and (ii) any (sub)manifold $W$ contained by either $V$ or $V'$ corresponds to one and only one (sub)manifold $W'$ contained by the other. He then says that manifolds $V$ and $V'$ are *homeomorphic* if there is a homeomorphism between the two manifolds. Poincaré's characterization of homeomorphism is thus consistent with the familiar definition of such a relation as a continuous bijection from $V$ to $V'$ with continuous inverse.

Now Poincaré had conjectured in the *Analysis* that a 3-manifold with trivial homology would be homeomorphic to the 3-sphere, which has both trivial homology and fundamental group. However, in the *Fifth Complement* to his *Analysis*, he set out to show that this was false by constructing a manifold with trivial homology but non-trivial fundamental group. He constructed the manifold not as a polyhedral manifold but rather by the method of a Heegard decomposition, which consisted in identifying the boundaries of two genus-2 handle-bodies (i.e. a solid tori with two holes) via a certain homeomorphism. Upon analyzing the relations between the pertinent loops of the resulting space, he was able to determine its fundamental group. It is an amazing fact that the group had a homomorphic image (moding out by the center) defined by the icosahedral relations that Hamilton had invented. Poincaré doesn't appear to have been aware of this fact. That such relations were found to hold may have suggested to later topologists that the space that he discovered — known today as the Poincaré homology sphere — could possibly be obtained by conjugating the faces of the polyhedron associated to the given relations. That polyhedron was the dodecahedron.

For several decades after Poincaré's effort to establish the field, the history of topology becomes complex due to an increased number of individuals making discoveries or working on topological problems that just happened to be focused on the same topological object that was appearing in different guises. Poul Heegaard (1871-1948) and Max Dehn (1878-1952) had introduced Poincaré's *Analysis* to the German-speaking mathematical community by 1907 (see James,1999) and not long after that German mathematicians were relating Poincaré's results to two polyhedral manifolds that took the dodecahedron as the polyhedron for the construction of two distinct three-manifolds. Gordon's (ibid., pp. 462- 464) account of the development of three-dimensional manifolds — and, in particular, of homology three spheres —

reviews the origins of the constructions given during this time. He explains that, in addition to Poincaré's homology sphere, there was Dehn's 1910 construction of a homology sphere using surgery techniques on a right-handed trefoil knot, as well as Kneser's 1929 construction of a spherical dodecahedral space. Although Kneser claimed that the spherical dodecahedral space was homeomorphic to Dehn's homology sphere, Gordon notes that no two of these spaces were known (by proof) to be homeomorphic before 1933. In that year, Weber and Seifert finally proved that they were all equivalent. In "Two Dodecahedral Spaces," Weber and Seifert (1933) provide two examples of dodecahedral spaces, one of which was equivalent to the three equivalent descriptions of the Poincaré homology sphere. The history may suddenly appear to pertain more to the Poincaré homology sphere than the Platonic solids, but we are able once more to observe an historically significant use of one of the solids in this historical context of topology. To appreciate the appearance of the dodecahedron in this context, I consider Seifert and Weber's dodecahedral space.

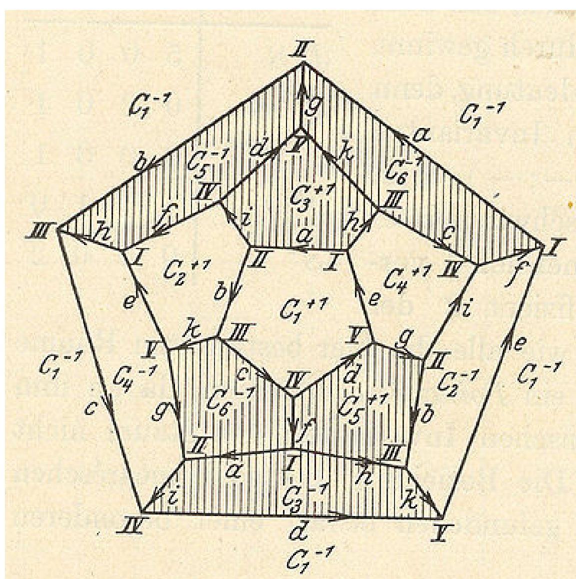

Figure 4.1: Seifert and Threllfal's representation of the dodecahedral space.

Seifert and Threlfall (1934) recapitulate the construction of the Poincare dodecahedral space via the method of conjugation of opposite faces of the dodecahedron, as illustrated in figure 4.1. The conjugation requires a $\pi/5$ rotation of a face that gets identified to the opposite face. In contrast to Poincaré's first example of a polyhedral manifold using a cube, the dodecahedral space is found to have five non-equivalent vertices, labeled $I$, $II$, $III$, $IV$ and $V$. The authors then specify and label the closed path classes based at the point $I$ that generate the fundamental group of the space:

$$
\begin{aligned}
A &= aa^{-1}, & B &= abh^{-1}, \\
C &= hcf, & D &= f^{-1}d\left(d^{-1}f\right), \\
E &= \left(f^{-1}d\right)e, & F &= f^{-1}f, \\
G &= \left(f^{-1}d\right)ga^{-1}, & H &= hh^{-1}, \\
J &= aif, & K &= hk\left(d^{-1}f\right).
\end{aligned}
$$

They then consider six products of these classes that entail relations that the generators satisfy. By simplifying these six relations and letting $U = BC^{-1}$, they find that $B^5 = (BU)^2 = U^3 = 1$, precisely the same relations that Hamilton used to define the icosians and which were found to correspond to rotations of the icosahedron. Thus just as $\lambda$ and $\kappa$ generate the group of rotations of the icosahedron, $B$ and $U$ generate the fundamental group of the dodecahedral space. Understanding the mathematical significance of the relations, Seifert and Threlfall subsequently interpret $B$ as a rotation $2\pi/5$ radians about a vertex of the icosahedron and $U$ as a $2\pi/3$ rotation about the midpoint of a face containing the vertex. As $B$ and $U$ are non-equivalent loops in the resulting space, they are the generators for the non-trivial fundamental group of the dodecahedral space, which (as Seifert and Weber showed) is homeomorphic to the Poincaré homology sphere.

## 4.5 The dodecahedral space is a manifold

Recall that $S^3$ is a compact 3-manifold that can be described by the group of unit quaternions. The binary icosahedral manifold, $I$ (that is, the inverse image of the icosahedral group under the map $\varphi$), is a finite subgroup of $S^3$, and he quotient $S^3/I$ is another description of the dodecahedral space (see Montesinos 1985; Tapp 2005). The group $I$ is a subgroup of $S^3$, and is in fact the inverse image of the icosahedral group under the map $\varphi$ discussed in section 3.6. Since $\varphi$ is two-to-one and the icosahedral group has sixty elements, this inverse image consists of one hundred and twenty elements. As $I$ is a compact manifold, and $S^3$ a compact Lie group, it follows that the given quotient is a manifold. This algebraic approach to describing the dodecahedral space builds on the results of the final sections of the previous two chapters, which further illustrates the interconnectedness of the relevant geometry, algebra and topology: Pacioli's construction for the purpose of specifying the Cartesian coordinates of the icosahedron; the general linear and group-theoretic relations between the quaternions and rotations of the sphere, as motivated by Hamilton and Klein; and, finally, the topological properties of dodecahedral space $S^3/I$.

Hamilton's introduction of the imaginary units $j$ and $k$ compelled him to "permit" — or to *imagine* — a fourth dimension by which the quaternions $\mathbb{H}$ could be interpreted geometrically. When viewed as a vector space, $\mathbb{H}$ is linearly isomorphic to $\mathbb{E}^4$. This allows us to interpret the set of unit quaternions as the 3-sphere, $S^3$, which embeds in $\mathbb{E}^4$.

In Poincaré, we find that the Platonic solids permit the construction of locally 3-dimensional manifolds, of which the dodecahedral space is a historically and theoretically significant example. Does the dodecahedral space embed in Euclidean $n$-space for some specific integer value $n$? Yes, it does. And this is because it is a manifold. Every compact $n$-manifold can be embedded in $\mathbb{E}^{2n+1}$ (Munkres 2000, p.314). It follows that $S^3/I$, a compact 3-manifold, can be embedded in $\mathbb{E}^7$ — the Euclidean space in which the dodecahedral space "lives."

# Chapter 5

# Conclusion

What I have attempted to show in my thesis is the fact that the different appearances of the Platonic solids in geometry, modern algebra, and topology were important for the historical role they played in connecting these areas of pure mathematics. The final sections of each chapter further showed that the icosahedron (and dodecahedron) is an especially significant solid in this respect, since its different theoretical aspects illustrate the interconnectness of the pertinent areas of mathematics. While this part of the historical investigation indeed involved mathematical theory, we can nevertheless conclude that the Platonic solids are just as much an historical as a mathematical phenomenon. The conclusion prompts a few philosophical questions: What consequences might this history have for our understanding of the Platonic solids in mathematics? How does the history enhance our understanding of the solids? What might this history, parameterized by one complex of mathematical ideas, say about the history of mathematics in general?

Ludwig Wittgenstein took mathematics to consist in the use of concepts. Assuming that the Platonic solids are concepts, or a complex of concepts, the questions of interest naturally shift to questions about the human mind. Poincaré suggested that "by studying the process of geometric thought we may hope to arrive at what is most essential in the human mind" (1914, p. 46). This view of geometric thought as a kind of process is compelling, since we can see the process at work in the history of the Platonic solids: e.g. in the individual human mind, as in Hamilton's own account of his work on the quaternions and icosians. As a result, the history of the Platonic solids that I have attempted to write could undermine any uncritical or naive modern Platonist view of the ontological status of mathematical ideas: viz. as non-changing and existing in an eternal mind-independent realm of forms that await to be discovered. While I do not ascribe to such a Platonist view of mathematics, I do agree that mathematical ideas are alive in a sense. They are alive as thoughts in the mind of the mathematical thinker, a mind that has both internal and external features: e.g. the cognitive processes that underlie the abilities necessary for certain cognitive tasks, such as visualization of shape or numerical computation; or mathematical invention and discovery and the historical record that preserves these aspects. The historical record may aslo function to propel mathematics forward in directions with no pre-determined definite end-points, but in directions which are bound to continue to yield surprises of mathematical facts that might not have otherwise been believed to be theoretically necessary — as illustrated by the historical direction of the Platonic solids that led to the construction of the dodecahedral space.

The necessary, static facts of mathematics are one aspect of the subject's history, and the contingent contexts of discoveries or inventions of mathematical concepts another. Both aspects are reflected by the historical record, however much they might appear to diverge with respect to temporality. The Platonic solids are perhaps exemplary of this mathematico-historical phenomenon.

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