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Dual Diophantine Approximation on Planar Curves

General Hausdorff theory

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Abstract

The general Hausdorff theory of Dual Diophantine approximation on manifolds was initiated by the work of Beresnevich, Dickinson, and Velani in [4], in which they established that the set of ψ-approximable points on a manifold has full measure when a certain sum diverges. A decade later in [13], Hussain established the convergence counterpart to the above result in the case of the parabola. Not long after in [11], Huang proved a convergence result for all planar curves with regards to the Hausdorff s-measure. In this thesis, we extend Huang's convergence result to the Hausdorff g-measure for all planar curves.

To Carly.

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Contents

Notation

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. The positive integers, integers, rational numbers, and real numbers.

$\mathbb{Z}^*$ The set $\mathbb{Z} \setminus \{0\}$.

$\|\cdot\|$ The distance to the nearest integer.

$|\cdot|$ The ℓ^∞ norm, i.e. if $\mathbf{q} = (q_1, \ldots, q_n)$, then $|\mathbf{q}| = \max\{q_1, \ldots, q_n\}$.

$\exists^\infty$ Denotes "there exist infinitely many".

$\ll$ and $\gg$ Vindogradov's notation for inequality with an unspecified positive constant multiple.

$a \asymp b$ Means $a \ll b$ and $a \gg b$.

1 Introduction

Classical Diophantine approximation begins with Dirichlet's theorem, which states that for any real number α and any $Q > 1$, there exist integers p and q such that $1 \leq q < Q$ and

$$|\alpha q - p| \leq \frac{1}{Q}. \tag{1.1}$$

An equivalent restatement of equation (1.1) to be found in [3] is

$$\min_{q<Q} \|\alpha q\| \leq \frac{1}{Q}.$$

Here and throughout $\|\cdot\|$ refers to the distance to the nearest integer. Then, as q ranges through $\{1, 2, \ldots, Q-1\}$, the real numbers $\alpha, 2\alpha, \ldots, (Q-1)\alpha$ always come within $1/Q$ units of an integer.

Dirichlet's theorem gives a formal description of the density of $\mathbb{Q}$ in $\mathbb{R}$ in terms of the size of the denominator q. In particular, notice that inequality (1.1) implies that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{qQ} < \frac{1}{q^2}.$$

As an example of the applications of Dirichlet's theorem, a corollary states that a real number α is irrational if and only if there exist infinitely many pairs of relatively prime integers p and q satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

For more details regarding the classical theory of rational approximation to real numbers and related subjects, see [3], [10], and [20].

Many questions in metric Diophantine approximation concern what happens if one replaces the power function obtained from Dirichlet's theorem with a general 'approximation function' $\psi(q)$. In this thesis, we study approximation functions $\psi : \mathbb{N} \to (0, \infty)$ that are assumed to be monotone decreasing. Then one wishes to measure the size of the set of real numbers α satisfying

$$\|\alpha q\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}. \tag{1.2}$$

We say any real number α satisfying (1.2) is ψ-approximable.

The set of ψ-approximable number is defined

$$A_1(\psi) = \{\alpha \in \mathbb{R} : \exists^\infty\ q \in \mathbb{N} \text{ such that } \|\alpha q\| < \psi(q)\}.$$

The Lebesgue measure of $A_1(\psi)$ is determined in the following classic theorem (see [3] and the references therein). Note a set is said to have 'full measure' if its complement has measure zero.

Theorem 1 (Khinchin)**.** *Let $\psi : \mathbb{N} \to (0, \infty)$ be a function. Then*

$$|A_1(\psi)| = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty \\ \text{Full}, & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty \text{ and } \psi(q) \text{ is decreasing.} \end{cases}$$

Remark. The decreasing condition in the second part above is known to be necessary, the Duffin-Schaeffer counterexample gives an example of a non-monotonic approximation function ψ such that $\sum_q \psi(q) = \infty$ for which $|A_1(\psi)|$ has measure zero. Restating the divergence result in Khinchin's theorem for a non-monotonic ψ is the subject of the famous Duffin-Schaeffer conjecture

Conjecture (Duffin & Schaeffer). *Let* $\psi : \mathbb{N} \to (0, \infty)$ *be a function. Then the Lebesgue measure of* $A_1(\psi)$ *is full if the sum*

$$\sum_{q=1}^{\infty} \psi(q) \frac{\varphi(q)}{q}$$

diverges, where φ *is Euler's totient function.*

See [8] and the references therein for further discussion of non-monotonic approximation functions.

This kind of "zero-full" result in Theorem 1 is common in metric number theory. Khinchin's theorem gives a strikingly simple condition to determine if almost-none or almost-all real numbers are ψ-approximable. To study the more delicate structure of A_1 and its higher-dimensional analogues, we can make use of Hausdorff measures.

2 Hausdorff measures

Hausdorff measures generalize the Lebesgue measure and have numerous applications. They are famously useful for determining non-integer dimensions of fractal sets, and they can be used to discriminate between sets with Lebesgue measure zero. For an introduction see [19], which defines Hausdorff measures in general metric and topological spaces. Note that for our purposes it is convenient to use Hausdorff outer measures, without restricting to the σ-algebra of sets satisfying the Carathéodory measurablility condition. We will make ample use of the countable sub-additivity of these outer measures.

First we define the Hausdorff s-measure $\mathcal{H}^s$. Let E be a subset of $\mathbb{R}^n$. For a fixed $\rho > 0$ a 'ρ-cover' of E is a collection of balls $\{B_i\}$ in $\mathbb{R}^n$ with diameters $\mathrm{diam} B_i < \rho$ and $E \subseteq \bigcup_i B_i$.

For $s > 0$, define

$$\mathcal{H}^s_\rho(E) = \inf\{\sum_i (\mathrm{diam} B_i)^s\},$$

where the infimum is over all countable ρ-covers of E. Then the 'Hausdorff s-measure' is defined

$$\mathcal{H}^s(E) = \lim_{\rho \to 0} \mathcal{H}^s_\rho(E).$$

where the limit is defined in the extended real numbers since $\mathcal{H}^s_\rho$ decreases monotonically as ρ goes to zero.

Notice that $\mathcal{H}^0$ is the counting measure. $\mathcal{H}^n$ is a rescaling of the n-dimensional Lebesgue measure for all n in $\mathbb{N}$.

The 'Hausdorff dimension' of $E \subseteq \mathbb{R}^n$ is defined

$$\dim_{\mathcal{H}} E = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\}.$$

See [11], [3], and the references therein for a discussion of the progress towards determining the Hausdorff dimension of sets arising in Diophantine approximation on manifolds.

We say that g is a 'dimension function' (also known as 'gauge function') if $g\colon (0,\infty) \to (0,\infty)$ is a continuous increasing function satisfying $\lim_{r\to 0^+} g(r) = 0$ and $\lim_{r\to\infty} g(r) = \infty$. Replacing the power function x^s in the definition of $\mathcal{H}^s_\rho$

with a general dimension function $g(x)$ we define

$$\mathcal{H}^g_\rho(E) = \inf\{\sum_i g(\operatorname{diam} B_i)\},$$

where again the infimum is over all countable ρ-covers of E.

Then the 'Hausdorff g-measure' is defined

$$\mathcal{H}^g(E) = \lim_{\rho \to 0} \mathcal{H}^g_\rho(E).$$

Once the Hausdorff dimension of E, say $\dim_{\mathcal{H}} E = s_0$, has been determined, there are three cases: $\mathcal{H}^{s_0}(E)$ is zero, infinity, or it is positive and finite. In the cases in which $\mathcal{H}^{s_0}(E)$ is zero or infinity, we may still ask if there exists a dimension function g satisfying

$$0 < \mathcal{H}^g(E) < \infty.$$

Such a dimension function g is called the 'exact dimension function' of E. A set $E \subseteq \mathbb{R}^n$ is said to be 'dimensionless' if $\mathcal{H}^g(E)$ is zero or infinity for all dimension functions g.

Determining the exact dimension functions of particular sets is a challenging and recently active area of research (see [19] for an introduction). Dimension functions often take the form of iterated logarithms composed with and multiplied by powers. For example, in [18] the exact dimension functions of some Cantor sets are determined and applied to yield results in metric number theory.

3 Diophantine approximation in $\mathbb{R}^n$

When generalizing Dirichlet's theorem to handle Diophantine approximation in higher dimensions, one considers systems of inequalities. This can be done as in the following theorem (see [3] and [9]).

Theorem 2. *Let $R = (r_{ij})$ be an m by n matrix with real entries. For all $N > 1$, there is a $(q_i) = \mathbf{q} \in \mathbb{Z}^m$ with $1 \leq |\mathbf{q}| < N$ such that*

$$\|q_1 r_{1j} + \cdots + q_m r_{mj}\| < N^{-\frac{m}{n}}, \text{ for all } 1 \leq j \leq n.$$

As in the one-dimensional case introduced above, one wishes to study the Diophantine inequalities obtained by replacing the bound in Dirichlet's theorem with an approximation function ψ. Notice that one may define ψ in many ways due to the vector argument. In this thesis we treat the two-dimensional case in which $\mathbf{q} = (q_1, q_2)$, and we define $q = |\mathbf{q}| = \max\{|q_1|, |q_2|\}$ and $\psi(q) : \mathbb{N} \to (0, \infty)$ (where as above we assume ψ is decreasing). Note the definition of $|\mathbf{q}|$ for $\mathbf{q} \in \mathbb{Z}^n$ when $n > 2$ is similar. For convenience I will refer to functions depending on $|\mathbf{q}|$ simply as 'approximation function' for the remainder.

In Theorem 2, we will focus on two cases, particularly when the matrix R is a real $1 \times n$ matrix, or when R is a real $n \times 1$ matrix. These settings are referred to as dual and simultaneous approximation, respectively. This thesis concerns dual approximation, but for completeness I will describe and contrast the two.

First, regarding the dual setting let ψ be an approximation function and $\boldsymbol{\alpha} \in \mathbb{R}^n$. Then we say $\boldsymbol{\alpha}$ is dually ψ-approximable if there exists infinitely many $\mathbf{q}$

in $\mathbb{Z}^n$ such that

$$\|\mathbf{q} \cdot \boldsymbol{\alpha}\| = \|q_1\alpha_1 + \cdots + q_n\alpha_n\| < \psi(|\mathbf{q}|).$$

Define $A_n(\psi)$ as the set of all dually ψ-approximable n-dimensional real numbers, i.e.

$$A_n(\psi) = \{\boldsymbol{\alpha} \in \mathbb{R}^n : \exists^\infty \ \mathbf{q} \in \mathbb{Z}^n \text{ such that } \|\mathbf{q} \cdot \boldsymbol{\alpha}\| < \psi(|\mathbf{q}|)\}. \tag{3.1}$$

Geometrically, dual approximation concerns the use of rational hyperplanes to approximate points in n-dimensional space. In the two-dimensional case, this simply means using lines with rational slopes to approximate planar curves.

Second, in the simultaneous setting again let ψ be an approximation function and $\boldsymbol{\alpha} \in \mathbb{R}^n$. Then $\boldsymbol{\alpha}$ is simultaneously ψ-approximable if there exist infinitely many q in $\mathbb{N}$ such that

$$\max\{\|q\alpha_1\|, \ldots, \|q\alpha_n\|\} < \psi(q).$$

Define the set $S_n(\psi)$

$$S_n(\psi) = \{\boldsymbol{\alpha} \in \mathbb{R}^n : \exists^\infty \ q \in \mathbb{N} \text{ such that } \max(\|q\alpha_1\|, \ldots, \|q\alpha_n\|) < \psi(q)\}. \tag{3.2}$$

The geometric description of simultaneous approximation is measuring the distance (with respect to the supremum metric) from the point $\boldsymbol{\alpha}$ to some rational $\frac{\mathbf{p}}{q}$, where $\mathbf{p} \in \mathbb{Z}^n$.

It should be noted that the dual and simultaneous Diophantine approximation are related to each other through Khinchin's transference principle, which is described in [3]. In some contexts solving a problem in one setting guarantees the existence of a solution in the other setting. However, no such transference principle holds in general for approximation on manifolds.

When generalizing Theorem 1 to handle approximating points in n-dimensional space, the solution is referred to as the Khinchin-Groshev theorem. In fact Groshev proved a broader theorem (see Section 1.3.4 of [3]) concerning the approximation of real matrices in line with Theorem 2. A general treatment of the Lebesgue measure of the sets $A_n(\psi)$ and $S_n(\psi)$ is given in the following theorem, as presented in [8].

Theorem 3 (Khinchin-Groshev)**.** *Let $\psi : \mathbb{N} \to (0, \infty)$ be an approximation function. Then*

$$|A_n(\psi)| = \begin{cases} 0, & \textit{if } \sum_{q=1}^{\infty} \psi(q) q^{n-1} < \infty \\ \textit{Full}, & \textit{if } \sum_{q=1}^{\infty} \psi(q) q^{n-1} = \infty \textit{ and } \psi(q) \textit{ is monotonic in the case } n = 1, \end{cases}$$

and

$$|S_n(\psi)| = \begin{cases} 0, & \textit{if } \sum_{q=1}^{\infty} \psi(q)^n < \infty \\ \textit{Full}, & \textit{if } \sum_{q=1}^{\infty} \psi(q)^n = \infty \textit{ and } \psi(q) \textit{ is monotonic in the case } n = 1. \end{cases}$$

This theorem effectively completes the Lebesgue theory of dual and simultaneous approximation of n-dimensional Euclidean space. As noted above, the monotonicity condition on ψ is necessary in the case $n = 1$, where $A_1(\psi) = S_1(\psi)$, but as stated in Theorem 3 for $n \geq 2$ the approximation function need not be monotonic. See the paper [8] for a nuanced discussion of the role of monotonicity assumption on ψ and its relation to the conjectures of Catlin and of Duffin & Schaeffer.

The Hausdorff theory of Diophantine approximation in $\mathbb{R}^n$ is considered in [9], in which a Hausdorff g-measure version of the Khinchin-Groshev theorem is established for linear forms. Remarkably, Dickinson and Velani's result implies that the sets of under consideration are dimensionless, e.g. applied to the dual case $A_n(\psi)$ this means there is no dimension function g such that $0 < \mathcal{H}^g(\psi) < \infty$.

In [4], Beresnevich, Dickinson, and Velani consider a generalization of the approximation described above. For any compact metric space (Ω, d) with a non-atomic probability measure μ and for any positive decreasing ψ they develop a notion of ψ-approximability. Their results have important implications in the study of dual approximation on manifolds. In fact, the divergence counterpart of the main result proven in this thesis is first presented and proven in section 12.7 of [4] (Theorem 18).

4 Approximation on manifolds

Restricting attention to points on a manifold of $\mathbb{R}^n$, determining if those points satisfy the same approximability properties as the points in $\mathbb{R}^n$ is an area of ongoing research in which a multitude of techniques are used. The problem of extending to manifolds Khinchin-Groshev type theorems, as well as theorems concerning Hausdorff measures which are known as Jarník type theorems, represents a major challenge in metric number theory. This area of research began in 1932 with a conjecture by Kurt Mahler concerning the Veronese curve $V_n = \{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\} \subseteq \mathbb{R}^n$. V.G. Sprindzuk solved Mahler's conjecture in 1965, and studying this curve led to development of the general Lebesgue and Hausdorff theory of Diophantine approximation on manifolds. For more details regarding the early history of this subject, see [3] and the recent preprint [14] for a thorough time-line.

When working on problems in metric Diophantine approximation on manifolds, the major challenge often lies in showing that the worst-case scenario occurs infrequently, which is accomplished by counting the number of rational points near the manifold in question. As discussed below, Huxley's lemma (see [15]) provides the best known bound on the number of rational points near a planar curve, and that result is used in the proof presented here regarding planar curves. However, very little is known regarding counting rational points near manifolds with low codimension in dimensions higher than two, which means that proving many theorems for those manifold using current methods is out of reach.

Kleinbock and Margulis [17] built on Sprindzuk's work using tools from dynamical systems and inspired new interest in the subject around the turn of the century, which in turn led to generalizations of the Khinchin's theorem to handle dual approximation on nondegenerate manifolds. Thus the Lebesgue theory of dual approximation on manifolds is essentially complete. See [14] and the references therein.

Recall a 'hypersurface' in n-dimensional space is a manifold with dimension $n-1$ (i.e. with codimension 1). We say a C^2 planar curve is 'non-degenerate' if it has a non-vanishing second derivative. Non-degeneracy for higher-dimensional manifolds has a similar definition, which guarantees that the manifold is curved enough to deviate from all hyperplanes. See [3] and [14] for definitions and further discussion of manifolds in $\mathbb{R}^n$.

The Hausdorff theory of dual approximation on manifolds is the subject of the so-called 'Generalized Baker-Schmidt Problem' (GBSP). The preprint [14] states the GBSP is great generality for inhomogeneous approximation and multivariable functions.

The result presented here is an extension of Jing-Jing Huang's result in [11], and the method of proof is very similar. The main result proved in [11] is restated here.

Theorem 4 (Huang). *Let ψ be an approximation function. Let $\mathcal{C}$ be a C^2 planar curve non-degenerate everywhere except possibly a set of Hausdorff s-measure* 0*. Then*

$$\mathcal{H}^s\left(A_2(\psi)\cap\mathcal{C}\right)=\begin{cases}0, & \text{if } \sum_{q=1}^{\infty}\left(\frac{\psi(q)}{q}\right)^s q^2<\infty\\ \mathcal{H}^s\left(\mathcal{C}\right), & \text{if } \sum_{q=1}^{\infty}\left(\frac{\psi(q)}{q}\right)^s q^2=\infty.\end{cases}$$

The major difficulty in extending this result is in adapting Huxley's counting result to handle the Hausdorff g-measure, which necessitates the regularity condition (6.1) in the statement of Theorem 5.

5 Auxiliary lemmas and discussion

Note that one Lemma 5 remains in the body of the proof for convenience.

The following special case of the Borel-Cantelli lemma applied to the measure $\mathcal{H}^g$ will be one of the main tools in the proof of Theorem 5. This special case is referred to as the Hausdorff-Cantelli Lemma.

Lemma 1 (Hausdorff-Cantelli). *Let g be a dimension function, let $\{H_i\}$ be a sequence of intervals in $\mathbb{R}$, and suppose that*

$$\sum_i g\left(|H_i|\right)<\infty.$$

Then $\mathcal{H}^g\left(\limsup H_i\right) = 0.$

Above the limit supremum set is defined as usual

$$\limsup H_i = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} H_j = \{x : \exists^{\infty}\ i \text{ such that } x \in H_i\}.$$

For completeness and to demonstrate an application of the Hausdorff g-measure, we present a short proof of Lemma 1.

Proof. Denote

$$E = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} H_j.$$

By definition $\{H_j\}_{j=i}^{\infty}$ is a cover of E, for all i. Let $\rho > 0$. Then there exists an N in $\mathbb{N}$ such that $|H_j| < \rho$ for all $j > N$, because by hypothesis $\sum_j g\left(|H_j|\right)$ converges, so $\lim_{j \to \infty} g\left(|H_j|\right) = 0$. By definition $\lim_{x \to 0^+} g(x) = 0$ and g is increasing. Therefore $\lim_{j \to \infty} |H_j| = 0$. Hence $\{H_j\}_{j=i}^{\infty}$ is a ρ-cover of E for all $i > N$.

Next, for all $\varepsilon > 0$, there exists an M in $\mathbb{N}$ such that

$$\sum_{j=m}^{\infty} g\left(|H_j|\right) < \varepsilon$$

for all $m > M$. Then choose some $k > \max\{M, N\}$. Then $\{H_j\}_{j=k}^{\infty}$ is a ρ-cover of E and

$$\sum_{j=k}^{\infty} g\left(|H_j|\right) < \varepsilon.$$

Therefore, as the infimum of all such expressions, $\mathcal{H}_{\rho}^{g}(E) < \varepsilon$, and notice $\mathcal{H}_{\rho}^{g}$ is decreasing in ρ, so finally

$$\mathcal{H}^g(E) = \lim_{\rho \to 0} \mathcal{H}^g_\rho(E) < \varepsilon.$$

so $\mathcal{H}^g(E) = 0$. □

The following lemma, due to Pyartly, will be used several times throughout the argument. See [10] for a proof.

Lemma 2. *Let $h(x) \in C^2(I)$ and define*

$$\delta_1 = \min_{x \in I} |h'(x)| \quad \textit{and} \quad \delta_2 = \min_{x \in I} |h''(x)|.$$

For $\eta > 0$, define $E(\eta) = \{x \in I : |h(x)| < \eta\}$. Then

$$|E(\eta)| \ll \min\left\{\frac{\eta}{\delta_1}, \sqrt{\frac{\eta}{\delta_2}}\right\}.$$

The following two lemmas allow the number of rational points near a curve to be estimated. The first lemma is due to [15] and the second, being a consequence of the first, is due to [11]. Note that the counting result is only necessary in the most challenging case, i.e. the Θ_2 case when $p = p_0$.

Lemma 3 (Huxley)**.** *Suppose ϕ has a continuous second derivative on a bounded interval Γ which is bounded away from 0, and let $\delta \in \left(0, \frac{1}{4}\right)$. Then for any $\varepsilon > 0$ and $R \geq 1$,*

$$\sum_{R \leq r < 2R} \sum_{\substack{\frac{t}{r} \in \Gamma \\ \left\| r\phi\left(\frac{t}{r}\right) \right\| < \delta}} 1 \ll_\varepsilon \delta^{1-\varepsilon} R^2 + R \log(2R).$$

Lemma 4. *Under the same conditions as Lemma 3, for* $\lambda \in (0,1)$ *and* $R \geq 1$,

$$\sum_{R \leq r < 2R} \sum_{\substack{\frac{t}{r} \in \Gamma \\ \left\| r\phi\left(\frac{t}{r}\right)\right\| \geq \delta}} \left\| r\phi\left(\frac{t}{r}\right)\right\|^{-\lambda} \ll R^2 + \delta^{-\lambda} R \log(2R).$$

The technique employed in this proof, inspired by the techniques in [11], adapt Huxley's counting result (see [15]) into an analytic form useful for Diophantine approximation. The intuition behind this technique is that lines close to a curve correspond to points close to the dual curve. See [11] for further discussion. Note that counting the number of rational points near a manifold is an area of independent interest, see [12].

See the preprint [14] for a very recent breakthrough in the GBSP. The authors manage to prove the inhomogeneous GBSP for nondegenerate hypersurfaces in all dimensions $n \geq 3$, and even for multivariable approximation functions $\boldsymbol{\psi}$, so this thesis represents the completion of the homogeneous problem on hypersurfaces for single variable approximation functions, and is the only result of its kind for manifolds with co-dimension 1. Note that the argument in [14] makes no use of Huxley's or any counting result.

A natural question is whether the techniques used here can be adapted to treat high-codimension manifolds in $\mathbb{R}^n$, but this appears to require novel counting results, since Huxley is specific to $\mathbb{R}^2$. Even in the case of space curves very little is known. I hope to return to this problem in the future.

6 Statement and proof of main result

Theorem 5. *Let ψ be an approximation function. Let g be a dimension function, and assume that there are $0 < s_2 \leq s_1 \leq 1$ with*

$$5s_1 < 6s_2 \text{ and } x^{s_1} \ll g(x) \ll x^{s_2}, \tag{6.1}$$

and further assume that for all $k > 1$,

$$\frac{g(kx)}{g(x)} \text{ is bounded.} \tag{6.2}$$

Let $\mathcal{C}$ be a C^2 planar curve non-degenerate everywhere except possibly a set of Hausdorff g-measure 0. Then

$$\mathcal{H}^g\left(A_2(\psi) \cap \mathcal{C}\right) = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^2 < \infty \\ \mathcal{H}^g\left(\mathcal{C}\right), & \text{if } \sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^2 = \infty. \end{cases}$$

The divergence is established in [4], with no need of the regularity conditions (6.1) and (6.2) on the dimension function g. In this thesis the convergence part is established.

Remark. Regarding the regularity condition (6.1), if $s_1 \geq s_2 > 1$ then $\mathcal{H}^g\left(\mathcal{C}\right) = 0$. A regularity condition similar to (6.2) is used in the proof of the GBSP on hypersurfaces in [14].

Before beginning the proof we make reductions and introduce some notation.

$\mathcal{C}$ is locally represented by graphs of the form $\mathcal{C}_f = \{(x, f(x)) : x \in I\}$, where I is a compact interval and $f\colon I \to \mathbb{R}$ is C^2. Let $\boldsymbol{\gamma}\colon I \to \mathcal{C}_f$ be the coordinate map $\boldsymbol{\gamma}(x) = (x, f(x))$. Since f is differentiable, $\boldsymbol{\gamma}$ is bi-Lipschitz, so for any $E \subseteq I$, $\mathcal{H}^g(E) \asymp \mathcal{H}^g(\boldsymbol{\gamma}(E))$.

The curve $\mathcal{C}$ is defined to be non-degenerate except on a set of Hausdorff-g measure 0. As in [11], define $B = \{x \in I : f''(x) = 0\}$. By hypothesis $\mathcal{H}^g(B) = 0$. Since f'' is continuous, B is closed as the preimage of the closed set $\{0\}$. So $I \setminus B$ is a countable union of open intervals I_i. Each interval I_i can further be decomposed into a countable union of bounded open intervals I_i' where the curve has second derivative bounded away from zero, i.e.

$$0 < c_1 \leq |f''(x)| \leq c_2 < \infty \tag{6.3}$$

for all $x \in I_i'$ where c_1 and c_2 are positive constants depending on the choice of I_i'.

Therefore, by the countable sub-additivity of $\mathcal{H}^g$, is suffices to consider arcs that satisfy (6.3) on a bounded interval I.

Let $\psi : \mathbb{N} \to (0, \infty)$ be an approximation function. Let $(q_1, q_2, p) \in \mathbb{Z}^3$ and denote $q = \max\{|q_1|, |q_2|\}$. Define the sets

$$\mu\,(q_1, q_2, p) = \{x \in I : |q_1 x + q_2 f(x) - p| < \psi(q)\} \tag{6.4}$$

and

$$\mu\,(\psi) = \{x \in I : \exists^\infty\,(q_1, q_2, p) \in \mathbb{Z}^3 \text{ such that } x \in \mu\,((q_1, q_2, p))\}. \tag{6.5}$$

Comparing the definition of $A_2\,(\psi)$ to that of $\mu\,(\psi)$, notice that the arc $\mathcal{C}_f$ is a one-dimensional subset of $\mathbb{R}^2$, so the intersection under consideration in

Theorem 5, $A_2(\psi) \cap \mathcal{C}$, has Hausdorff dimension between zero and one. Further, by the argument above concerning the coordinate map, $\mathcal{H}^g(\mu(\psi)) = 0$ if and only if $\mathcal{H}^g(A_2(\psi) \cap \mathcal{C}_f) = 0$. Hence, to prove convergence in Theorem 5, is suffices to prove

Proposition 1. *Let g be a dimension function subject to (6.1) and (6.2), and assume that*

$$\sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^2 < \infty.$$

Then $\mathcal{H}^g(\mu(\psi)) = 0$.

Proof. By definition $\mu(\psi)$ is the limit supremum of the sets $\mu(q_1, q_2, p)$, i.e. it is the set of x in I such that $x \in \mu(q_1, q_2, p)$ for infinitely many $(q_1, q_2, p) \in \mathbb{Z}^3$. This allows the Hausdorff-Cantelli lemma to be applied to $\mu(\psi)$.

Observe that the set $\mu(q_1, q_2, p)$ consists of all those x in I such that $(x, f(x))$ is within $\psi(q)$ units of the line $y = \frac{p}{q_2} - \frac{q_1}{q_2}x$. Therefore, the assumption that the curvature of f never changes sign on I implies that $\mu(q_1, q_2, p)$ is either an interval or the union of two intervals.

The assumption of Proposition 1 is

$$\sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^2 < \infty. \tag{6.6}$$

Next, applying the Hausdorff-Cantelli lemma, to prove that $\mathcal{H}^g(\mu(\psi)) = 0$ we must show the convergence of the series

$$\sum_{\substack{q_1, q_2, p \in \mathbb{Z} \\ (q_1, q_2) \neq (0,0)}} g(|\mu(q_1, q_2, p)|). \tag{6.7}$$

We apply the standard reduction argument to the approximation function. Without loss of generality we may assume that

$$g\left(\frac{\psi(q)}{q}\right) \geq q^{-3-\varepsilon}, \text{ for all } \varepsilon > 0. \tag{6.8}$$

If this is not satisfied, then because g increases to infinity, we can always enlarge the approximation function ψ to another approximation function $\hat{\psi}$ satisfying

$$g\left(\frac{\hat{\psi}(q)}{q}\right) = \max\{g\left(\frac{\psi(q)}{q}\right), q^{-3-\varepsilon}\}.$$

Then $g\left(\frac{\hat{\psi}(q)}{q}\right)$ satisfies (6.8) as the maximum of two convergent series (each with nonnegative terms), and further we define the set $\hat{\mu}\left(q_1, q_2, p\right)$ in the same way as $\mu\left(q_1, q_2, p\right)$, but with ψ replaced with $\hat{\psi}$. By definition $\psi \leq \hat{\psi}$, so $\mu\left(q_1, q_2, p\right) \subseteq \hat{\mu}\left(q_1, q_2, p\right)$. Therefore if we prove that the limsup set $\mu\left(\hat{\psi}\right)$ satisfies $\mathcal{H}^g\left(\mu\left(\hat{\psi}\right)\right) = 0$, then it will follow that $\mathcal{H}^g\left(\mu\left(\psi\right)\right) = 0$. So without loss of generality we assume (6.8) for the rest of the argument.

Fix some (q_1, q_2) in $\mathbb{Z}^2 \setminus (0,0)$, and consider the sets $\mu\left(q_1, q_2, p\right)$, for all p in $\mathbb{Z}$. Changing p by one just shifts the approximating line up or down by q_1^{-1} units, it follows that there are only finitely many p such that $\mu\left(q_1, q_2, p\right)$ is nonempty. Referring to the definitions (6.4) and (6.5), it will be useful to notice that any such p satisfies

$$|p| \leq Cq = C\max\{|q_1|, |q_2|\}, \tag{6.9}$$

where C is defined

$$C = \max_{x \in I}\{|x| + |f(x)| + 1\}. \tag{6.10}$$

Note that in general C is well-defined because f is continuous on a compact interval by definition. Choosing C in this way guarantees that we consider all lines $y = \frac{p}{q_2} - \frac{q_1}{q_2}x$ that pass within at least one unit of the arc $\mathcal{C}_f$.

Now to prove the proposition, we must show the series (6.7) converges. Given a choice of $(q_1, q_2) \neq (0, 0)$ and for any p in $\mathbb{Z}$ the corresponding approximating line has slope $-\frac{q_1}{q_2}$, so we will partition the choices of (q_1, q_2) into those for which the approximating line has slope similar to some point of $\mathcal{C}_f$, and those for which it does not.

Define

$$M = 1 + \max_{x \in I} |f'(x)|,$$

and partition $\mathbb{Z}^2 \setminus (0, 0)$ into the following two sets.

$$\Theta_1 = \{(q_1, q_2) \in \mathbb{Z}^2 : |q_1| > 2M|q_2|\}$$
$$\Theta_2 = \mathbb{Z}^2 \setminus \Theta_1 \cup \{(0, 0)\}.$$

First, for the Θ_1 case, let (q_1, q_2) be in Θ_1. Then $|q_1| > 2M|q_2|$ by definition, and note $M \geq 1$, so

$$q = \max\{|q_1|, |q_2|\} = |q_1|.$$

In view of Lemma 2, define $h(x) = q_1 x + q_2 f(x) - p$ and $\eta = \psi(|q_1|)$, and notice that $E(\psi(|q_1|)) = \mu(q_1, q_2, p)$. Hence, applying Lemma 2 with the first derivative term yields

$$|\mu(q_1, q_2, p)| \ll \frac{\psi(|q_1|)}{\delta_1},$$

where

$$\delta_1 = \min_{x \in I} |q_1 + q_2 f'(x)|.$$

Further

$$|q_1 + q_2 f'(x)| \geq |q_1| - M|q_2| \geq \frac{|q_1|}{2}.$$

Therefore, $\delta_1 \geq \frac{|q_1|}{2}$. So in conclusion,

$$|\mu(q_1, q_2, p)| \ll \frac{\psi(|q_1|)}{|q_1|}.$$

Next, recall the dimension function g is monotonic increasing, so by (6.2)

$$g(|\mu(q_1, q_2, p)|) \ll g\left(\frac{\psi(|q_1|)}{|q_1|}\right).$$

Finally, notice that $q_1 \neq 0$, so for any given q_1 in $\mathbb{Z}^*$ (6.10) implies that there are order q_1 choices for p (that make $\mu(q_1, q_2, p) \neq \emptyset$). Also, by the definition of Θ_1 there are order q_1 choices for q_2. So in total for a fixed q_1 there are order q_1^2 choices for q_2 and p. Therefore,

$$\sum_{\substack{(q_1,q_2)\in\Theta_1 \\ p\in\mathbb{Z}}} g(|\mu(q_1, q_2, p)|) \ll \sum_{q_1\in\mathbb{Z}^*} g\left(\frac{\psi(|q_1|)}{|q_1|}\right)|q_1|^2,$$

which converges by (6.6). This completes the Θ_1 case.

It remains to consider the Θ_2 case. Let (q_1, q_2) be in Θ_2. Then $|q_1| \leq 2M|q_2|$ and $(q_1, q_2) \neq (0, 0)$. Therefore $|q_2| \asymp q$.

The arc $\mathcal{C}_f$ may be extended from I to all of $\mathbb{R}$ by taking second order Taylor expansions at the endpoints of $I = [a, b]$. That is, define

$$f(x) = \begin{cases} f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2, \text{ when } x > b \\ f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2, \text{ when } x < a. \end{cases} \tag{6.11}$$

This extension of f is in $C^2(\mathbb{R})$ and has second derivative bounded away from zero, in particular $0 < c_1 \leq |f''(x)| \leq c_2$ for all x in $\mathbb{R}$. Moreover, this implies that f' is strictly monotonic on $\mathbb{R}$ and that the range of f is $\mathbb{R}$. Then define $l : \mathbb{R} \to \mathbb{R}$ to be the inverse function of $-f'(x)$, and define

$$x_0 = l\left(\frac{q_1}{q_2}\right).$$

Notice that x_0 is the unique number satisfying

$$q_1 + q_2 f'(x_0) = 0.$$

Define the interval $J = [-2M, 2M]$, and define $I' = l(J)$. Observe that I' contains I by the definition of the constant M, and

$$\left|\frac{q_1}{q_2}\right| \leq 2M \quad \text{implies} \quad \frac{q_1}{q_2} \in J,$$

so $x_0 \in I'$. We may bound l' using the inverse function theorem

$$l'(y) = \frac{-1}{f''(l(y))},$$

so

$$\frac{1}{c_2} \leq |l'(y)| \leq \frac{1}{c_1}. \tag{6.12}$$

Applying this inequality with the mean value theorem yields a bound on how much the function l stretches the length of the interval J.

$$|I'| = |l(J)| \leq \frac{1}{c_1}|J| = \frac{4M}{c_1}. \tag{6.13}$$

The following lemma, due to [11], provides a bound on the rate of change of the function $F(x) = q_1x + q_2f(x)$. It should be noted that the q_2 appearing in the lemma need not be an integer for this lemma to hold true, but it is always an integer in the context of this argument.

Lemma 5.

$$|F'(x)| \asymp |q_2 (F(x) - F(x_0))|,$$

where the $\asymp$ constants depend only on c_1 and c_2.

Note that the function F has its global maximum or minimum at x_0.

Let p_0 be the unique integer satisfying

$$-\frac{1}{2} < F(x_0) - p_0 \leq \frac{1}{2}. \tag{6.14}$$

We will first consider the case $p \neq p_0$. Let $x \in \mu(q_1, q_2, p)$. Then

$$\begin{aligned}|F(x) - F(x_0)| &= |p - p_0 + F(x) - p - F(x_0) + p_0| \\ &\geq |p - p_0| - |F(x) - p| - |F(x_0) - p_0| \\ &\geq |p - p_0| - \psi(q) - \frac{1}{2},\end{aligned}$$

where the last inequality comes from definition (6.4) and (6.14). Further, recall that ψ decreases monotonically to zero, so there is a positive integer q_0 with

$$\psi(q) \leq \frac{1}{8} \text{ for all } q \geq q_0. \tag{6.15}$$

Hence, applying this to the bound on $|F(x) - F(x_0)|$ above,

$$|F(x) - F(x_0)| \geq |p - p_0| - \frac{5}{8} > |p - p_0| - \frac{2}{3} \geq \frac{1}{3}|p - p_0|.$$

In conclusion,

$$|F(x) - F(x_0)| > \frac{1}{3}|p - p_0| \text{ for all } q \geq q_0. \tag{6.16}$$

Fix some $p \neq p_0$ and (q_1, q_2) in Θ_2 satisfying $q = \max\{|q_1|, |q_2|\} > q_0$. In terms of the notation used in the statement of Lemma 2, take $h(x) = F(x) - p$, and $\eta = \psi(q)$. Note that here, as in the Θ_1 case, the first derivative term may be used when applying Lemma 2. Thus, by Lemma 2

$$|\mu(q_1, q_2, p)| \ll \frac{\psi(q)}{|F'(x)|},$$

and by Lemma 5

$$\frac{\psi(q)}{|F'(x)|} \ll \psi(q)|q_2(F(x) - F(x_0))|^{-\frac{1}{2}},$$

Then by (6.16)

$$\psi(q)|q_2(F(x) - F(x_0))|^{-\frac{1}{2}} < \sqrt{3}\psi(q)|q_2(p - p_0)|^{-\frac{1}{2}}.$$

Combining the above three inequalities gives

$$|\mu(q_1, q_2, p)| \ll \psi(q)|q_2(p - p_0)|^{-\frac{1}{2}}. \tag{6.17}$$

Before applying the dimension function g, sum over all $p \neq p_0$ and apply the fact that

$$\sum_{n \leq N} \frac{1}{\sqrt{n}} \ll \sqrt{N}. \tag{6.18}$$

So starting from inequality (6.17),

$$\sum_{p\neq p_0}|\mu\left(q_1,q_2,p\right)| \underset{(6.9)}{\ll} \frac{\psi(q)}{\sqrt{q}}\sum_{\substack{|p|\leq Cq\\ p\neq p_0}}\frac{1}{\sqrt{|p-p_0|}} \underset{(6.18)}{\ll} \psi(q). \tag{6.19}$$

Notice that Lemma 5 also provides an upper bound on the derivative of F,

$$|F'(x)| \ll |q_2\left(F(x)-F\left(x_0\right)\right)|.$$

So by the mean value theorem

$$|\mu\left(q_1,q_2,p\right)| \gg \frac{\psi(q)}{|q_2\left(F(x)-F\left(x_0\right)\right)|}. \tag{6.20}$$

For an upper bound on $|F(x)-F\left(x_0\right)|$, proceed as above in (6.16), i.e.

$$\begin{aligned}|F(x)-F\left(x_0\right)| &= |p-p_0+F(x)-p-F\left(x_0\right)+p_0|\\ &\leq |p-p_0|+|F(x)-p|+|F\left(x_0\right)-p_0|\\ &\leq |p-p_0|+\psi(q)+\frac{1}{2}\\ &\leq |p-p_0|+\frac{5}{8}.\end{aligned}$$

So we have

$$|F(x)-F\left(x_0\right)| \ll |p-p_0|. \tag{6.21}$$

Combining (6.20) and (6.21) yields

$$|\mu\left(q_1,q_2,p\right)| \gg \frac{\psi(q)}{q}. \tag{6.22}$$

Recall that there are order q choices for p that make $|\mu\left(q_1,q_2,p\right)|$ nonempty.

Thus comparing the two inequalities (6.19) and (6.22) it is evident that most of the sets will satisfy

$$|\mu(q_1, q_2, p)| \ll \frac{\psi(q)}{q}.$$

There may be some $|\mu(q_1, q_2, p)|$ that are much larger than this, but in that case we may always manually split the large sets into a finite collection of sets, say $N_1, N_2, \ldots N_n$, each satisfying

$$|N_i| \leq \frac{\psi(q)}{q}.$$

Notice that the upper bound on the sum (6.19) means that breaking some of the sets into smaller subsets will not affect the final result, and further the limit supremum set of the broken sets will be contained in the limit supremum set of the unbroken sets. So without loss of generality for the rest of the $p \neq p_0$ case we assume

$$|\mu(q_1, q_2, p)| \ll \frac{\psi(q)}{q}.$$

Then by (6.2),

$$g(|\mu(q_1, q_2, p)|) \ll g\left(\frac{\psi(q)}{q}\right),$$

so as noted above

$$\sum_{p \neq p_0} g(|\mu(q_1, q_2, p)|) \ll g\left(\frac{\psi(q)}{q}\right) q,$$

and finally for a fixed value of q, there are order q choices of (q_1, q_2) in Θ_2. Therefore,

$$\sum_{\substack{(q_1,q_2)\in\Theta_2 \\ q \geq q_0 \\ p \neq p_0}} g(|\mu(q_1, q_2, p)|) \ll \sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^2 < \infty,$$

which completes the $p \neq p_0$ case.

The last case to consider is when (q_1, q_2) is in Θ_2 and $p = p_0$. Consider two final sub-cases based off of the size of the distance of $F(x_0)$ from zero, i.e. $\|F(x_0)\|$. Namely, by (6.14),

$$|F(x_0) - p_0| = \|F(x_0)\|. \tag{6.23}$$

Then for x in $\mu(q_1, q_2, p_0)$,

$$\begin{aligned}|F(x_0) - F(x)| &= |F(x_0) - p_0 + p_0 - F(x)| \\ &\geq |F(x_0) - p_0| - |F(x) - p| \underset{(6.23)}{\geq} \|F(x_0)\| - \psi(q).\end{aligned}$$

The two cases to consider are as follows.

On the first hand if $\|F(x_0)\| \geq 2\psi(q)$, then

$$|F(x_0) - F(x)| \geq \frac{1}{2}\|F(x_0)\|,$$

so by Lemma 5 and then the above inequality

$$|F'(x)| \gg |q_2(F(x) - F(x_0))|^{\frac{1}{2}} \gg (q\|F(x_0)\|)^{\frac{1}{2}} > 0.$$

Hence, the first derivative may be used in Lemma 2, and doing so yields

$$|\mu(q_1, q_2, p_0)| \ll \frac{\psi(q)}{\min_{x\in I}|F'(x)|} \ll \frac{\psi(q)}{\sqrt{q}}\|F(x_0)\|^{-\frac{1}{2}}.$$

On the second hand if $\|F(x_0)\| < 2\psi(q)$, then in Lemma 2 the second derivative must be used, so

$$|\mu(q_1, q_2, p_0)| \ll \sqrt{\frac{\psi(q)}{\min_{x\in I}|F''(x)|}} \ll \sqrt{\frac{\psi(q)}{q}},$$

where the last inequality is a consequence of the fact that

$$\min_{x\in I}|F''(x)| = \min_{x\in I}|q_2 f''(x)| \asymp q.$$

In summary,

$$\|F(x_0)\| \geq 2\psi(q) \quad \text{implies} \quad |\mu(q_1, q_2, p_0)| \ll \frac{\psi(q)}{\sqrt{q}}\|F(x_0)\|^{-\frac{1}{2}}, \tag{6.24}$$

and

$$\|F(x_0)\| < 2\psi(q) \quad \text{implies} \quad s|\mu(q_1, q_2, p_0)| \ll \sqrt{\frac{\psi(q)}{q}}. \tag{6.25}$$

Define the dual curve $f^*(y) = yl(y) + f(l(y))$ to the curve $f(x)$, which satisfies

$$\frac{d}{dy}f^*(y) = l(y)$$

and recall $l(y)$ is defined as the inverse of $-f'(x)$. Moreover,

$$q_2 f^*\left(\frac{q_1}{q_2}\right) = q_1 l\left(\frac{q_1}{q_2}\right) + q_2 f\left(l\left(\frac{q_1}{q_2}\right)\right) = q_1 x_0 + q_2 f(x_0) = F(x_0). \tag{6.26}$$

Recall that in the statement of Theorem 5 the growth condition on g states that there exist $0 < s_1 \leq s_2 \leq 1$ with

$$5s_1 < 6s_2 \quad \text{and} \quad x^{s_1} \ll g(x) \ll x^{s_2}. \tag{6.27}$$

It will be useful to notice that this growth condition implies

$$x^{s_2} \ll g(x)^{\frac{s_2}{s_1}}, \tag{6.28}$$

and further

$$g\left(\sqrt{x}\right)^{3-\varepsilon} \ll g(x) \quad \text{for all} \quad \varepsilon > 0. \tag{6.29}$$

Now we are equipped to prove the proposition by showing the convergence of the series

$$\sum_{(q_1,q_2)\in\Theta_2} g\left(\left|\mu\left(q_1, q_2, p_0\right)\right|\right).$$

First, return to the case (6.25). Since g is monotonic increasing and by (6.2)

$$g\left(\left|\mu\left(q_1, q_2, p_0\right)\right|\right) \ll g\left(\sqrt{\frac{\psi(q)}{q}}\right).$$

Let k be a natural number with $2^k \geq q_0$, and consider a dyadic range for the values of $|q_2|$, i.e. $2^k \leq |q_2| < 2^{k+1}$. Then applying the duality principle (6.26),

$$\begin{aligned}
&\sum_{\substack{(q_1,q_2)\in\Theta_2 \\ 2^k\leq|q_2|<2^{k+1} \\ \|F(x_0)\|<2\psi(q)}} g\left(\left|\mu\left(q_1, q_2, p_0\right)\right|\right) \\
&\ll \sum_{2^k\leq q_2<2^{k+1}} \sum_{\substack{\frac{q_1}{q_2}\in J \\ \left\|q_2 f^*\left(\frac{q_1}{q_2}\right)\right\|<2\psi\left(2^k\right)}} g\left(\sqrt{\frac{\psi\left(2^k\right)}{2^k}}\right) \\
&\underset{(\text{Lemma } 3)}{\ll} \left[\left(2\psi\left(2^k\right)\right)^{1-\varepsilon_0} 2^{2k} + 2^k \log\left(2^{k+1}\right)\right] g\left(\sqrt{\frac{\psi\left(2^k\right)}{2^k}}\right) \\
&\underset{(6.29)}{\ll} \left(\psi\left(2^k\right)^{1-\varepsilon_0} 2^{2k} + k2^k\right) g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{\frac{1}{3-\varepsilon_1}}.
\end{aligned}$$

Note that in the above inequalities ε_0 comes from the application of Lemma 3 and ε_1 from the derived growth condition (6.29). By the Cauchy condensation test, it remains to show

$$\left(\psi\left(2^k\right)^{1-\varepsilon_0} 2^{2k} + k2^k\right) g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{\dfrac{1}{3-\varepsilon_1}} \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right) 2^{3k}. \tag{6.30}$$

Regarding the first term of (6.30), namely

$$g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{\frac{1}{3-\varepsilon_1}} \psi\left(2^k\right)^{1-\varepsilon_0} 2^{2k},$$

notice that $\dfrac{1}{3-\varepsilon_1} > \dfrac{1}{3}$, so there is an $\varepsilon_2 > 0$ (depending on ε_1) such that

$$1 - \frac{1}{3-\varepsilon_1} = \frac{2}{3} - \varepsilon_2.$$

Hence, the desired inequality

$$g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{\dfrac{1}{3-\varepsilon_1}} \psi\left(2^k\right)^{1-\varepsilon_0} 2^{2k} \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right) 2^{3k}$$

holds if and only if

$$\frac{\psi\left(2^k\right)^{1-\varepsilon_0}}{2^k} \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{\frac{2}{3}-\varepsilon_2}.$$

Note that

$$\frac{\psi\left(2^k\right)^{1-\varepsilon_0}}{2^k} \ll \left(\frac{\psi\left(2^k\right)}{2^k}\right)^{1-\varepsilon_0} \ll \frac{\psi\left(2^k\right)}{2^k},$$

so applying (6.27), it suffices to show that

$$\frac{\psi\left(2^k\right)}{2^k} \ll \left(\frac{\psi\left(2^k\right)}{2^k}\right)^{s_1\left(\frac{2}{3}-\varepsilon_2\right)}.$$

Recall $s_1 < \frac{3}{2}s_2$, so

$$\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{s_1\left(\frac{2}{3}-\varepsilon_2\right)} \gg \left(\frac{\psi\left(2^k\right)}{2^k}\right)^{s_2-\frac{3}{2}\varepsilon_2}.$$

So we wish to show

$$\frac{\psi\left(2^k\right)}{2^k} \ll \left(\frac{\psi\left(2^k\right)}{2^k}\right)^{s_2-\frac{3}{2}\varepsilon_2},$$

which reduces to $1 \geq s_2 - \frac{3}{2}\varepsilon_2$, and this is always true because $s_2 \leq 1$ by definition.

In the second term of (6.30), we wish to show that

$$g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{\dfrac{1}{3-\varepsilon_1}} k2^k \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right) 2^{3k}.$$

Applying the reduction assumption (6.8), for all $\varepsilon_2 > 0$

$$g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{\dfrac{1}{3-\varepsilon_1}} k2^k \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right) k\left(2^k\right)^{(-3-\varepsilon_2)\left(-1+\frac{1}{3-\varepsilon_1}\right)+1},$$

and since $k \ll \left(2^k\right)^{\varepsilon_3}$ for all $\varepsilon_3 > 0$, the rightmost term above is

$$\ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right)\left(2^k\right)^{(-3-\varepsilon_2)\left(-1+\frac{1}{3-\varepsilon_1}\right)+1+\varepsilon_3}.$$

Therefore, it remains to show

$$(-3-\varepsilon_2)\left(-1+\frac{1}{3-\varepsilon_1}\right)+1+\varepsilon_3\leq 3,$$

or equivalently

$$\varepsilon_2+\varepsilon_3-\frac{3+\varepsilon_2}{3-\varepsilon_1}\leq -1.$$

Observe that ε_1 comes from (6.29), ε_2 comes from (6.8), and lastly ε_3 comes from the factor of k above, so in particular they can always be chosen to satisfy this inequality.

Finally, returning to the case (6.24), g is monotonic increasing, so by 6.2

$$g\left(\left|\mu\left(q_1,q_2,p_0\right)\right|\right)\ll g\left(\frac{\psi(q)}{\sqrt{q\|F\left(x_0\right)\|}}\right).$$

Again take k in $\mathbb{N}$ such that $2^k\geq q_0$ and consider a dyadic range $2^k\leq |q_2|<2^{k+1}$. Apply (6.26) and recall the definition of the interval J. Then

$$\begin{aligned}\sum_{\substack{(q_1,q_2)\in\Theta_2\\ 2^k\leq|q_2|<2^{k+1}\\ \|F(x_0)\|\geq 2\psi(q)}} g\left(\left|\mu\left(q_1,q_2,p_0\right)\right|\right) &\ll \sum_{2^k\leq q_2<2^{k+1}}\ \sum_{\substack{\frac{q_1}{q_2}\in J\\ \left\|q_2 f^*\left(\frac{q_1}{q_2}\right)\right\|\geq 2\psi\left(2^k\right)}} g\left(\frac{\psi\left(2^k\right)}{\sqrt{2^k\|q_2 f^*\left(q_1/q_2\right)\|}}\right)\\ &\underset{(6.27)}{\ll} \sum_{2^k\leq q_2<2^{k+1}}\ \sum_{\substack{\frac{q_1}{q_2}\in J\\ \left\|q_2 f^*\left(\frac{q_1}{q_2}\right)\right\|\geq 2\psi\left(2^k\right)}} \frac{\left(\frac{\psi\left(2^k\right)}{2^{\frac{k}{2}}}\right)^{s_2}}{\left\|q_2 f^*\left(\frac{q_1}{q_2}\right)\right\|^{s_2/2}}\\ &\underset{(\text{Lemma 4})}{\ll} \left(2^{2k}+\psi\left(2^k\right)^{-s_2/2}2^k k\right)\left(\frac{\psi\left(2^k\right)}{2^{\frac{k}{2}}}\right)^{s_2}.\end{aligned}$$

So again applying the Cauchy condensation test, it remains to show that

$$\left(2^{2k} + \psi\left(2^k\right)^{-s_2/2} 2^k k\right)\left(\frac{\psi\left(2^k\right)}{2^{\frac{k}{2}}}\right)^{s_2} \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right)2^{3k}. \tag{6.31}$$

This can be handled term by term. For the first term of (6.31),

$$\left(\frac{\psi\left(2^k\right)}{2^{\frac{k}{2}}}\right)^{s_2} 2^{2k} \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{s_2/s_1}\left(2^k\right)^{\frac{s_2}{2}+2},$$

before applying (6.8), write the desired inequality as follows:

$$g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{s_2/s_1}\left(2^k\right)^{\frac{s_2}{2}-1} \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right), \tag{6.32}$$

or equivalently, in the case that $s_1 \neq s_2$,

$$\left(2^k\right)^{\left(\frac{s_2}{2}-1\right)\left(\frac{1}{1-s_2/s_1}\right)} \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right).$$

Then applying (6.8) implies that it suffices to show that for some $\varepsilon_0 > 0$,

$$\left(2^k\right)^{\left(\frac{s_2}{2}-1\right)\left(\frac{1}{1-s_2/s_1}\right)} \ll \left(2^k\right)^{-3-\varepsilon_0}.$$

Comparing the exponents in the above inequality, we wish to show

$$\left(\frac{s_2}{2}-1\right)\left(\frac{s_1}{s_1-s_2}\right) \leq -3-\varepsilon_0.$$

Recall that $s_1 > s_2$ (where the inequality is strict due to the previous assertion that we are considering the case $s_1 \neq s_2$), so this inequality may be rewritten

$$\frac{s_2}{2} - 1 \leq \left(\frac{s_2}{s_1} - 1\right)(3 + \varepsilon_0),$$

and (6.27) implies that

$$\left(\frac{5}{6} - 1\right)(3 + \varepsilon_0) < \left(\frac{s_1}{s_2} - 1\right)(3 + \varepsilon_0),$$

and further, ε_0 can be chosen to satisfy

$$\left(\frac{5}{6} - 1\right)(3 + \varepsilon_0) < -\frac{1}{2} < \left(\frac{s_1}{s_2} - 1\right)(3 + \varepsilon_0).$$

Thus the goal is to show that

$$\frac{s_2}{2} - 1 \leq -\frac{1}{2},$$

which always holds by (6.27). Note that the above calculations admit the case $s_1 = 1$. Now in the case that $s_1 = s_2$, observe that inequality (6.32) is trivially true when $s_1 = s_2$.

Lastly, the second term of (6.31) satisfies

$$\left(\frac{\psi\left(2^k\right)}{2^{\frac{k}{2}}}\right)^{s_2} \psi\left(2^k\right)^{-\frac{s_2}{2}} 2^k k \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{s_2/(2s_1)} 2^k k.$$

Once again applying (6.8), for all $\varepsilon_0 > 0$

$$g\left(\frac{\psi\left(2^k\right)}{2^k}\right)^{s_2/(2s_1)} 2^k k \ll g\left(\frac{\psi\left(2^k\right)}{2^k}\right) k \left(\left(2^k\right)^{-3-\varepsilon_0}\right)^{-1+s_2/(2s_1)},$$

and recall $k \ll \left(2^k\right)^{\varepsilon_1}$ for all $\varepsilon_1 > 0$, so in view of (6.31) we wish to show

$$(-3-\varepsilon_0)\left(-1+\frac{s_2}{2s_1}\right)+\varepsilon_1 \leq 3,$$

or equivalently,

$$3s_2\left(1+\frac{\varepsilon_0}{3}\right) \geq 2s_1\left(\varepsilon_0+\varepsilon_1\right),$$

which follows by (6.27).

This completes the $p = p_0$ case, so we have shown that the series (6.7) converges. Therefore, by the Hausdorff-Cantelli lemma $\mathcal{H}^g\left(\mu\left(\psi\right)\right) = 0$.

□

7 Future work

While this thesis employs purely analytical techniques, many theorems in Diophantine approximation are established using ergodic theory (see [17]). It is the author's hope that the techniques used here will inspire further research into the general Hausdorff theory of planar curves and low codimension manifolds.

Modifying the results proven here and in [11] to handle multiplicative or general multivariable approximation functions is a challenging open problem.

Additionally, the following questions are of particular interest.

1. Can the techniques applied here be adapted to determine the upper box-counting or Assouad dimensions of the sets under consideration?
2. Can we find additional hypotheses that guarantee $A_2 \cap \mathcal{C}$ or $S_n \cap \mathcal{C}$ are countable or even finite?

3. Can the regularity conditions (6.1) and (6.2) be weakened or removed?

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