Abstract

This thesis conducts a number of numerical experiments using massively parallel GPU computations to study a new continuous data assimilation algorithm. We test the algorithm on two-dimensional incompressible fluid given by the Navier-Stokes equations. In this context, observations of the Eulerian velocity field given at a finite resolution of nodal points in space may be used to recover the exact velocity field over time. We also consider nodal measurements of the vorticity field and stream function. The main difference between this new algorithm and previous continuous data assimilation methods is the inclusion of a relaxation parameter μ that controls the rate at which the approximate solution is forced toward the observational measurements. If μ is too small, the approximate solution obtained by data assimilation may not converge to the reference solution; however, if μ is too large then high frequency spill-over from the observations may contaminate the approximate solution. Our focus is on the resolution of the nodal points necessary for the algorithm to recover the exact velocity field and how best to choose the parameter μ.

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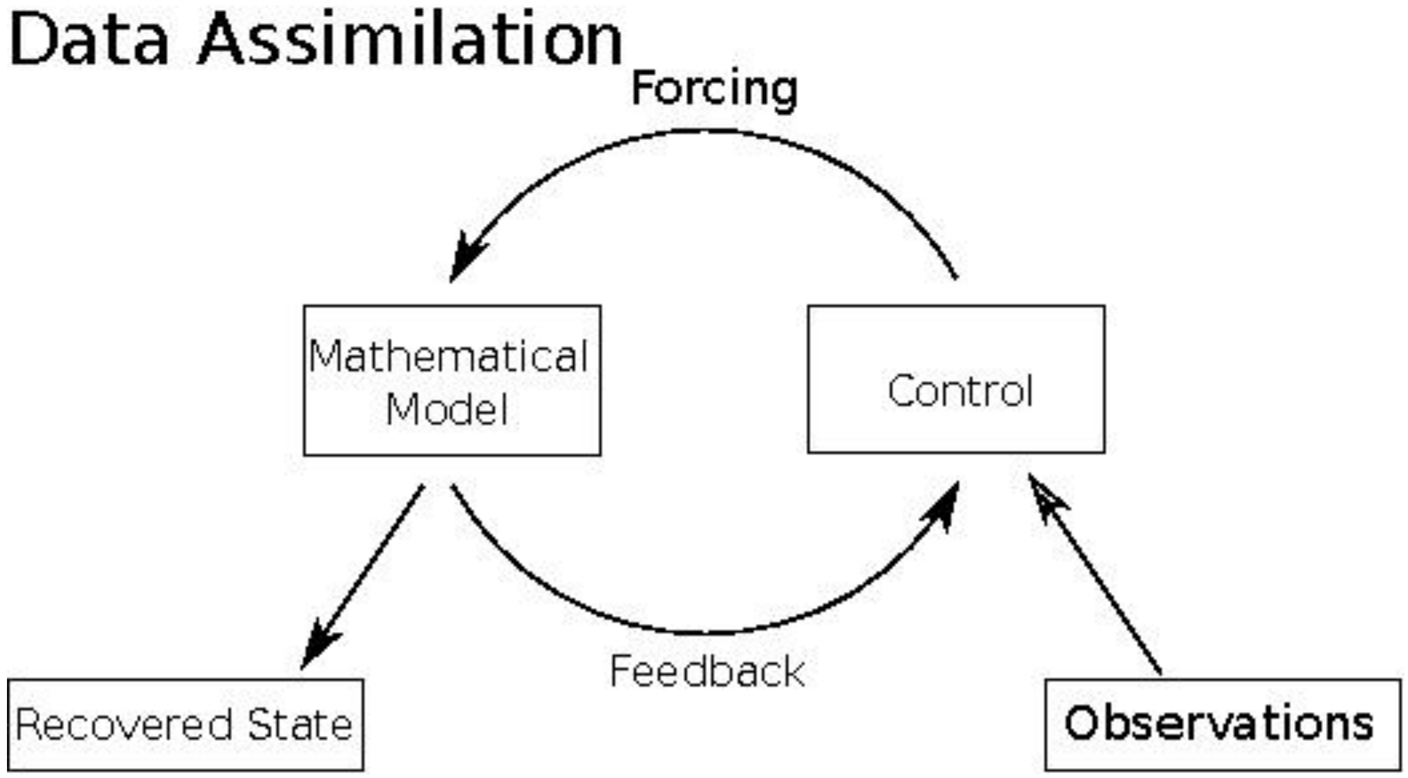
A Numerical Study of Continuous Data Assimilation for 2D-NS Equations Using Nodal Points

Reference

- **A. Azouani, E. Olson, and E.S. Titi:**
  – Continuous data assimulation using general interplant observables.
- **A.J. Chorin and J.E. Marsden:**
  – A mathematical introduction to fluid mechanics.
- **K. Hayden:**
  – Modeling of dynamical systems.
- **Nvidia:**
  – CUDA C programming guide.
- **C. Basdevant:**
  – Technical improvements for direct numerical simulation of homogeneous three-dimensional turbulence.

Data Assimilation



Data Assimilation Algorithm

- **Exact Solution:**

$$\dot{U} = F(U, t)$$

- **Approximating Solution:**

$$\dot{V} = F(V, t) + \mu(I_h(U) - I_h(V))$$

where $I_h(U)$ **is an interpolation of the observation of** U**.**

μ **is a relaxation parameter.**

Lorenz System

- **Exact Solution:**

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} \sigma(Y - X) \\ X(\rho - Z) - Y \\ XY - \beta Z \end{bmatrix}$$

- **Approximating Solution:**

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y - x) \\ x(\rho - z) - y \\ xy - \beta z \end{bmatrix} + \mu\left(\begin{bmatrix} X \\ C_2 \\ C_3 \end{bmatrix} - \begin{bmatrix} x \\ C_2 \\ C_3 \end{bmatrix} \right)$$

where $I_h(X, Y, Z) = (X, C_2, C_3)$

Theorem

- Let J, K be bounds such that

$$(\rho+\sigma-Z)^2 \leq K \quad \text{and} \quad Y^2 \leq J$$

and suppose

$$\mu \geq \max(K, J/\beta)$$

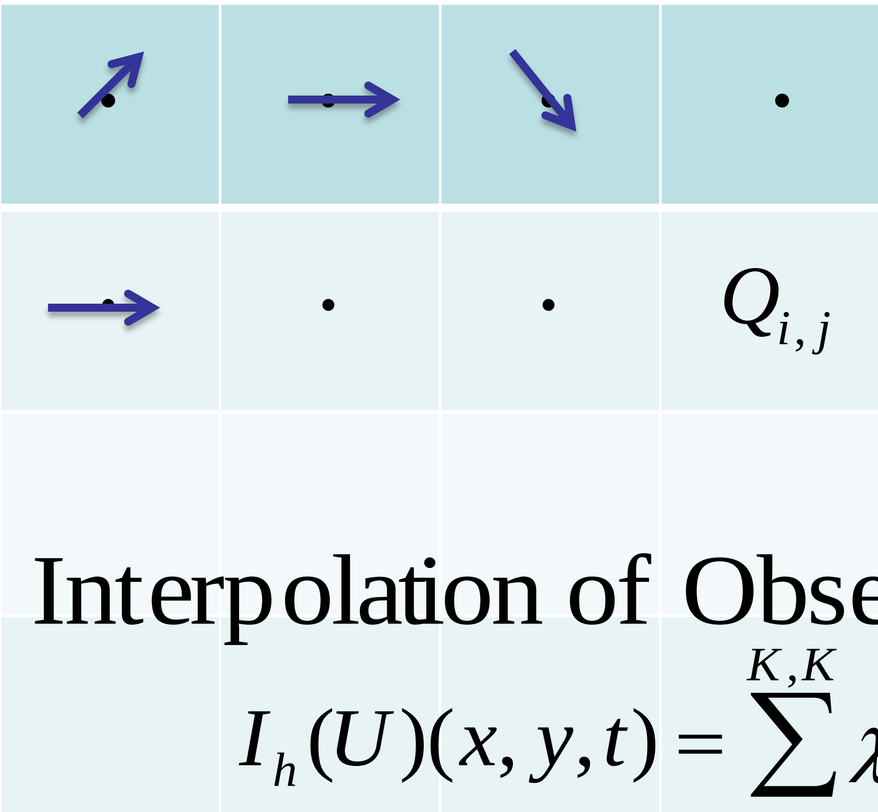
Then

$$\|(X,Y,Z)-(x,y,z)\| \rightarrow 0 \ \text{ as } \ t \rightarrow \infty.$$

Measuring Speed of Wind & Direction

- **Measurement**
  - Speed
  - Direction

Observation of the Velocity



- **Dimension:** $2\pi \times 2\pi$
- **Boundary: Periodic**
- **Measure:**
  - Velocity
  - Direction

Interpolation of Observation

$$I_h(U)(x,y,t) = \sum_{i,j=1}^{K,K} \chi_{Q_{i,j}}(x,y)U(x_{i+1/2}, y_{j+1/2}, t)$$

where $h = 2\pi / K$, $Q_{i,j} = [\xi_{i-1}, \xi_i] \times [\eta_{j-1}, \eta_j]$ and $\xi_i = ih, \eta_j = jh.$

2D-NS Equations

- **Exact Solution:**

$$\begin{cases} \dfrac{\partial U}{\partial t} - \nu \Delta U + U \cdot \nabla U + \nabla P_1 = F \\ \nabla \cdot U_1 = 0 \end{cases}$$

- **Approximating Solution:**

$$\begin{cases} \dfrac{\partial V}{\partial t} - \nu \Delta V + V \cdot \nabla V + \nabla P_2 = F + \mu (I_h(U) - I_h(V)) \\ \nabla \cdot V_2 = 0 \end{cases}$$

Theorem: Azouani, Olson, Titi

$$I_h(U) = \sum_{i,j=1}^{K,K} \chi_{Q_{i,j}}(x,y)\,U(x_{i+1/2}, y_{j+1/2}, t)$$

with previous definitions,

$$3\nu\lambda(2c\log(2c^{3/2}) + 8c\log(1+G))G \le \mu \le c_2\nu K^2$$

Then:

$$\|U - V\|_{L^2} \to 0 \text{ as } t \to \infty.$$

Note: $G = \|f\|_{L^2} / \nu^2 = 5{,}000{,}000$

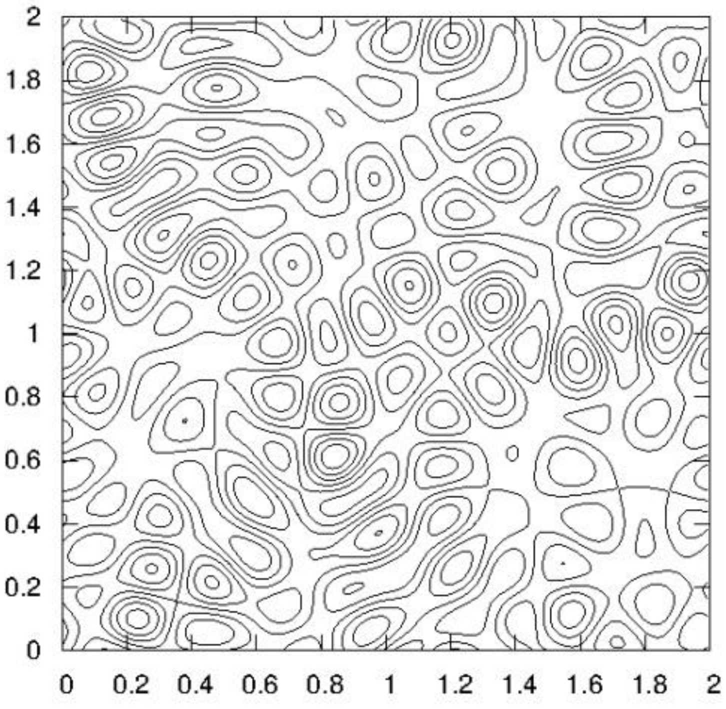
Motivation to CFD

- **Confirm the theorem.**
- **Test sharpness of analytical bounds on μ (relaxation parameter) and**

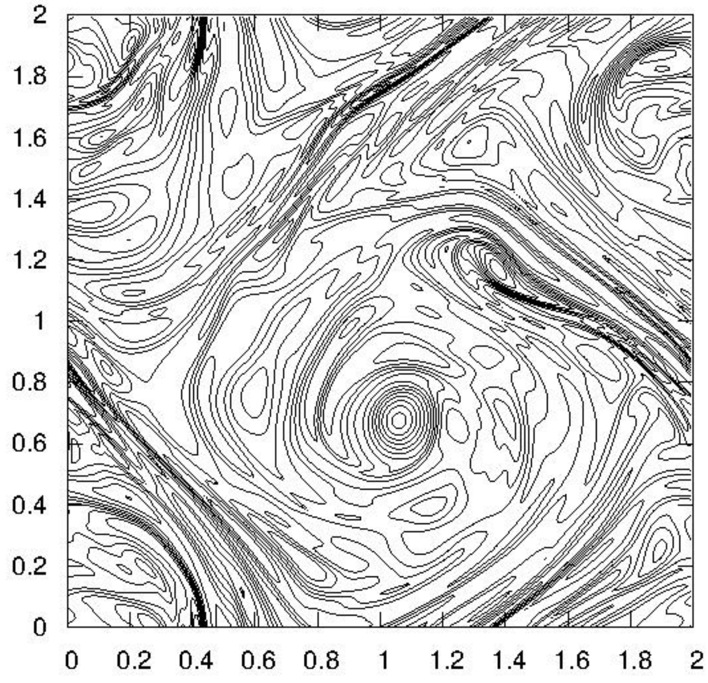
  h (resolution of observation).
- **Find practical values of μ and h.**
- **Experiment and look for new and interesting results.**

Forcing Function & Initial Condition

f_{64}

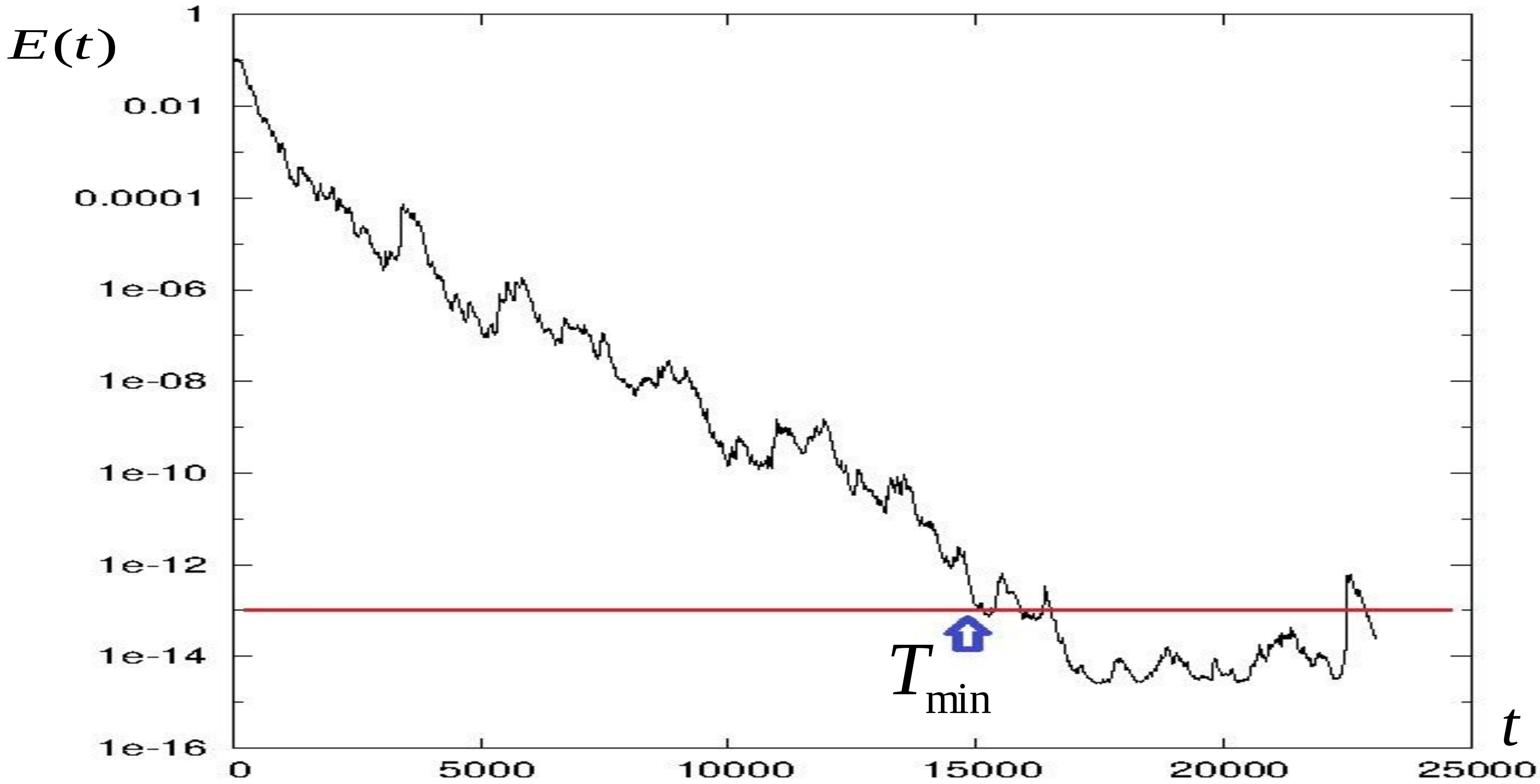


with $\lambda = 64$



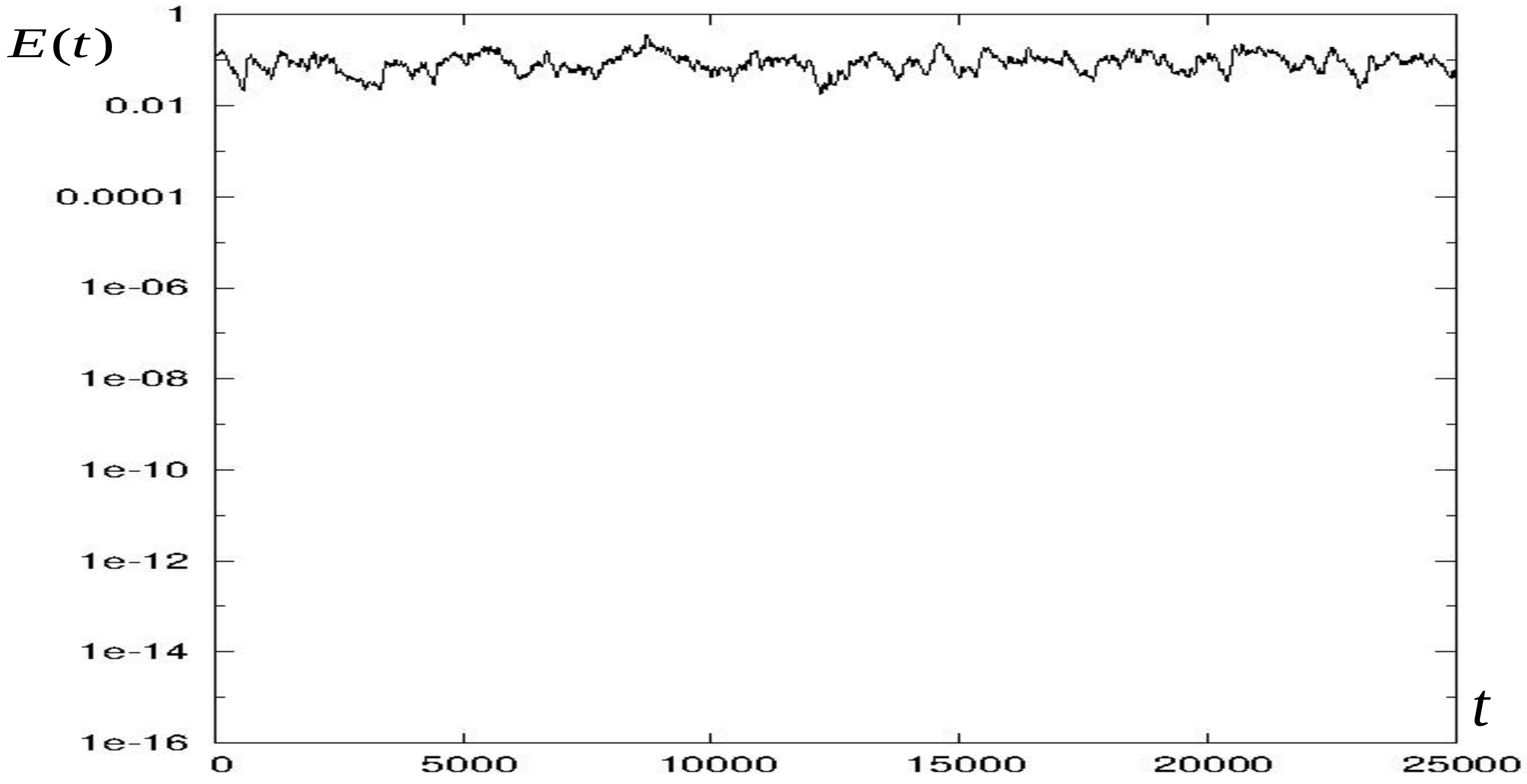
Result: Success

$$G = 5{,}000{,}000;\ \nu = 0.0001;\ \lambda = 64;\ K = 7;\ \mu = 150;\ T_{mim} = 22890$$



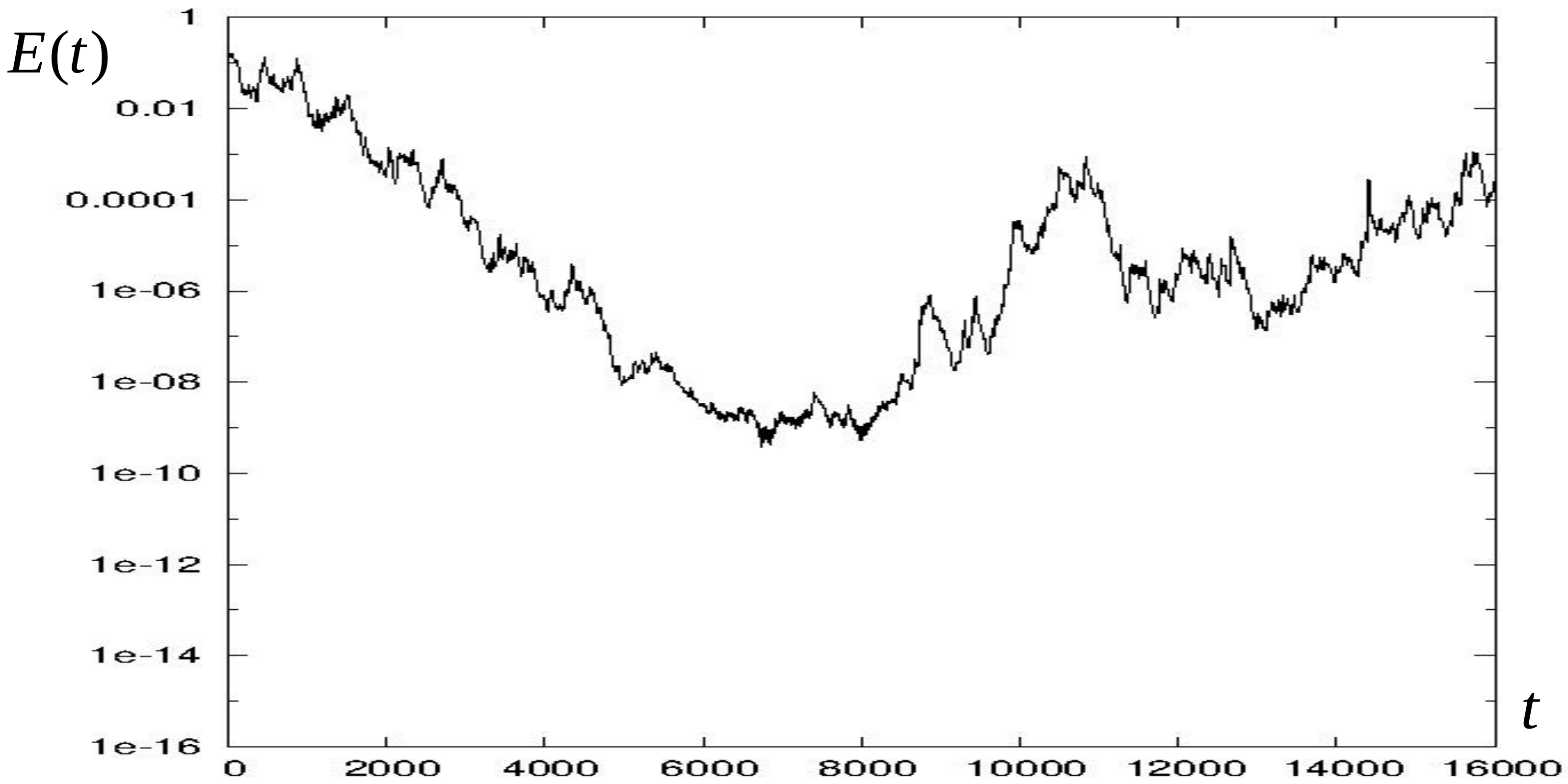
Results: Fail

$$G = 5{,}000{,}000;\ \nu = 0.0001;\ \lambda = 64;\ K = 7;\ \mu = 200;\ T_{mim} = \infty$$



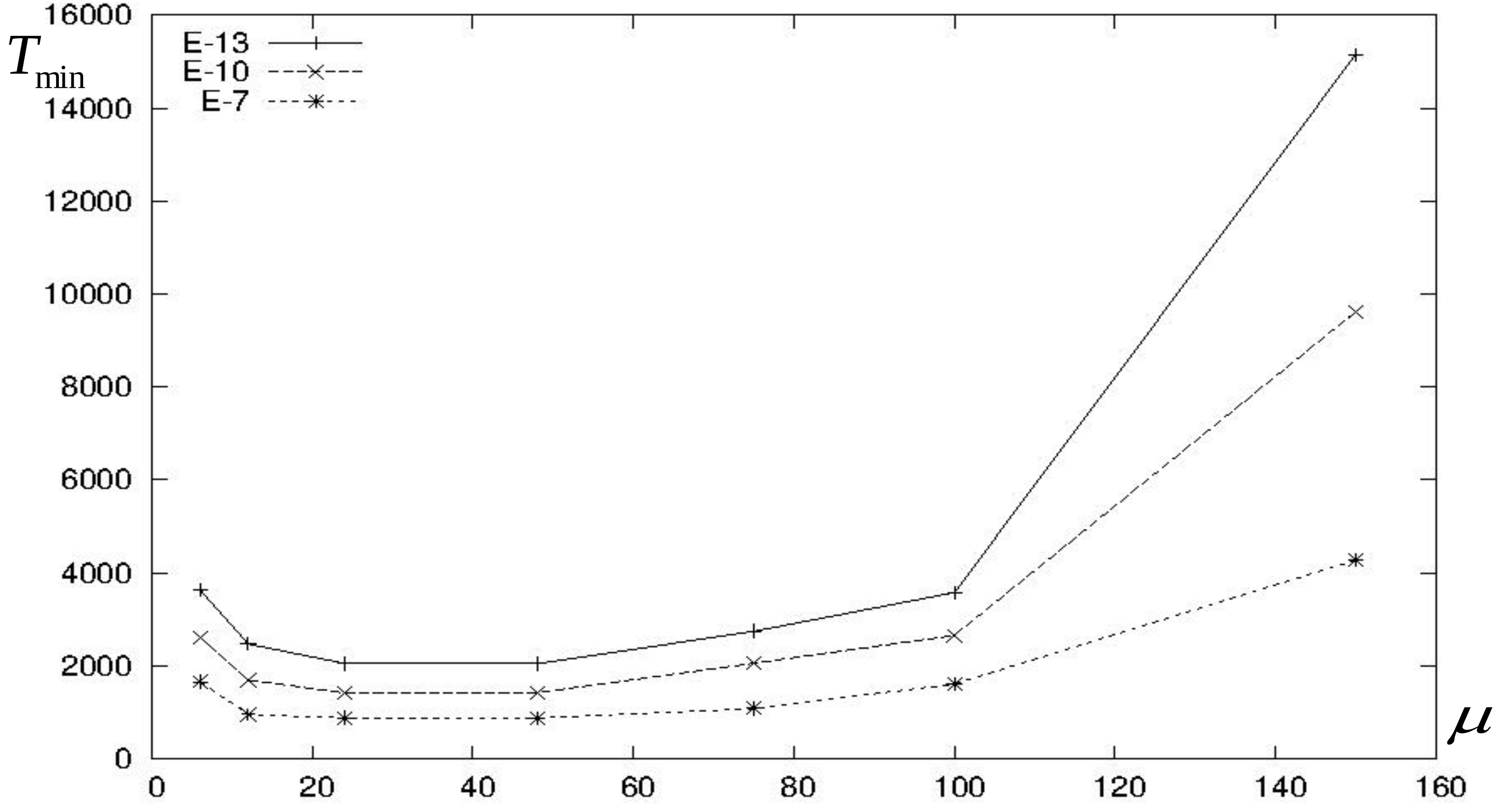
Results: Partial Convergence

$$G = 5{,}000{,}000;\ \nu = 0.0001;\ \lambda = 64;\ K = 6;\ \mu = 5;\ T_{mim} = \infty$$



Conclusion: Optimal μ

$$G = 5{,}000{,}000;\ \lambda = 64;\ K = 7$$



Conclusion:

Study for $K_{\min}$

Force	G	μ	$T_{\min}$	$K_{\min}$
f_{25}	5,000,000	6.0	22440	4
f_{64}	5,000,000	6.5	18200	6
f_{121}	5,000,000	20.0	14030	8

Study for μ_{opt} where $K = 7$

Initial Condition	μ_{opt}	$T_{\min}$
u_0	48	2046
u_1	24	2264
u_2	48	2159

Summary

- **There were choices of μ and K such that $\|U - V\| \to 0$ to with in machine epsilon.**

- **The value of $K_{\min}$ is comparable to the number of numerically determining modes.**

- **There is a wide range of values μ near μ_{opt} for which the algorithm works well.**

Movie:

1. **Movie ….**

Future Work

1: μ_{opt} for f_{25} and f_{121}

2: High Resolution Simulation

3: Random Error in Observations

4: Dependency of $T_{\min}$ and μ_{opt} on Initial Conditions

Fin