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Cantor Minimal Systems and Dimension Groups

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

by

Cyrus Anthony Luciano

Dr. Bruce E. Blackadar/Thesis Advisor

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CYRUS ANTHONY LUCIANO

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Bruce E. Blackadar, Ph.D., Advisor

Alex Kumjian, Ph.D., Committee Member

Eelke Folmer, Ph.D., Graduate School Representative

Marsha H. Read, Ph.D., Dean, Graduate School

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Abstract

The theory of general dynamical systems evolved originally in the context of modeling movement in physical systems. Consequently, traditional dynamics is viewed through the lens of differential and difference equations using notions such as state space, trajectory and attractors. Topological dynamics is a generalization and abstraction of these concepts in the context of topological spaces and homeomorphisms. Dimension groups provide a classification up to strong orbit equivalence of minimal $\mathbb{Z}^d$-actions on a Cantor set. The range of such dimension groups for $d > 1$ is still an open question. It appears as if symbolic dynamical systems may be a fruitful approach to this question. In this work, the relationship is explored between Cantor minimal systems, symbolic dynamical systems and dimension groups using properly ordered Bratteli diagrams and their associated Bratteli-Vershik systems. In order to illustrate this relationship we develop the examples of general odometer systems and irrational rotations on the Cantorized circle. Both the K_0 and K^0 groups are calculated using Bratteli diagrams and directed graphs, respectively. Furthermore, the substitution dynamical system associated to a specified irrational rotation is identified showing the method by which one may move between Cantor minimal systems and symbolic dynamical systems.

To my beautiful wife, Josie.

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There are countless people who have helped me along the way. I would like to thank my advisor, Professor Bruce Blackadar, for his amazing wealth of knowledge and patience; my wife, Josie, for her unending support during those many long nights; my daughter, Coco, for providing so much joy and light; my good friend, Dirk, for teaching me to live life without hesitation; and to my family, for bringing me to be the person that I am.

Contents

Chapter 1

Introduction

In the work [8], the authors provide a classification up to strong orbit equivalence of minimal $\mathbb{Z}^d$-actions on a Cantor set. This was a big accomplishment in that it sheds light on the study of multi-dimensional topological dynamical systems. One of the results in [8] gives that every minimal free $\mathbb{Z}^d$-action on a Cantor set is orbit equivalent to a $\mathbb{Z}$-action. In [11], the authors calculate the range of the dimension group for $\mathbb{Z}$-actions on a Cantor set. As explained in [8], this question is still open for the more general $\mathbb{Z}^d$-actions on a Cantor set where $d > 1$.

The work to follow is motivated by this question. In particular, the original arc of this work set out to examine this question through the context of substitution minimal systems. Unfortunately, definitive results for this question were not obtained; however, this work does contribute to the general knowledge of the subject in that it explicitly calculates the Kakutani-Rohlin partitions for the adding machine and irrational rotation systems. Furthermore, this work explicitly calculates the K^0 and K_0 groups for these systems, correcting in several cases, the work of [14]. Finally, this work examines the relationship between substitution minimal systems and Cantor minimal systems through Bratteli diagrams and dimension groups in order to shed light on an approach to the original motivating question.

This work proceeds by first running through an exposition of the fundamental ideas upon which the this work relies. We then introduce the concepts of general topological dynamical systems, building up to the definitions of a Cantor minimal system and a Bratteli-Vershik system. From here, we introduce the concept of substitution dynamical systems, looking in particular at the relationship between substitutions and Bratteli diagrams. Finally, we introduce dimension groups and illuminate their fundamental importance as algebraic invariants of Cantor minimal systems.

Chapter 2

Preliminary Material

Before we get into the meat of the subject at hand, we must walk through some material which will set the stage for the main work. In particular, we must introduce the concepts which underly the fundamental examples traced through this work.

2.1 Continued Fractions

In this section we would like to introduce the basic concepts used in the study of continued fractions. The work in this section references material from [13]. Continued fractions have been around for quite some time. In fact, their roots may be traced all the way back to Euclid's algorithm for determining the greatest common divisor. Among their many applications, continued fractions have become useful in the study of symbolic and topological dynamics. In the context of the work presented here, continued fractions have been found to be very useful in the study of irrational rotations on the circle.

We will say that an *infinite continued fraction* is an expression of the form

$$\alpha = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \ddots}}},$$

Where, in general, $a_j, b_j \in \mathbb{C}$, and care must be taken to always ensure a nonzero denominator. If $b_{j_0} = 0$ for some j_0, we say that the continued fraction is *finite*. Otherwise, we say that it is *infinite*. If $b_j = 1$ for all j, $a_0 \in \mathbb{Z}$ and $a_j \in \mathbb{N}$ for $j > 0$, we say that the continued fraction is *simple*, and we may represent it in the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} = [a_0, a_1, a_2, a_3 \ \ldots]$$

From hereon forward, we will be concerned primarily with simple continued fractions. Consequently, it is not necessary to go into great depth in the general context. Note, that in the case of a simple continued fraction, we no longer worry about a zero denominator. From this point forward, all continued fractions will be assumed simple and infinite unless otherwise specified. Now, it follows from above that finite continued fractions can be represented in the form

$$[a_0] = a_0, \quad [a_0, a_1] = a_0 + \frac{1}{a_1}, \quad [a_0, a_1, a_2] = a_0 + \frac{1}{[a_1, a_2]} = [a_0, [a_1, a_2]],$$

And in general,

$$[a_0, a_1, \ \ldots \ , a_k] = a_0 + \frac{1}{[a_1, a_2, \ \ldots \ , a_k]} = [a_0, [a_1, \ \ldots \ , a_k]].$$

From this, it is natural to refer to the a_j as *partial quotients*, and to define the k^{th} *convergent* of the continued fraction as the truncation of the continued fraction after k partial quotients. Thus, for the continued fraction $[a_1, a_2, a_3, \ ...]$, the k^{th} convergent would be the finite continued fraction $[a_1, a_2, \ ... \ , a_k]$.

It is not difficult to see that every finite continued fraction is a rational number, but to be thorough, we will give a short proof by induction. Representing the finite continued fraction as $\alpha = [a_1, a_2, \ \ldots \ , a_k]$, we will begin with the case $k = 0$. Clearly, then, $\alpha = a_0 \in \mathbb{Q}$. Remember, we are operating under the assumption that a continued fraction is simple. That is $a_j \in \mathbb{N}$ for all $j > 0$ and $a_0 \in \mathbb{Z}$. Proceeding to the induction step, we will show that this statement is true for $k = n + 1$ given that it's true for $k \leq n$. Thus, from above, we have that

$$\alpha = [a_0, a_1, \ \ldots \ , a_{n+1}] = [a_0, [a_1, a_2, \ \ldots \ , a_{n+1}]]$$

Denoting $\beta = [b_0, b_1, \ \ldots \ , b_n] = [a_1, a_2, \ \ldots \ , a_{n+1}]$, where $b_j = a_{j+1}$ for $0 \leq j \leq n$, it follows by assumption that $\beta \in \mathbb{Q}$. Consequently, $\beta = p/q$ for some $p, q \in \mathbb{Z}$. Therefore,

$$\alpha = [a_0, \beta] = a_0 + \frac{1}{\beta} = \frac{a_0 p + q}{p} \in \mathbb{Q}.$$

From this result it follows that every finite continued fraction, and, by inclusion, every k$^{\text{th}}$ convergent, can be expressed in the form

$$[a_0, a_1, \ \dots \ , a_k] = \frac{p_k(a_0, a_1, \ \dots \ , a_k)}{q_k(a_0, a_1, \ \dots \ , a_k)},$$

Where both p_k and q_k are integer valued functions on k variables.

Definition 2.1.1. Let $a_0, a_1, \ \dots \ \in \mathbb{Z}$. Then we say that their *upper* and *lower* k^{th} *convergents*, $p_k(a_0, \ \dots \ , a_k)$ and $q_k(a_0, \ \dots \ , a_k)$, respectively, are defined by the recurrences

$$\begin{array}{lllll} p_{-1} = 1, & p_0 = a_0, & p_1 = 1 + a_1 a_0, & \dots & p_k = p_{k-2} + a_k p_{k-1}. \\ q_{-1} = 0, & q_0 = 1, & q_1 = a_1, & \dots & q_k = q_{k-2} + a_k q_{k-1}. \end{array}$$

Proposition 2.1.1. Given the partial quotients $a_1, a_2, \ \dots \ , a_k$ for some $k \geq 0$ along with their upper and lower k$^{\text{th}}$ convergents, p_k, q_k, it follows that

(a) $[a_0, a_1, \ \dots \ , a_k] = p_k/q_k$,

(b) $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k+1}$,

(c) $p_{k+1} q_{k-1} - p_{k-1} q_{k+1} = (-1)^{k+1} a_{k+1}$.

Proof. We will proceed by induction. Beginning with (a), we have for $k = 0$, $[a_0] = a_0 = p_0/q_0$. By the induction hypothesis, we assume that $[a_0, a_1, \ \dots \ , a_k] = p_k/q_k$.

Letting $a'_k = [a_k, a_{k+1}] = a_k + 1/a_{k+1}$, we have that

$$\begin{aligned}
[a_0, a_1, \ \ldots \ , a_{k+1}] &= [a_0, a_1, \ \ldots \ , [a_k, a_{k+1}]] \\
&= [a_0, a_1, \ \ldots \ , a_{k-1}, a'_k] \\
&= \frac{p'_k}{q'_k} \\
&= \frac{p_{k-2} + a'_k p_{k-1}}{q_{k-2} + a'_k q_{k-1}} \\
&= \frac{p_{k-2} + a_k p_{k-1} + \frac{p_{k-1}}{a_{k+1}}}{q_{k-2} + a_k q_{k-1} + \frac{p_{k-1}}{a_{k+1}}} \\
&= \frac{p_k + \frac{p_{k-1}}{a_{k+1}}}{q_k + \frac{q_{k-1}}{a_{k+1}}} \\
&= \frac{p_{k-1} + a_{k+1} p_k}{q_{k-1} + a_{k+1} q_k} \\
&= \frac{p_{k+1}}{q_{k+1}},
\end{aligned}$$

As required. Moving on to (b) and (c), we have for $k = 0$ that,

$$p_0 q_{-1} - p_{-1} q_0 = -1, \quad \text{and} \quad p_1 q_{-1} - p_{-1} q_1 = -a_1.$$

Proceeding by the induction hypothesis, we have that

$$\begin{aligned}
p_{k+1} q_k - p_k q_{k+1} &= (p_{k-1} + a_{k+1} p_k) q_k - p_k (q_{k-1} + a_{k+1} q_k) \\
&= -(p_k q_{k-1} - p_{k-1} q_k) \\
&= -(-1)^{k+1} \\
&= (-1)^{k+2},
\end{aligned}$$

And

$$\begin{aligned} p_{k+2}q_k - p_k q_{k+2} &= (p_k + a_{k+2}p_{k+1})q_k - p_k(q_k + a_{k+2}q_{k+1}) \\ &= a_{k+2}(p_{k+1}q_k - p_k q_{k+1}) \\ &= (-1)^{k+2}a_{k+2} \end{aligned}$$

□

2.2 p-adic Integers

The subject of p-adic numbers and the resulting p-adic analysis is a fascinating study in its own right. The work in this section references the work of [13]. We will focus our attention on a specific subset of the p-adic numbers. Specifically, we will be looking at the p-adic integers and moving on to the more general **p**-adic integers. Note, we will not be moving on the most general case of the **a**-adic integers since we will ultimately want to work with maximal ideals. These concepts are important because they generate some very fundamental examples in topological dynamical systems; namely, the odometer systems which will form one of our main objects of study. Thus, we may begin with the definition of the p-adic integers.

Definition 2.2.1. Let $p \in \mathbb{Z}$ be prime. Then we define the set of p-adic integers, $\mathbb{Z}_{(p)}$, to be the inverse limit

$$\mathbb{Z}_{(p)} = \lim_{\leftarrow} \mathbb{Z}/p^j\mathbb{Z} = \left\{ x \in \prod_{j=1}^{\infty} \mathbb{Z}/p^j\mathbb{Z} : \forall\, j, \operatorname{mod}_{p^j}(x_{j+1}) = x_j \right\}$$

Note that the zero element of $\mathbb{Z}_{(p)}$ is the element such that $x_j = 0$ for all j, and that addition and subtraction are defined by the formulas

$$(x+y)_j = \text{mod}_{p^j}(x_j + y_j), \quad (-x)_j = p^j - x_j.$$

Furthermore, it is not difficult to see that $n \in \mathbb{N}$ are represented in $\mathbb{Z}_{(p)}$ by sequences which are eventually constant, and negative integers, $-n$, are represented by sequences for which eventually $x_j = p^j - n$. Now, in order to give a representation of $\mathbb{Z}_{(p)}$ which is easy to work with, we consider the product space

$$X_p = \prod_{j=1}^{\infty} \{0, 1, \ ... \ , p-1\}_j$$

From this perspective, $n \in \mathbb{N}$ is represented by a sequence $x \in X_p$ with finitely many nonzero entries, x_j,

$$n = (x_1, x_2, \ ... \ , x_N, 0, 0, \ ... \ ),$$

Where $x_j \in \{0, 1, \ ... \ , p-1\}$. Similarly, n such that $-n \in \mathbb{N}$ are represented by sequences with finitely many entries, x_j, different from $p-1$. For example,

$$-1 = (p-1, p-1, p-1, \ ... \ ).$$

We define addition on X_p by coordinatewise addition with carry to the right. That is, denoting c_j to be the carry from the j^{th} position to the $(j+1)^{\text{st}}$ position, then we have that $c_0 = 0$, and for $j > 0$ and $x, y \in X_p$,

$$(x+y)_j = \text{mod}_p(x_j + y_j + c_{j-1}), \quad \text{and} \quad c_j = \left\lfloor \frac{x_j + y_j + c_{j-1}}{p} \right\rfloor$$

Thus, it follows that

$$(x+y)_j + c_j p^j = x_j + y_j + c_{j-1}.$$

Now, if we suppose that $0 \leq c_{j-1} \leq 1$, then

$$0 \leq x_j + y_j + c_{j-1} \leq 2(p^j - 1) + c_{j-1} \leq 2p^j - 1,$$

In which case, we have that $0 \leq c_j \leq 1$. We have just shown by induction that $c_j \in \{0, 1\}$ for all j. From here, we would like to show that X_p and $\mathbb{Z}_{(p)}$ are isomorphic as groups. Consequently, we can use the more simple X_p representation when working with $\mathbb{Z}_{(p)}$. With this in mind, define $\varphi : X_p \to \mathbb{Z}_{(p)}$ and $\psi : \mathbb{Z}_{(p)} \to X_p$ by

$$\varphi(x)_j = \sum_{l=1}^{j} x_l p^{l-1} \quad \text{and} \quad \psi(y)_j = \left\lfloor \frac{y_j}{p^{j-1}} \right\rfloor.$$

It is necessary to begin by showing that $\varphi(x)_j < p^j$ for $x \in X_p$. Letting $j = 1$, it follows that $\varphi(x)_1 = x_1 < p$ by definition. Proceeding to the induction step, we suppose that $\varphi(x)_{j-1} < p^{j-1}$. It follows that

$$\begin{aligned}
\varphi(x)_j &= \varphi(x)_{j-1} + x_j p^{j-1} \\
&< p^{j-1} + (p-1)p^{j-1} \\
&= p^j.
\end{aligned}$$

Consequently, we have

$$\psi(\varphi(x))_j = \left\lfloor \frac{\varphi(x)_j}{p^{j-1}} \right\rfloor = \left\lfloor \frac{\varphi(x)_{j-1}}{p^{j-1}} + x_j \right\rfloor = x_j.$$

On the other hand, for $y \in \mathbb{Z}_{(p)}$, we have

$$\begin{aligned}
\varphi(\psi(y))_j &= \sum_{l=1}^{j} \left\lfloor \frac{y_l}{p^{l-1}} \right\rfloor p^{l-1} \\
&= \sum_{l=2}^{j} \left\lfloor \frac{y_l}{p^{l-1}} \right\rfloor p^{l-1} + y_1 \\
&= \sum_{l=2}^{j} \left\lfloor \frac{y_l}{p^{l-1}} \right\rfloor p^{l-1} + \mathrm{mod}_p(y_2) \\
&= \sum_{l=3}^{j} \left\lfloor \frac{y_l}{p^{l-1}} \right\rfloor p^{l-1} + \left\lfloor \frac{y_2}{p} \right\rfloor p + \mathrm{mod}_p(y_2) \\
&= \sum_{l=3}^{j} \left\lfloor \frac{y_l}{p^{l-1}} \right\rfloor p^{l-1} + y_2 \\
&\vdots \\
&= \left\lfloor \frac{y_j}{p^{j-1}} \right\rfloor p^{j-1} + \mathrm{mod}_{j-1}(y_j) \\
&= y_j.
\end{aligned}$$

We may conclude that $\psi = \varphi^{-1}$. It suffices to show that φ preserves the group operation. Thus, let $x, y \in X_p$ and $z = x + y$. Then we have that

$$x_j + y_j + c_{j-1} = z_j + c_j p.$$

It follows that

$$\sum_{l=1}^{j}(x_l + y_l)p^{l-1} + \sum_{l=1}^{j} c_{l-1}p^{l-1} = \sum_{l=1}^{j} z_l p^{l-1} + \sum_{l=1}^{j} c_l p^l,$$

So that consequently,

$$\sum_{l=1}^{j} x_l p^{l-1} + \sum_{l=1}^{j} y_l p^{l-1} = \sum_{l=1}^{j} z_l p^{l-1} + c_j p^j.$$

We may conclude that $\varphi(z)_j = \text{mod}_{p^j}(\varphi(x)_j + \varphi(y)_j)$. In order to generalize to the **p**-adic integers, in which case we do not specify a prime p, we must make slight changes to the definition of both X_p and $\mathbb{Z}_{(p)}$. With this in mind we define, for a sequence $\mathbf{n} = \{n_j\}_{j\geq 0}$ of integers greater than 1,

$$X_{\mathbf{n}} = \prod_{j=0}^{\infty}\{0, 1, \ \dots\ , n_j - 1\}_j.$$

Apart from the obvious change, the structure for $X_{\mathbf{n}}$ mirrors that of X_p defined above very closely. In fact, it is easy to see that $X_{\mathbf{n}}$ is exactly X_p when $n_j = p$ for all j. Similarly, if we let $\mathbf{p} = \{p_j\}_{j\geq 0}$ be an increasing sequence satisfying $p_0 > 1$ and p_j divides p_{j+1} and p_{j+1}/p_j prime. We will use the convention that $p_{-1} = 1$. Then we may define

$$\mathbb{Z}_{(\mathbf{p})} = \left\{ x \in \prod_{j=0}^{\infty} \mathbb{Z}/p_j\mathbb{Z} : \forall\ j,\ \operatorname{mod}_{p_j}(x_{j+1}) = x_j \right\}$$

Again, the structure of $\mathbb{Z}_{(\mathbf{p})}$ mirrors the structure of $\mathbb{Z}_{(p)}$ very closely as $\mathbb{Z}_{(\mathbf{p})}$ is exactly $\mathbb{Z}_{(p)}$ when $p_j = p^{j+1}$ for all j. Making slight index and notation changes, the proof that $X_{\mathbf{n}}$ and $\mathbb{Z}_{(\mathbf{p})}$ are isomorphic as groups is nearly identical. In fact, we will show eventually that $X_{\mathbf{n}}$ and $\mathbb{Z}_{(\mathbf{p})}$ are isomorphic as topological groups. In order to see this, we must introduce topologies on $X_{\mathbf{n}}$ and $\mathbb{Z}_{(\mathbf{p})}$.

We will begin with the topology on $X_{\mathbf{n}}$. Let each set, $\{0,\ \dots, n_j - 1\}_j$ be given the metric $d(j,k)$ where

$$d(j,k) = \begin{cases} 1, & \text{if } j \neq k, \\ 0, & \text{if } j = k. \end{cases}$$

Then it follows that the topology induced by this metric is compact and has the discrete topology since each both is both open and closed. If we proceed to put the product topology on $X_{\mathbf{n}}$, it follows from Tychonoff's theorem for products of compact spaces that $X_{\mathbf{n}}$ is compact. One can see that sets of the form

$$C_n = [c_0,\ \dots, c_n] = \{x \in X_{\mathbf{n}} : x_j = c_j \text{ for } 0 \leq j \leq n\},$$

Form a countable basis of clopen sets in this topology. We will call the sets C_n *cylinder* sets. We will study the properties of the topology of $X_{\mathbf{n}}$ in greater detail later on. For now, we will shift our attention to define the topology on $\mathbb{Z}_{(\mathbf{p})}$.

In a similar manner, we may give each group, $\mathbb{Z}/p^j\mathbb{Z}$ the discrete topology, in which case each of these topological groups is compact since they are finite. We then give $\mathbb{Z}_{(\mathbf{p})}$ the product topology, and Tychonoff's theorem again gives that $\mathbb{Z}_{(\mathbf{p})}$ is compact. It follows that sets of the form

$$C_n' = [c_0', \ \dots, c_n'] = \{x \in \mathbb{Z}_{(\mathbf{p})} : \ \text{such that} \ x = c_0' + \cdots + c_n' p_n + \sum_{j=n+1}^{\infty} x_j p_j\},$$

Form a countable basis of clopen sets in this topology. From this viewpoint, we can now see that the isomorphism defined above between $X_{\mathbf{n}}$ and $\mathbb{Z}_{(\mathbf{p})}$ is actually a homeomorphism, so that these sets are not only isomorphic as groups, but homeomorphic as topological spaces. From this point forward, we will always refer to $X_{\mathbf{n}}$ as the set of **p**-adic Integers since it is easier to work with the cylinder sets arising in this form. Before, we close this section, however, it is necessary to give one more result. Namely, we would like to make a note of the maximal ideals of the **p**-adic Integers. For this result, it is actually more natural if we work with $\mathbb{Z}_{(\mathbf{p})}$.

Thus, define the set

$$\mathcal{M}_0 = \{x = (x_0, x_1, x_2, \ \dots) \in \mathbb{Z}_{(\mathbf{p})} : x_0 = 0\},$$

And define the map $\pi_0 : \mathbb{Z}_{(\mathbf{p})} \to \mathbb{Z}/p_0\mathbb{Z}$ by

$$\pi_0(x) = x_0.$$

It is clear that π_0 is a surjective homomorphism and that $\ker(\pi) = \mathcal{M}_0$. Consequently, it is a well known result commonly referred to as the *First Isomorphism*

Theorem for Rings that $\mathcal{M}_0$ is an ideal of $\mathbb{Z}_{(\mathbf{p})}$ and that the induced map $\mathbb{Z}_{(\mathbf{p})}/\mathcal{M}_0 \to \mathbb{Z}/p_0\mathbb{Z}$ is an isomorphism. Furthermore, since p_0 is prime, it follows that $\mathbb{Z}/p_0\mathbb{Z}$ is a field, in which case $\mathcal{M}_0 = p_0\mathbb{Z}_{(\mathbf{p})}$ is a maximal ideal. Similarly, we may define

$$\mathcal{M}_n = \{x = (0, 0, \ \dots \ , 0, x_n, x_{n+1}, \ \dots \ ) \in p_n\mathbb{Z}_{(\mathbf{p})} : x_n = 0\}$$

Together with the map $\pi_n : p_{n-1}\mathbb{Z}_{(\mathbf{p})} \to p_{n-1}\mathbb{Z}/p_n\mathbb{Z}$ defined by

$$\pi_n(x) = p_n \cdot \text{mod}_{p_{n-1}}(x_n).$$

Again, the First Isomorphism Theorem for Rings gives that $\mathcal{M}_n = p_n\mathbb{Z}_{(\mathbf{p})}$ is an ideal of $p_{n-1}\mathbb{Z}_{(\mathbf{p})}$ and that the induced map $p_{n-1}\mathbb{Z}_{(\mathbf{p})}/\mathcal{M}_n \to p_{n-1}\mathbb{Z}/p_n\mathbb{Z}$ is an isomorphism. Now, clearly, $p_{n-1}\mathbb{Z}/p_n\mathbb{Z}$ is not a field. It is not even unital. However, since $p_{n-1}\mathbb{Z}_{(\mathbf{p})}$ is not unital to begin with, we can see that $\mathcal{M}_n$ is maximal since p_n/p_{n-1} is prime.

2.3 Bratteli Diagrams

Bratteli diagrams are the underpinning of the work to follow. This section references the work of [2], [9] and [11]. The concept of a Bratteli diagram was introduced by Ola Bratteli in his seminal work, [2], studying the classification of AF algebras. In fact, it is this work which underscores the classification of Cantor minimal systems as studied here. We will see that Bratteli diagrams are the tool by which we can move between topological dynamical systems and dimension groups. We will discuss AF algebras later on, including the manner in which a Bratteli diagram can represent an AF algebra. In this section, we will focus on the structure of a Bratteli diagram itself without mention of its algebraic implications. Thus, we begin with a definition.

Definition 2.3.1. A *Bratteli diagram* is an infinite directed graph (V, E) such that the vertex set V and the edge set E satisfy the following conditions:

1. V can be partitioned into disjoint sets, $V = V_0 \cup V_1 \cup \cdots$, such that V_n is finite and non-empty and $V_0 = \{v_0\}$ is a singleton set.

2. E can be partitioned into disjoint sets, $E = E_1 \cup E_2 \cup \cdots$, such that each E_n is finite and non-empty.

3. There exist maps $r, s : E \to V$, respectively the range and source maps, such that $r(E_n) \subseteq V_n$ and $s(E_n) \subseteq V_{n-1}$ for $n \in \mathbb{N}$. Furthermore, $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \backslash V_0$.

There is a natural notion of isomorphism between Bratteli diagrams (V, E) and (V', E'). Precisely, (V, E) and (V', E') are isomorphic if there exist a pair of bijections between V and V' and between E and E' preserving the gradings and intertwining the respective source and range maps. From the above definition there are several ways we may present a Bratteli diagram. The most natural presentation of a Bratteli diagram is in the form of a graph:

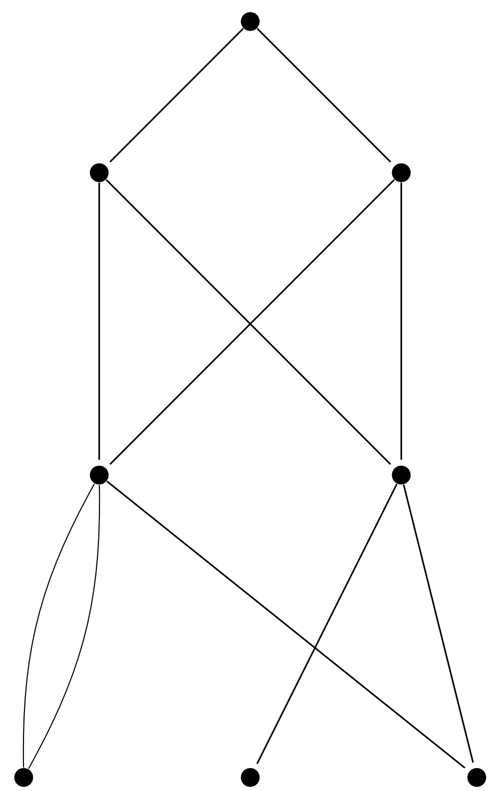


Figure 2.3.1

Where, in this case, we are only presenting the sets V_n for $n = 0, 1, 2, 3$, and the sets E_n for $n = 1, 2, 3$. A more convenient, and in some cases more practical, representation of a Bratteli diagram is through the use of an *incidence* matrix. Specifically, we say that E_n defines an $m_n \times m_{n-1}$ matrix called an incidence matrix, which we will denote as A_n, where $|V_{n-1}| = m_{n-1}$, $|V_n| = m_n$, and the ij^{th} entry is the number of edges between $v_i \in V_n$ and $v_j \in V_{n-1}$. Consequently, it follows that the respective incidence matrices determined by E_1, E_2, and E_3 from Figure 2.3.1 above are

$$A_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Given a Bratteli diagram, it becomes important to speak of so-called contraction, or telescoping, of the diagram. That is, the idea of removing sets of vertices while maintaining prior established paths. Formally, letting $k, l \in \mathbb{Z}^+$ with $k < l$, we may denote all paths from V_k to V_l as $E_{k+1} \circ E_{k+2} \circ \cdots \circ E_l$. That is,

$$E_{k+1} \circ \cdots \circ E_l = \{(e_{k+1}, \ \dots \ , e_l) \ : \ e_n \in E_n, \, n \in [k+1, l] \text{ and} \cdots$$
$$\cdots r(e_n) = s(e_{n+1}), \ n \in [k+1, l-1]\}.$$

Note that we have implicitly defined the notion of a path above. From this definition, letting $p_{kl} = (e_{k+1}, \ \dots \ , e_l)$, denote a path from V_k to V_l, we may define the range and source maps in the natural manner,

$$r(p_{kl}) = r(e_l), \ \text{ and } \ s(p_{kl}) = s(e_{k+1}).$$

At this point we may formally define that which is meant by telescoping.

Definition 2.3.2. Given a Bratteli diagram (V, E) and a sequence $m_0 = 0 < m_1 < m_2 < \cdots$ in $\mathbb{Z}^+$, we define the *telescoping* of (V, E) to $\{m_n\}_{n=0}^{\infty}$ as (V', E'), where $V'_n = V_{m_n}$ for $n \geq 0$ and $E'_n = E_{m_{n-1}+1} \circ \cdots \circ E_{m_n}$ and $r' = r$ and $s' = s$, as above.

Given this definition, we can see that a possible telescoping of Figure 2.3.1 is given by the diagram

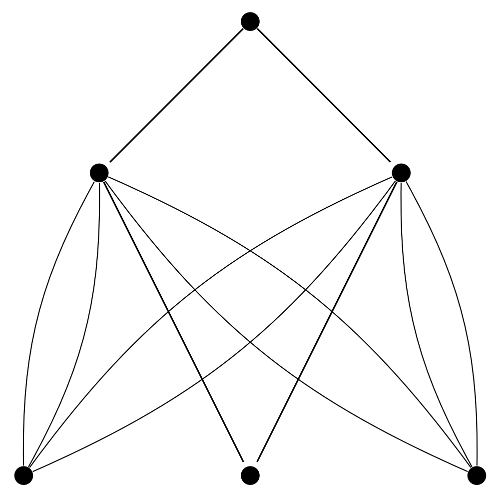


Figure 1.3.2

Note that the telescoped incidence matrix, A_2' is the product of the two combined incidence matrices A_2 and A_3. Now, the reason we have gone through all this trouble is because we wish to work with simple Bratteli diagrams, which we are now positioned to define.

Definition 2.3.3. We say that a Bratteli diagram (V, E) is *simple* if there exists a telescoping (V', E') of (V, E) such that each of the incidence matrices of (V', E') have strictly positive entries.

This accomplished, we would now like to endow a Bratteli diagram with a partial order. The motivation for this lies in that we ultimately would like to construct a dynamical system from a Bratteli diagram. This, however, will not be undertaken until we have given an introduction to topological dynamical systems. For now, we will define that which is meant by an ordered Bratteli diagram.

Definition 2.3.4. An *ordered* Bratteli diagram $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ on E such that the edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. That is, there exists a linear order on each set $r^{-1}(\{v\})$, where $v \in V \backslash V_0$.

It is not difficult to see that the set of paths, as defined above, has an induced lexicographic order. That is, if we let $k, l \in \mathbb{Z}^+$ with $k < l$, then

$$(e_{k+1}, e_{k+2}, \; .. \; , e_l) > (f_{k+1}, \; f_{k+2}, \; ... \; , f_l),$$

If and only if there is some i such that $k+1 \leq i \leq l$ with $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. It is important to note that this idea is also extended naturally to the telescoping of a Bratteli diagram and to isomorphisms between Bratteli diagrams.

2.4 AF Algebras

Although approximately finite dimensional, or AF, algebras will not be used explicitly in this work, they do hold a close relationship and form the foundation of the work to be done. Consequently, this section will introduce the notion of an AF algebra along with some definitions and results. As a reference we are using the work of [1], [2], [3] and [5]. In order to properly define an AF algebra, we must first develop the general context of C*-algebras.

A *Banach* algebra is a complex normed algebra $\mathcal{A}$ which is complete as a topological space and which satisfies

$$||AB|| \leq ||A|| \cdot ||B|| \quad \text{for all} \quad A, B \in \mathcal{A}.$$

If we endow a Banach algebra, $\mathcal{A}$, with a conjugate linear involution, $*$, that is an anti-isomorphism we say that $\mathcal{A}$ is a Banach $*$-algebra. Furthermore, we will call $*$ the *adjoint*. Thus, for $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we have that

$$(A+B)^* = A^* + B^*,$$

$$(\lambda A)^* = \overline{\lambda} A^*,$$

$$A^{**} = A,$$

$$(AB)^* = B^* A^*.$$

Definition 2.4.1. A *C^*-algebra*, $\mathcal{A}$, is a Banach $*$-algebra with the additional norm condition that

$$||A^*A|| = ||A||^2 \quad \text{for all} \quad A \in \mathcal{A}.$$

Some examples of C*-algebras follow:

1. The algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ with the usual adjoint operation. Indeed, it is well known that

$$||A^*A|| = \sup_{||x||=||y||=1} |(A^*Ax, y)| = \sup_{||x||=||y||=1} |(Ax, Ay)| = ||A||^2.$$

2. The set of all continuous functions on a locally compact Hausdorff space, X, vanishing at infinity, $C_0(X)$, forms a C*-algebra taking complex conjugation as the adjoint operation. We have that

$$||\overline{f}f||_X = \sup_{x \in X} |\overline{f(x)}f(x)| = \sup_{x \in X} |f(x)|^2 = ||f||_X^2.$$

Definition 2.4.2. A C*-algebra, $\mathcal{A}$, is an *AF algebra* if $\mathcal{A}$ is an inductive limit of a sequence of finite-dimensional C*-algebras.

It can be shown that a finite-dimensional C*-algebra is simply a direct sum of matrix-algebras. Consequently, it follows that an AF algebra is an inductive limit of direct sums of matrix-algebras. Due to this fact, most AF algebras are defined in terms of direct limits. In what follows, we will denote $M_p(A)$ to be the algebra of $p \times p$ matrices with entries in A. In particular, we let $M(p) = M_p(\mathbb{C})$, and more generally, we denote the multimatrix-algebra $M(p_1, \ .. \ , p_n) = M(p_1) \oplus \cdots \oplus M(p_n)$. Furthermore, note that the maps used in the direct limits defining AF algebras denote the multiplicity of the embedding of each component matrix-algebra from one step to the next. Some examples of AF-algebras follow:

1. The Canonical Anticommutation Relations, or CAR, algebra given by

$$M(2) \xrightarrow{[2]} M(4) \xrightarrow{[2]} \cdots ,$$

2. The above example can also be expressed as the direct limit,

$$M(1,1) \xrightarrow{A} M(2,2) \xrightarrow{A} M(4,4) \xrightarrow{A} \cdots ,$$

Where A at each step is defined as,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

3. The Fibonacci algebra given by

$$M(1,1) \xrightarrow{A} M(2,1) \xrightarrow{A} M(3,2) \xrightarrow{A} \cdots$$

Where A at each step is defined as,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Before we close this section, we would like to make a note concerning Bratteli diagrams. As they were originally defined in [2], and used in the context of AF algebras, Bratteli diagrams were constructed so as to encode and preserve the dimension of the finite dimensional C*-algebras at each step. Thus, each vertex in a Bratteli diagram traditionally encodes the dimension of the component matrix-algebra at that level while the edges connecting the vertices encode the embedding information from one level to the next. Note that this implies that the dimension of a matrix-algebra at a given vertex in this construction must be sufficiently large to accommodate the dimension, including multiplicity, of each matrix-algebra to be embedded. It should be noted as well that if the embeddings at each level are unital, then this information becomes redundant and is therefore conventionally not recorded.

Since we will not be explicitly working with AF-algebras, we will not be encoding this dimension information. More to the point, the Bratteli diagrams used in the work to follow will be assumed to represent a copy of $\mathbb{Z}$ at each vertex, and the edges will denote linear maps from $\mathbb{Z}^n$ to $\mathbb{Z}^m$ instead of the multiplicity of embeddings in matrix-algebras.

Chapter 3

Topological Dynamical Systems

3.1 The Basics

The theory of general dynamical systems evolved originally in the context of modeling movement in physical systems; most notably, in the development of our understanding of planetary motion. Consequently, traditional dynamics is viewed through the lens of differential and difference equations using notions such as state space, trajectory and attractors. Topological dynamics is a generalization and abstraction of these concepts in the context of topological spaces and homeomorphisms. We will begin at a fundamental level with a basic definition of a dynamical system. Note, the primary reference for this chapter is [13].

Definition 3.1.1. A *dynamical system* is a nonempty set, X, together with a bijection $\varphi : X \to X$.

Through this perspective, we may think of ‘movement’ in a dynamical system as iterations of φ over a specified point, $x \in X$. That is we may look at $\varphi^n : X \to X$ for $n \in \mathbb{Z}$. Note that this makes sense since we have defined φ to be a bijection, in which case, φ is invertible. In this case, we may note that φ induces a group representation of $\mathbb{Z}$. In a more general case, we might consider representations of groups other than $\mathbb{Z}$, for instance $\mathbb{R}$ or a lattice $\mathbb{Z}^d$. In the context of the work to follow, however, we will restrict our attention to group actions of $\mathbb{Z}$ on X. With this in mind, unless otherwise specified we will assume a dynamical system to be a representation of $\mathbb{Z}$, and we will define an orbit in this dynamical system as an orbit of the group action over a specified point in X. Formally,

Definition 3.1.2. Given a dynamical system, (X, φ), we say that the *orbit* of $x \in X$ under φ is the set $\mathcal{O}_x = \{\varphi^n(x) : n \in \mathbb{Z}\}$.

Similarly, we may look at orbits of subsets of X.

Definition 3.1.3. Given a dynamical system (X, φ), we will say that the orbit of a subset $E \subseteq X$ of X under φ is the set $\mathcal{O}_E = \{\varphi^n(E) : n \in \mathbb{Z}\}$.

Some examples of dynamical systems follow:

1. Let X be such that $|X| < \infty$ and φ a permutation on X. This dynamical system is known as a *finite system.*

2. Let X be the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with the map $\varphi(x) = x + 1$. Note, this is an example of a finite system.

3. On the other hand, let X be the group of integers with the map $\varphi(x) = x + 1$. This is a dynamical system which is not finite.

4. Let X be the group of $\mathbf{p}$-adic integers $\mathbb{Z}_{(\mathbf{p})}$ defined in the first chapter with the map $\varphi(x) = x + 1$. This dynamical system is known as a generalized adding machine.

5. Let X be the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the map $\varphi_\alpha(x) = \mod {}_1(x + \alpha)$, where $\alpha \in \mathbb{R}$. This system can be viewed as rotations on the circle. This example along with the general adding machine described above turn out to be quite important examples which we will visit throughout the work to follow.

One of the main areas of interest in dynamical systems is the behavior of the orbits of points and subsets of X under φ. This is referred to as the *asymptotic* behavior of the dynamical system. It turns out that questions arising from this line of thought both motivate and enable a classification of dynamical systems. It is this which is studied in the work to follow. At this point, we may restrict our attention to a particular class of dynamical systems.

Definition 3.1.4. We say that the dynamical system (X, φ) is a *topological dynamical system* if X is a compact metrizable topological space and φ is a homeomorphism.

Note that the assumption of compactness ensures that the dynamical system resembles, to some extent, a finite dynamical system. When this assumption is taken away, the dynamics become considerably more complicated. In the scope of the work done here, we will be asserting even stronger conditions on the topological spaces under consideration.

It is not difficult to see that examples (2) and (3) above are examples of a topological dynamical systems if we give X, in both cases, the discrete topology. In the case of example (4) above, we may give X the topology as defined in section 1.2, and example (5) is an example of a topological dynamical system if we give X the usual topology.

3.2 Notions of Equivalence

In order to properly round out an introduction to topological dynamical systems, we must discuss notions of equivalence between topological dynamical systems. The natural question which arises is that of a concept of an isomorphism between topological dynamical systems. That is, under what conditions may we compare these systems? The answer lies in the ideas of conjugacy and orbit equivalence.

Definition 3.2.1. We say that topological dynamical systems (X_1, φ_1) and (X_2, φ_2) are *conjugate* if there exists a homeomorphism $F : X_1 \to X_2$ such that $F \circ \varphi_1 = \varphi_2 \circ F$.

Definition 3.2.2. We say that topological dynamical systems (X_1, φ_1) and (X_2, φ_2) are *orbit equivalent* if there exists a homeomorphism $F : X_1 \to X_2$ such that $F(\mathcal{O}_{x_1}) = \mathcal{O}_{F(x_1)}$ for each $x_1 \in X_1$. We will use the term *orbit map* in reference to F.

Letting (X_1, φ_1), (X_2, φ_2), and F be as in the above definition. Then, for each point $x \in X_1$, there exists an integer $n(x)$ such that $F \circ \varphi_1(x) = \varphi_2^{n(x)} \circ F(x)$. Similarly, there exists an integer $m(x)$ such that $F \circ \varphi_1^{m(x)}(x) = \varphi_2 \circ F(x)$. If (X_1, φ_1) and (X_2, φ_2) are minimal systems (as defined in the proceeding section), one can see that m and n are uniquely defined integer-valued functions on X_1. Consequently, we define m and n to be the *orbit cocycles* associated to the orbit map F.

Definition 3.2.3. We say that topological dynamical systems (X_1, φ_1) and (X_2, φ_2)

are *strong orbit equivalent* if they are orbit equivalent and if there exists an orbit map $F : X_1 \to X_2$ such that the associated orbit cocycles $m, n : X_1 \to \mathbb{Z}$ each have at most one point of discontinuity.

It is clear from these definitions that conjugacy is the strongest relation. In fact, it is not difficult to see that conjugacy implies orbit equivalence. It is because of this that we will say that topological dynamical systems are *isomorphic* if they are conjugate. In the work to follow, however, we will focus primarily on strong orbit equivalence. In particular we will be studying algebraic objects which remain invariant under strong orbit equivalence. Note that in the section on Kakutani-Rohlin partitions below, we introduce one more notion of equivalence between topological dynamical systems.

3.3 Minimal Dynamical Systems

Definition 3.3.1. Given a topological dynamical system (X, φ), we say that an orbit $\mathcal{O}_x$ of the system is *minimal* if $\mathcal{O}_x$ is dense in X.

For an example of a minimal orbit, we need only look so far as example (5) above. Let $x \in \mathbb{T}$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then $\mathcal{O}_x$ is a minimal orbit of the system $(\mathbb{T}, \varphi_\alpha)$. On the other hand, if $\alpha \in \mathbb{Q}$, then $\mathcal{O}_x$ is not minimal for any $x \in \mathbb{T}$. We may extend the concept of minimality from single orbits to entire systems as follows.

Definition 3.3.2. We say that a topological dynamical system (X, φ) is *minimal* if $\mathcal{O}_x$ is minimal for every $x \in X$.

We show that $(\mathbb{Z}_{(\mathbf{p})}, \varphi)$ and $(\mathbb{T}, \varphi_\alpha)$ when $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, as described above, are both minimal topological dynamical systems beginning with the adding machine, $(\mathbb{Z}_{(\mathbf{p})}, \varphi)$. Define the metric, $d_{\mathbf{p}}$, on $\mathbb{Z}_{(\mathbf{p})}$ by $d_{\mathbf{p}}(x, y) = 1/2^N$ where $N = \max\{n : x_j = y_j$ for $i < n\}$. Now, we choose a point $x \in \mathbb{Z}_{(\mathbf{p})}$. We must show that for any point $y \in \mathbb{Z}_{(\mathbf{p})}$ and any $\epsilon > 0$, there is a point $x_0 \in \mathcal{O}_x$ such that $d_{\mathbf{p}}(x_0, y) < \epsilon$. With this in mind, choose $y \in \mathbb{Z}_{(\mathbf{p})}$ and let $\epsilon > 0$. Since $1/2^n \to 0$ as $n \to \infty$, we may choose N_0 such that $1/2^{N_0} < \epsilon$. Note that if we restrict x to the finite expression $x|_n = (x_0, x_1, \ \ldots \ , x_n)$ then $\mathcal{O}_x$ becomes transitive and periodic with period p_n. Consequently, it follows that there exists some $m \leq N_0$ such that $\varphi^m(x)|_{N_0} = y|_{N_0}$. Letting $x_0 = \varphi^m(x)$, we have that $d_{\mathbf{p}}(x_0, y) \leq 1/2^{N_0} < \epsilon$. We may conclude that the orbit $\mathcal{O}_x$ is minimal in $(\mathbb{Z}_{(\mathbf{p})}, \varphi)$; and since the choice of x is arbitrary, that $(\mathbb{Z}_{(\mathbf{p})}, \varphi)$ is a minimal topological dynamical system.

We may now move our attention to $(\mathbb{T}, \varphi_\alpha)$ and begin by showing that for $\alpha \in \mathbb{Q}$, $\mathcal{O}_x$ is periodic for every $x \in \mathbb{T}$. That is, given $\alpha \in \mathbb{Q}$, then for each point $x \in \mathbb{T}$ there exists an $n \in \mathbb{N}$ such that $x = \varphi_\alpha^n(x)$. This result is fairly immediate, as we shall see. Since $\alpha \in \mathbb{Q}$, we may represent α as p/q with $p, q \in \mathbb{Z}$ relatively prime. Consequently, it follows that

$$\varphi_\alpha^q(x) = \text{mod}_1(x + q\alpha) = \text{mod}_1(x + p) = x.$$

Consequently, we have shown that $\mathcal{O}_x$ is periodic and therefore finite. Thus, it follows that $\mathcal{O}_x$ is not minimal for any $x \in \mathbb{T}$ when $\alpha \in \mathbb{Q}$. This accomplished, we can move our attention to the case when $\alpha \in \mathbb{R}/\mathbb{Q}$. Since α is irrational, then we have for any $n \in \mathbb{Z}$, $n\alpha \notin \mathbb{Z}$. Consequently, we can see that $\mathcal{O}_x$ cannot be periodic for any $x \in \mathbb{T}$. With this in mind, we will show that $\mathcal{O}_0$ is minimal. Thus, let $\epsilon > 0$, and choose $n \in \mathbb{N}$ such that $1/n < \epsilon$. Since α is irrational, we have just shown that the points $\{\varphi_\alpha^j(0)\}_{j=0}^n$ are distinct. Therefore, since $d(\mathbb{T}) = 1$, it follows that there exist $0 \leq j_1 < j_2 < n$ such that $d(\varphi_\alpha^{j_1}(0), \varphi_\alpha^{j_2}(0)) < 1/n$. We then have that

$$\begin{aligned} d(0, \varphi_\alpha^{j_2 - j_1}(0)) &= \text{mod}_1((j_2 - j_1)\alpha) \\ &= \text{mod}_1(j_2\alpha) - \text{mod}_1(j_1\alpha) \\ &= d(\varphi_\alpha^{j_1}(0), \varphi_\alpha^{j_2}(0)) \\ &\leq \frac{1}{n} \\ &< \epsilon. \end{aligned}$$

It is not difficult to see that $\varphi_\alpha^{j_2 - j_1} = \varphi_{\alpha_0}$ is a rotation with either $\alpha_0 \in (0, 1/n)$ or $\alpha_0 \in (1 - 1/n, 1)$. It follows that the rotation φ_{α_0} will iterate the point 0 through $\mathbb{T}$ in increments less than $1/n$. Consequently, given $x \in \mathbb{T}$ it follows that there exists $n_x \in \mathbb{N}$ such that

$$d(\varphi_\alpha^{n_x(j_2 - j_1)}(0), x) = d(\varphi_{\alpha_0}^{n_x}(0), x) \leq \frac{1}{n} < \epsilon.$$

Since this holds for any $x \in \mathbb{T}$, we may conclude that $\mathcal{O}_0$ is dense. A similar argument shows that $\mathcal{O}_x$ is dense for any $x \in \mathbb{T}$, from which we can see that $(\mathbb{T}, \varphi_\alpha)$ is a minimal topological dynamical system for irrational α.

3.4 Cantor Minimal Systems

In the work that follows, we will be focusing our attention on *Cantor minimal systems*. That is, topological dynamical systems in which the topological space is a

Cantor space. We begin with the definition of a Cantor space.

Definition 3.4.1. A *Cantor space* is a topological space which is compact, metrizable, totally disconnected, and which contains no isolated points.

The prototypical example of a Cantor space is the standard middle thirds Cantor set endowed with the relative topology. It is a standard result that every Cantor space is homeomorphic to any other Cantor space. In order to look at some interesting Cantor minimal systems, however, we would like to develop some examples of Cantor spaces which are more interesting than the usual middle thirds Cantor set. We will begin by showing that $\mathbb{Z}_{(\mathbf{p})}$ with the topology defined in section 2.2 is a Cantor space.

In order to show that it is metrizable, we simply need to show that the metric $d_{\mathbf{p}}$ on $\mathbb{Z}_{(\mathbf{p})}$, defined above as $d_{\mathbf{p}}(x, y) = 1/2^N$, where $N = \max\{n : x_j = y_j \text{ for } i < n\}$, induces the topology. Indeed, one can see that this metric yields the clopen base of cylinder sets as described in section 2.2.

In order to show that $\mathbb{Z}_{(\mathbf{p})}$ contains no isolated points, we choose a point $x \in \mathbb{Z}_{(\mathbf{p})}$ along with some $\epsilon > 0$. We may choose $N \in \mathbb{N}$ such that $1/2^N < \epsilon$. It follows that this choice of N defines a cylinder set C_N such that $x \in C_N$. A choice of $y \in C_N$ such that $x \neq y$ yields $d_{\mathbf{p}}(x, y) \leq 1/2^N < \epsilon$ showing the desired result that $\mathbb{Z}_{(\mathbf{p})}$ contains no isolated points.

Since $\mathbb{Z}_{(\mathbf{p})}$ has a basis of clopen sets, it follows that $\mathbb{Z}_{(\mathbf{p})}$ is a zero dimensional space, in which case it is totally disconnected. We have already shown that $\mathbb{Z}_{(\mathbf{p})}$ is compact; consequently, it follows that $\mathbb{Z}_{(\mathbf{p})}$ is a Cantor space.

We may now shift our attention in order to develop another example of a Cantor space. Clearly, $\mathbb{T}$ is not a Cantor space. It is compact, metrizable and contains no isolated points; however, it is not totally disconnected. We would consequently like to *Cantorize* the circle. That is, we would like to construct a space which retains the properties of $\mathbb{T}$ while at the same time has the property of being totally disconnected.

We proceed by taking each point $\varphi_\alpha^n(0)$, for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and replacing it with its left and right limit points. We denote α_n^l to be the left limit point of $\varphi_\alpha^n(0)$ and α_n^r to be the right limit point of $\varphi_\alpha^n(0)$; and we define $\mathbb{T}_n$ to be $\mathbb{T}$ with the points $\{\varphi_\alpha^j(0)\}_{j=0}^n$ replaced with their left and right limit points. We also define the projection $\pi_j : \mathbb{T}_{j+1} \to \mathbb{T}_j$ as,

$$\pi_j(x) = \begin{cases} x, & \text{if } x \notin \{\alpha_{j+1}^l, \alpha_{j+1}^r\}, \\ \varphi_\alpha^j(0), & \text{otherwise.} \end{cases}$$

We may then define the Cantorized circle to be the inverse limit,

$$\mathbb{T}_{\mathcal{C}} = \lim_{\leftarrow} \mathbb{T}_j = \left\{ x \in \prod_{j=0}^{\infty} \mathbb{T}_j \ : \ \forall\ j,\ \pi_j(x_{j+1}) = x_j \right\}.$$

Since $\mathbb{T}_n$ is compact for each n, Tychonoff's theorem gives that $\mathbb{T}_{\mathcal{C}}$ is compact. Furthermore, we can see that it is metrizable since it is a countable inverse limit of compact metrizable spaces. We can see, from the construction of the inverse limit, that we may obtain a basis of open sets of the form $[\alpha_{n_i}^r, \alpha_{n_j}^l]$. In order to see that each of these open sets is also closed, we simply note that $\mathbb{T}_{\mathcal{C}}\backslash[\alpha_{n_i}^r, \alpha_{n_j}^l] = [\alpha_{n_j}^r, \alpha_{n_i}^l]$. Thus, it follows that $\mathbb{T}_{\mathcal{C}}$ is metrizable with an induced topology which is zero dimensional.

In order to show that $\mathbb{T}_{\mathcal{C}}$ contains no isolated points, it suffices to observe that $\mathbb{T}_{\mathcal{C}}$ does not contain any singleton subsets. It follows that $\mathbb{T}_{\mathcal{C}}$ is a Cantor space.

3.5 Dynamics on Ordered Bratteli Diagrams

Before we finish this chapter, we would like to develop a topological dynamical system associated to an ordered Bratteli diagram. This stated, we will jump right into it. Let $\mathcal{B} = (V, E, \geq)$ be an ordered Bratteli diagram. Recall that there is an associated infinite path space associated to $\mathcal{B}$. Specifically, denoting this space $X_{\mathcal{B}}$,

$$X_{\mathcal{B}} = \{(e_1, e_2, \ ... \ ) : e_j \in E_j, \ r(e_j) = s(e_{j+1}), \ j \in [1, \infty)\}.$$

In order that we may focus on the interesting dynamics associated with $X_{\mathcal{B}}$, we will exclude the trivial cases and henceforth assume that $X_{\mathcal{B}}$ is an infinite set. Now, we may, in the usual manner, define cylinder sets on $X_{\mathcal{B}}$ as

$$[c_1 c_2 \cdots c_n] = \{(e_1, e_2, \ ... \ ) \in X_{\mathcal{B}} : e_j = c_j, \ j \in [1, k]\}.$$

We note that, as shown in similar cases above, $X_{\mathcal{B}}$ is compact, metrizable and totally disconnected. Furthermore, if we put the condition that $\mathcal{B}$ is a simple Bratteli diagram, it follows that $X_{\mathcal{B}}$ does not have isolated points, in which case it is a Cantor space.

Now, in order to define a homeomorphism on $X_{\mathcal{B}}$ so that we may define its dynamics, it is necessary to introduce some notation. Let $x \in X_{\mathcal{B}}$ and define $x(n) = e_n$. That is, $x(n)$ is the n^{th} edge in the path x. With this, we may define $X_{\mathcal{B}}^{\max}$ to be the set of $x \in X_{\mathcal{B}}$ such that $x(n)$ is a maximal edge for each $n \in \mathbb{N}$. Similarly, we define $X_{\mathcal{B}}^{\min}$ to be the set of $x \in X_{\mathcal{B}}$ such that $x(n)$ is a minimal edge for each $n \in \mathbb{N}$. It is immediate that both $X_{\mathcal{B}}^{\max}$ and $X_{\mathcal{B}}^{\min}$ are nonempty. This accomplished, we must give a definition before describing a homeomorphism on $X_{\mathcal{B}}$. In particular, we would like to impose conditions on $\mathcal{B}$ so that we may define a homeomorphism which will yield a Cantor minimal system.

Definition 3.5.1. We say that an ordered Bratteli diagram, $(V, E, \geq)$ is *properly ordered*, or *simple ordered*, if

1. (V, E) is a simple Bratteli diagram.
2. $X_{\mathcal{B}}^{\max}$ and $X_{\mathcal{B}}^{\min}$ each consist of exactly one point, which we will denote $x_{\max}$ and $x_{\min}$, respectively.

We may now define a map $V_{\mathcal{B}} : X_{\mathcal{B}} \to X_{\mathcal{B}}$, called the *Vershik* or *Lexicographic* map, where $\mathcal{B}$ is a properly ordered Bratteli diagram. Specifically, let $V_{\mathcal{B}}(x_{\max}) = x_{\min}$. If $x \neq x_{\max}$, let N be the smallest number such that e_N is not a maximal edge. Define f_N to be the successor of e_N, and let $(f_1, f_2, \ldots, f_{N-1})$ be the unique path in $X_{\mathcal{B}}^{\min}$ from v_0 to $s(f_N)$. Then, we may define $V_{\mathcal{B}}(x) = (f_1, \ldots, f_N, e_{N+1}, \ldots)$, noting that since f_N and e_N are comparable, $r(f_N) = r(e_N) = s(e_{N+1})$.

Now, it is shown in [11] that, not only is the Vershik map a minimal homeomorphism, but that the resulting Cantor minimal system, which is generally referred to as *Bratteli-Vershik system*, is indeed very useful. Specifically, the authors showed that every Cantor minimal system is isomorphic to a Bratteli-Vershik system. First, however, we must also note that Herman, Putnam, and Skau were working with *essentially* minimal dynamical systems, and consequently had to work with *pointed* essentially minimal dynamcial systems. That is, they specified a point, y, such that $\mathcal{O}_y$ was minimal in the specified essentially minimal system. Since, minimal implies essentially minimal, we may rephrase their result in terms of strictly minimal dynamical systems.

Theorem 3.5.1. Let (X, φ) be a Cantor minimal system. Then there exists a properly ordered Bratteli diagram $\mathcal{B} = (V, E, \geq)$ such that (X, φ) is isomorphic to $(X_{\mathcal{B}}, V_{\mathcal{B}})$.

Though we do not present the proof of this result, we will come back to several of the methods developed within the proof which allow one to construct a Bratteli diagram from a given Cantor minimal system. Of particular importance are the Kakutani-Rohlin partitions of a Cantor minimal system.

The importance of this result cannot be overstated. It is this result which allows us to associate a Bratteli diagram, and consequently a Dimension group and substitution, to a given Cantor minimal system. In other words, this result lies at the heart of everything that is done herein.

3.6 Kakutani-Rohlin Partitions

In this section, we will present a method for constructing a properly ordered Bratteli diagram $\mathcal{B} = (V, E, \geq)$ from a Cantor minimal system (X, φ) satisfying the condition that (X, φ) and $(X_{\mathcal{B}}, V_{\mathcal{B}})$ are isomorphic. We will begin with a Lemma and Theorem from [11].

Lemma 3.6.1. Let (X, φ) be a Cantor minimal system with Z a clopen subset of X and let $\mathcal{P}$ be a partition of X. Then there exist positive integers $K,\ J(1),\ J(2),\ \dots,\ J(K)$ and clopen sets $Z(k, j)$ for $1 \le k \le K$ and $1 \le j \le J(k)$ satisfying

1. $\bigcup_k Z(k, J(k)) = Z$,

2. $\bigcup_k Z(k, 1) = \varphi(Z)$,

3. $\varphi(Z(k, j)) = Z(k, j+1)$ for $1 \le j < J(k)$,

4. $\{Z(k, j)\ :\ 1 \le k \le K \text{ and } 1 \le j \le J(k)\}$ is a partition of X which is finer than $\mathcal{P}$.

Definition 3.6.1. Let (X, φ) be a Cantor minimal system. A partition $\mathcal{P}$ constructed in the manner of Lemma 3.6.1 is called a *Kakutani-Rohlin* partition. Furthermore, we will refer to the set $\{Z(k, j)\ :\ 1 \le j \le J(k)\}$ as a *tower*.

The above result allows us to present the theorem from [11] which will explain how we may use Kakutani-Rohlin partitions inductively in order to construct a properly ordered Bratteli diagram. We must note that since the systems considered here are minimal, we simply need to choose some point $x \in X$ with which we will work. In [11], this choice had to be made carefully as the authors were working with essentially minimal systems.

Theorem 3.6.1. Let (X, φ) be a Cantor minimal system and choose a point $x \in X$. Then there are positive integers $K(n),\ J(n,1),\ \dots,\ J(n, K(n))$, and clopen sets $Z(n, k, j)$ with $1 \le k \le K(n)$ and $1 \le j \le J(n, k)$ for each $n \in \mathbb{N}$ satisfying

1. The sequence $\{\bigcup_k Z(n,k,J(n,k))\}_{n\geq 1}$ is a decreasing sequence of clopen sets with intersection $\{x\}$,

2. For each n, $\mathcal{P}_n = \{Z(n,k,j) \ : \ 1 \leq k \leq K(n) \,\text{and}\, 1 \leq j \leq J(n,k)\}$ is a Kakutani-Rohlin partition of (X,φ),

3. For each n, $\mathcal{P}_n \leq \mathcal{P}_{n+1}$,

4. $\bigcup_n \mathcal{P}_n$ generates the topology of X.

Thus, given a Cantor minimal system (X,φ), we may construct a sequence of Kakutani-Rohlin partitions around a point $x \in X$ as described in Theorem 2.6.1. We may define a properly ordered Bratteli diagram from this sequence of partitions in the following manner. By convention, we will define $K(0) = 1$ and $Z(0,1,1) = X$. Then, for each $n \in \mathbb{N} \cup \{0\}$, we have one vertex in V_n for each tower in the n^{th} Kakutani-Rohlin partition. In other words,

$$V_n = \{(n,1), \ \dots \ , (n,K(n))\}.$$

Furthermore, for each $n \in \mathbb{N}$, we have one edge E_n for each time that a tower in the n^{th} decomposition passes through a tower at level $n-1$. That is,

$$E_n = \{(n,k,k',j') \ : \ Z(n,k',j'+j) \subseteq Z(n-1,k,j) \ \forall \ 1 \leq j \leq J(n-1,k)\}.$$

It should be noted that it follows from the above theorem that it suffices to check that $Z(n,k',j'+1) \subseteq Z(n-1,k,1)$ in order to determine the set of edges. More importantly, the source and range maps for each edge are given by

$$s(n,k,k',j') = (n-1,k),$$
$$r(n,k,k',j') = (n,k').$$

An order on the edges going into a given vertex is obtained from the order in which the tower on the vertex passes through a tower in the $(n-1)^{\text{st}}$ level. Specifically, we have that $(n_1,k_1,k_1',j_1') \geq (n_2,k_2,k_2',j_2')$ if and only if $n_1 = n_2$, $k_1 = k_2$ and $j_1' \geq j_2'$. We have thus constructed the specified properly ordered Bratteli diagram $\mathcal{B}$. It is shown in [11] that the Bratteli-Vershik system obtained from $\mathcal{B}$ is isomorphic to (X,φ). Before moving on, we would like to give a definition which yields yet another notion of equivalence between Cantor minimal systems.

Definition 3.6.2. The Cantor minimal systems (X_1,φ_1) and (X_2,φ_2) are said to be *Kakutani orbit (Kakutani strong orbit) equivalent* if they are orbit (strong orbit) equivalent to the systems (Y_1,ψ_1) and (Y_2,ψ_2), respectively, where (Y_1,ψ_1) and (Y_2,ψ_2) yield isomorphic Bratteli diagrams.

It is shown in [9] that the notion of Kakutani orbit equivalence is, indeed, an equivalence relation.

3.6.1 The Adding Machine Examples

Since we have shown how this is done, we will give several explicit examples in order to illustrate this process in detail. We will start with the dyadic adding machine, $(\mathbb{Z}_{(2)},\varphi)$. In other words, we are taking the adding machine with $p_{j+1}/p_j = 2$ for all j. For simplicity, we will choose the point around which we will construct the Kakutani-Rohlin partitions to be 0. By convention, we have that $K(0) = 1$ and $Z(0,1,1) = \mathbb{Z}_{(2)}$. Moving on to the next iteration, we note that we would like to construct a partition of $\mathbb{Z}_{(2)}$ which is a Kakutani-Rohlin partition. Since this is the first partition, we may choose our clopen subset, Z, to be all $\mathbb{Z}_{(2)}$. We are now in the position where we may choose our partition of $\mathbb{Z}_{(2)}$ such that we satisfy the conditions of Lemma 3.6.1 given $Z = \mathbb{Z}_{(2)}$. Since $2\mathbb{Z}_{(2)}$ is a maximal ideal of $\mathbb{Z}_{(2)}$, we examine the partition $\{2\mathbb{Z}_{(2)},\ 2\mathbb{Z}_{(2)} + 1\}$. Since, in order to satisfy conditions 1 and 2, we need that $\bigcup_k Z(k, J(k)) = \mathbb{Z}_{(2)}$ and $\bigcup_k J(k,1) = \varphi(\mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}$, it follows that we must set $K(1) = 2$ and $J(1,1) = J(1,2) = 1$. Furthermore, in order to satisfy condition 3, we must have that $\varphi(Z(k,j)) = Z(k, j+1)$ for $1 \leq j \leq J(k)$. Given our assignment of $K(1)$, $J(1,1)$ and $J(1,2)$, this turns out to be trivially satisfied regardless of assignment. However, in order to satisfy condition 1 in Theorem 3.6.1, we will assign $Z(1,1,1) = 2\mathbb{Z}_{(2)} + 1$ and $Z(1,2,1) = 2\mathbb{Z}_{(2)}$. Note that we also trivially satisfy condition 4. We have consequently constructed a Kakutani-Rohlin partition on the system $(\mathbb{Z}_{(2)}, \varphi)$,

$$\mathcal{P}_1 = \{Z(1,1,1),\ Z(1,2,1)\}.$$

For the next iteration, we must be a bit more careful in our choices. We will set our clopen subset, Z, to be the maximal ideal, $2\mathbb{Z}_{(2)}$, of $\mathbb{Z}_{(2)}$ noting that $0 \in 2\mathbb{Z}_{(2)}$. Since $4\mathbb{Z}_{(2)}$ is a maximal ideal of $2\mathbb{Z}_{(2)}$, we examine the partition of $\mathbb{Z}_{(2)}$,

$$\{4\mathbb{Z}_{(2)},\, 4\mathbb{Z}_{(2)} + 1,\, 4\mathbb{Z}_{(2)} + 2,\, 4\mathbb{Z}_{(2)} + 3\}.$$

In order to satisfy conditions 1 and 2 of Lemma 3.6.1, we must set $K(2) = 2$ since $2\mathbb{Z}_{(2)}$ and $\varphi(2\mathbb{Z}_{(2)}) = 2\mathbb{Z}_{(2)} + 1$ each contain 2 elements of the above partition. Now, we note that

$$\begin{aligned}
\varphi(4\mathbb{Z}_{(2)}) &= 4\mathbb{Z}_{(2)} + 1, \\
\varphi^2(4\mathbb{Z}_{(2)}) &= 4\mathbb{Z}_{(2)} + 2, \\
\varphi^3(4\mathbb{Z}_{(2)}) &= 4\mathbb{Z}_{(2)} + 3, \\
\varphi^4(4\mathbb{Z}_{(2)}) &= 4\mathbb{Z}_{(2)}.
\end{aligned}$$

This observation, along with the fact that $0 \in 4\mathbb{Z}_{(2)}$, $2\mathbb{Z}_{(2)} = (4\mathbb{Z}_{(2)}) \cup (4\mathbb{Z}_{(2)} + 2)$ and $\varphi(2\mathbb{Z}_{(2)}) = (4\mathbb{Z}_{(2)}+1) \cup (4\mathbb{Z}_{(2)}+3)$ necessitates the choices of $J(2,1) = J(2,2) = 2$ so that $Z(2,1,1) = 4\mathbb{Z}_{(2)} + 1$, $Z(2,1,2) = 4\mathbb{Z}_{(2)} + 2$, $Z(2,2,1) = 4\mathbb{Z}_{(2)} + 3$, and $Z(2,2,2) = 4\mathbb{Z}_{(2)}$. Once can easily verify that we have constructed a Kakutani-Rohlin partition of $\mathbb{Z}_{(2)}$,

$$\mathcal{P}_2 = \{Z(2,1,1),\ Z(2,1,2),\ Z(2,2,1),\ Z(2,2,2)\},$$

Such that $\mathcal{P}_1 \leq \mathcal{P}_2$. Proceeding in this manner, we see that we may set $K(n) = 2$ since $2^n\mathbb{Z}_{(2)}$ will always have as a maximal ideal $2^{n+1}\mathbb{Z}_{(2)}$ for each $n \in \mathbb{N}$. Furthermore, we may always set $J(n,1) = J(n,2) = \cdots = J(n, 2^{n-1}) = 2^{n-1}$ for each $n \in \mathbb{N}$, in which case, it is a quick verification that the choice of

$$\begin{aligned}
Z(n,1,j) &= 2^n\mathbb{Z}_{(2)} + j, && \text{for } 1 \leq j \leq 2^{n-1}, \\
Z(n,2,j) &= 2^n\mathbb{Z}_{(2)} + j + 2^{n-1}, && \text{for } 1 \leq j \leq 2^{n-1} - 1, \\
Z(n,2,2^{n-1}) &= 2^n\mathbb{Z}_{(2)},
\end{aligned}$$

For each $n \in \mathbb{N}$ yields a sequence of Kakutani-Rohlin Partitions satisfying the conditions of Theorem 3.6.1. From here, we may now define the sets of vertices, V_n, and the sets of edges, E_n, which will describe the associated properly ordered Bratteli diagram. Thus, we have that

$$\begin{aligned} V_0 &= \{(0,1)\}, \\ V_1 &= \{(1,1),\ (1,2)\}, \\ V_2 &= \{(2,1),\ (2,2)\}, \\ &\vdots \end{aligned}$$

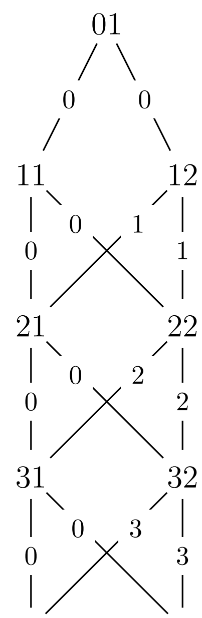
Where, as shown above, $V_n = \{(n,1),\ \dots\ ,(n,K(n)\}$ and

$$\begin{aligned} E_1 &= \{(1,1,1,0),\ (1,1,2,0)\}, \\ E_2 &= \{(2,1,1,0),\ (2,1,2,0),\ (2,2,1,1),\ (2,2,2,1)\}, \\ E_3 &= \{(3,1,1,0),\ (3,1,2,0),\ (3,2,1,2),\ (3,2,2,2)\}, \\ &\vdots \end{aligned}$$

Where, as shown above, $E_n = \{(n,k,k',j')\ :\ Z(n,k',j'+j) \subseteq Z(n-1,k,j)\ \forall\ 1 \leq j \leq J(n-1,k)\}$. Using the source and range maps

$$\begin{aligned} s(n,k,k',j') &= (n-1,k), \\ r(n,k,k',j') &= (n,k'), \end{aligned}$$

With the ordering described above produces the stationary properly ordered Bratteli diagram



From here, we would like to give this construction for the more generalized adding machine $(\mathbb{Z}_{(\mathbf{p})}, \varphi)$ as described in section 2.2 for some specified $\mathbf{p}$. We will proceed along in a similar manner. By convention, we have that $K(0) = 1$ and $Z(0,1,1) = \mathbb{Z}_{(\mathbf{p})}$. Moving on to the next iteration, we choose $Z = \mathbb{Z}_{(\mathbf{p})}$. Since $p_0\mathbb{Z}_{(\mathbf{p})}$ is a maximal ideal of $\mathbb{Z}_{(\mathbf{p})}$, we examine the partition

$$\{p_0\mathbb{Z}_{(\mathbf{p})},\ p_0\mathbb{Z}_{(\mathbf{p})} + 1,\ p_0\mathbb{Z}_{(\mathbf{p})} + 2,\ \ldots\ , p_0\mathbb{Z}_{(\mathbf{p})} + p_0 - 1\}.$$

In order to satisfy the conditions of Lemma 3.6.1 and Theorem 3.6.1, we set $K(1) = p_0$ and $J(1,1) = J(1,2) = \cdots = J(1,p_0) = 1$; and we choose

$$Z(1,1,1) = p_0\mathbb{Z}_{(\mathbf{p})} + 1, \quad Z(1,2,1) = p_0\mathbb{Z}_{(\mathbf{p})} + 2, \quad Z(1,p_0,1) = p_0\mathbb{Z}_{(\mathbf{p})}.$$

It is a quick verification that this is a Kakutani-Rohlin partition of $\mathbb{Z}_{(\mathbf{p})}$,

$$\mathcal{P}_1 = \{Z(1,1,1),\ Z(1,2,1),\ \dots,Z(1,p_0,1)\}.$$

For the next iteration, we set $Z = p_0\mathbb{Z}_{(\mathbf{p})}$, noting that $p_0\mathbb{Z}_{(\mathbf{p})}$ is a maximal ideal of $\mathbb{Z}_{(\mathbf{p})}$ and that $0 \in p_0\mathbb{Z}_{(\mathbf{p})}$. Since $p_1\mathbb{Z}_{(\mathbf{p})}$ is a maximal ideal of $p_0\mathbb{Z}_{(\mathbf{p})}$, we examine the partition of $\mathbb{Z}_{(\mathbf{p})}$,

$$\{p_1\mathbb{Z}_{(\mathbf{p})},\ p_1\mathbb{Z}_{(\mathbf{p})}+1,\ p_1\mathbb{Z}_{(\mathbf{p})}+2,\ \dots,p_1\mathbb{Z}_{(\mathbf{p})}+p_1-1\}.$$

For ease of notation, we will denote $q_n = p_n/p_{n-1}$. Thus, similar to the dyadic example, we may set $K(2) = q_1$ and $J(2,1) = J(2,2) = \cdots = J(2,p_0) = p_0$; and we choose

$$\begin{array}{lcl}
Z(2,1,1) = p_1\mathbb{Z}_{(\mathbf{p})}+1, & \dots & ,\ Z(2,1,p_0) = p_1\mathbb{Z}_{(\mathbf{p})}+p_0, \\
Z(2,2,1) = p_1\mathbb{Z}_{(\mathbf{p})}+p_0+1, & \dots & ,\ Z(2,2,p_0) = p_1\mathbb{Z}_{(\mathbf{p})}+2p_0, \\
\vdots & \vdots & \vdots \\
Z(2,q_1,1) = p_1\mathbb{Z}_{(\mathbf{p})}+p_1-p_0+1, & \dots & ,\ Z(2,q_1,p_0) = p_1\mathbb{Z}_{(\mathbf{p})}.
\end{array}$$

It is again a quick verification that this is a Kakutani-Rohlin partition of $\mathbb{Z}_{(\mathbf{p})}$,

$$\mathcal{P}_2 = \{Z(2,1,1),\ \dots,Z(2,1,p_0),\ \dots,Z(2,q_1,1),\ \dots,Z(2,q_1,p_0)\},$$

Such that $\mathcal{P}_1 \leq \mathcal{P}_2$. Proceeding in this manner, we see that we may set $K(n) = q_{n-1}$ since $p_{n-2}\mathbb{Z}_{(\mathbf{p})}$ will always have as a maximal ideal $p_{n-1}\mathbb{Z}_{(\mathbf{p})}$ for each natural number

$n > 1$. Furthermore, for such n, we may always set $J(n,1) = J(n,2) = \cdots = J(n,p_{n-2}) = p_{n-2}$. It follows that the choice of

$$
\begin{aligned}
&Z(n,j,k) = p_{n-1}\mathbb{Z}_{(\mathbf{p})} + k + p_{n-2}(j-1), &&\text{for } 1 \le j \le q_{n-1}-1 \text{ and } 1 \le k \le p_{n-2},\\
&Z\left(n,q_{n-1},k\right) = p_{n-1}\mathbb{Z}_{(\mathbf{p})} + k + p_{n-2}, &&\text{for } 1 \le k \le p_{n-2}-1,\\
&Z\left(n,q_{n-1},p_{n-2}\right) = p_{n-1}\mathbb{Z}_{(\mathbf{p})},
\end{aligned}
$$

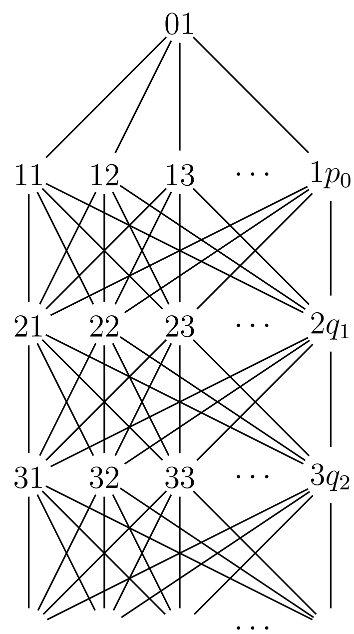
For each natural number $n > 1$ yields a sequence of Kakutani-Rohlin partitions satisfying the conditions of Theorem 3.6.1. Defining the sets of vertices and edges as above we have the sets of vertices,

$$
\begin{aligned}
V_0 &= \{(0,1)\},\\
V_1 &= \{(1,1),\ (1,2),\ \dots,(1,p_0)\},\\
V_2 &= \{(2,1),\ (2,2),\ \dots,(2,q_1)\},\\
V_3 &= \{(3,1),\ (3,2),\ \dots,(3,q_2)\},\\
&\ \vdots
\end{aligned}
$$

We also have the sets of edges,

$$
\begin{aligned}
E_1 &= \{(1,1,1,0),\ (1,1,2,0),\ \dots,(1,1,p_0,0)\},\\
E_2 &= \{(2,1,1,0),\ \dots,(2,1,q_1,0),(2,2,1,1),\ \dots,(2,2,q_1,1),\ \dots\\
&\qquad\qquad \dots,(2,p_0,1,p_0-1),\ \dots,(2,p_0,q_1,p_0-1)\},\\
E_3 &= \{(3,1,1,0),\ \dots,(3,1,q_2,0),(3,2,q_2,q_2),\ \dots,(3,2,q_2,q_2),\ \dots\\
&\qquad\qquad \dots,(3,q_1,1,q_2(q_1-1)),\ \dots,(3,q_1,q_2,q_2(q_1-1))\},\\
&\ \vdots
\end{aligned}
$$

With the ordering described above produces the stationary properly ordered Bratteli diagram



Note that the ordering has been left off the figure since it is quite busy as it is.

3.6.2 The Irrational Rotation Examples

We would like to illustrate the construction of the Kakutani-Rohlin partitions with one more class of examples before we close this chapter. Specifically, we would like to look at irrational rotations on the Cantorized circle, $(\mathbb{T}_{\mathcal{C}}, \varphi_\alpha)$, for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. In particular, we will work with the rotation given by $\alpha = \frac{1+\sqrt{5}}{2}$. Note that this is the irrational number ϕ with continued fraction decomposition $[1, 1, 1, \ ...]$. We will choose the point around which we construct our Kakutani-Rohlin partitions to be α^r_{-1}. As always, we set $K(0) = 1$ and $Z(0, 1, 1) = \mathbb{T}_{\mathcal{C}}$. Moving on to the next iteration, we choose $Z = \mathbb{T}_{\mathcal{C}}$, and we are in a position to choose a partition of $\mathbb{T}_{\mathcal{C}}$ with which we will work. In order to do this, we first calculate the upper and lower convergents of α as described in section 2.1,

$$p_{-1} = 1, \quad p_0 = 1, \quad p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \quad p_4 = 8, \quad \cdots ,$$
$$q_{-1} = 0, \quad q_0 = 1, \quad q_1 = 1, \quad q_2 = 2, \quad q_3 = 3, \quad q_4 = 5, \quad \cdots .$$

For each $n \in \mathbb{N} \cup \{0\}$, define $Q_n = \{\varphi_\alpha^m(0) \ : \ -p_{2(n-1)} \leq m < q_{2(n-1)}\}$. It follows that we may derive a partition, $\mathcal{P}_n$, from Q_n of $\mathbb{T}_\mathcal{C}$ consisting of $p_{2(n-1)} + q_{2(n-1)}$ sets of the form $[\alpha^r_{n_1}, \alpha^l_{n_2}]$ such that $n_1, n_2 \in \mathbb{Q}_n$. Consequently, it follows that

$$Q_1 = \{\varphi_\alpha^n(0) \ : \ -p_0 \leq n \leq q_0\} = \{\varphi_\alpha^n(0) \ : \ -1 \leq n \leq 1\},$$

Which yields the partition,

$$\mathcal{P}_1 = \{[\alpha^r_0, \alpha^1_{-1}], [\alpha^r_{-1}, \alpha^l_0]\}.$$

Setting $K(1) = 2$, $J(1,1) = J(1,2) = 1$ and

$$Z(1,1,1) = [\alpha^r_{-1}, \alpha^l_0], \quad Z(1,2,1) = [\alpha^r_0, \alpha^l_{-1}],$$

It is a quick check that $\mathcal{P}_1$ satisfies the conditions of a Kakutani-Rohlin partition. Continuing on in this manner, we set $Z = [\alpha^r_{-1}, \alpha^l_{-2}]$ and note that

$$Q_2 = \{\varphi_\alpha^n(0) \ : \ -p_2 \leq n \leq q_2\} = \{\varphi_\alpha^n(0) \ : \ -3 \leq n \leq 2\},$$

From which we calculate the order which defines the partition $\mathcal{P}_2$,

$$\varphi_\alpha^0(0) < \varphi_\alpha^{-3}(0) < \varphi_\alpha^{-1}(0) < \varphi_\alpha^1(0) < \varphi_\alpha^{-2}(0).$$

Again, we set $K(2) = 2$. This time, however, we set $J(2,1) = 3$, $J(2,2) = 2$, and

$$Z(2,1,1) = [\alpha_{-3}^r, \alpha_{-1}^l],\ Z(2,1,2) = [\alpha_{-2}^r, \alpha_0^l],\ \ Z(2,1,3) = [\alpha_{-1}^r, \alpha_1^l],$$
$$Z(2,2,1) = [\alpha_0^r, \alpha_{-3}^l],\quad Z(2,2,2) = [\alpha_{-1}^r, \alpha_{-2}^l],$$

From which we can verify that $\mathcal{P}_2$ is a Kakutani-Rohlin partition satisfying the condition that $\mathcal{P}_1 \leq \mathcal{P}_2$. Generalizing this approach, we note from above that

$$|Q_n| = |\mathcal{P}_n| = p_{2(n-1)} + q_{2(n-1)}.$$

Thus, with the choice of $Z = [\alpha_{-1}^r, \alpha_{-(p_{2(n-1)}-q_{2(n-1)}+1)}^l]$ for each $n \in \mathbb{N}$, we have that $K(n) = 2$, $J(n,1) = p_{2(n-1)}$ and $J(n,2) = q_{2(n-1)}$. Furthermore, given an interval $I_m = [\alpha_{N_m}^r, \alpha_{N_{m+1}}^l]$ in the partition $\mathcal{P}_n$, the consecutive interval in the partition will be $I_{m+1} = [\alpha_{N_{m+1}}^r, \alpha_{N_{m+2}}^l]$, where the N_m for $m > 1$ are given by

$$N_m = \begin{cases} N_{m-1} - p_{2(n-1)}, & \text{if } \ N_{m-1} \geq 0, \\ N_{m-1} - p_{2(n-1)} + p_{2n-1} & \text{otherwise.} \end{cases}$$

Note that it will always be the case that $N_1 = 0$. Finally, for each $n \in \mathbb{N}$, we have that $Z(n,2,1) \cup Z(n,1,1) = [\alpha_0^r, \alpha_{-(p_{2(n-1)}-q_{2(n-1)})}^l]$, in which case the intervals in $\mathcal{P}_n$

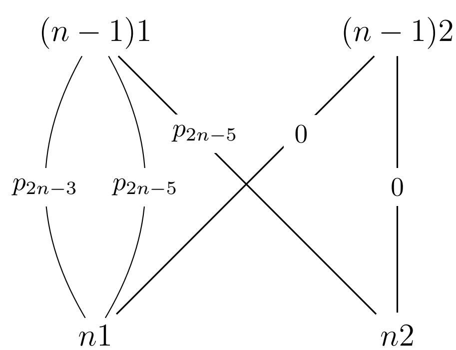
may be identified by iterating φ_α. This yields the sets of vertices,

$$V_0 = \{(0,1)\},$$
$$V_1 = \{(1,1),(1,2)\},$$
$$V_2 = \{(2,1),(2,2)\},$$
$$\vdots$$

As well as the sets of edges,

$$E_1 = \{(1,1,1,0),(1,1,2,0)\},$$
$$E_2 = \{(2,1,1,2),(2,1,1,1),(2,1,2,1),(2,2,1,0),(2,2,2,0)\},$$
$$\vdots$$
$$E_n = \{(n,1,1,p_{2n-3}),(n,1,1,p_{2n-5}),(n,1,2,p_{2n-5}),(n,2,1,0),(n,2,2,0)\},$$
$$\vdots$$

This produces the stationary properly ordered Bratteli diagram, noting that in this presentation, $n > 1$,



We now turn our attention to one last example. We would like to look at this construction with $\alpha = 1+\sqrt{2} = [2,2,2, \ ...]$. We will again choose the point around which we will construct our Kakutani-Rohlin partitions to be α^r_{-1}, and we set $K(0) = 1$ and $Z(0,1,1) = \mathbb{T}_{\mathcal{C}}$. Moving on to the calculation for $n = 1$, we calculate the upper and lower convergents of α,

$$p_{-1} = 1, \quad p_0 = 2, \quad p_1 = 5, \quad p_2 = 12, \quad p_3 = 29, \quad p_4 = 70, \quad \cdots,$$
$$q_{-1} = 0, \quad q_0 = 1, \quad q_1 = 2, \quad q_2 = 5, \quad q_3 = 12, \quad q_4 = 29, \quad \cdots.$$

Defining Q_n in the same manner as above, we have that

$$Q_1 = \{\varphi^n_\alpha(0) \ : \ -p_0 \le n \le q_0\} = \{\varphi^n_\alpha(0) \ : \ -2 \le n \le 1\},$$

Which yields the order defining the partition $\mathcal{P}_1$,

$$\varphi^0_\alpha(0) < \varphi^{-2}_\alpha(0) < \varphi^{-1}_\alpha(0).$$

From this, we choose our clopen subset $Z = [\alpha^r_{-1}, \alpha^l_{-2}]$, and it follows that $K(1) = 2$, $J(1,1) = 2, \ J(1,2) = 1$ and

$$Z(1,1,1) = [\alpha^r_{-2}, \alpha^l_{-1}], \quad Z(1,1,2) = [\alpha^r_{-1}, \alpha^l_0], \quad Z(1,2,1) = [\alpha^r_0, \alpha^l_{-2}].$$

It is a quick check that $\mathcal{P}_1$ satisfies the conditions of a Kakutani-Rohlin partition. Continuing on in this manner, we note that

$$Q_2 = \{\varphi_\alpha^n(0) \ : \ -p_2 \le n \le q_2\} = \{\varphi_\alpha^n(0) \ : \ -12 \le n \le 5\},$$

From which we calculate the order which defines the partition $\mathcal{P}_2$,

$$\begin{aligned} &\varphi_\alpha^0(0) < \varphi_\alpha^{-12}(0) < \varphi_\alpha^{-7}(0) < \varphi_\alpha^{-2}(0) < \varphi_\alpha^3(0) < \varphi_\alpha^{-9}(0) < \cdots \\ &\qquad \cdots < \varphi_\alpha^{-4}(0) < \varphi_\alpha^1(0) < \varphi_\alpha^{-11}(0) < \varphi_\alpha^{-6}(0) < \varphi_\alpha^{-1}(0) < \varphi_\alpha^4(0) < \cdots \\ &\qquad\qquad \cdots < \varphi_\alpha^{-8}(0) < \varphi_\alpha^{-3}(0) < \varphi_\alpha^2(0) < \varphi_\alpha^{-10}(0) < \varphi_\alpha^{-5}(0). \end{aligned}$$

Motivating the choice $Z = [\alpha_{-1}^r, \alpha_{-8}^l]$. We then set $K(2) = 2$, $J(1) = 12$ and $J(2) = 5$. Finally, we set $Z(2,1,1) = [\alpha_{-12}^r, \alpha_{-7}^l]$ and $Z(2,2,1) = [\alpha_0^r, \alpha_{-12}^l]$, from which the rest of the intervals in $\mathcal{P}_2$ may be identified by iterating φ_α. It follows that $\mathcal{P}_2$ is a Kakutani-Rohlin partition satisfying $\mathcal{P}_1 \le \mathcal{P}_2$.

As can be gathered, the general case in this example follows the general case in the example above. Consequently, we have the sets of vertices,

$$\begin{aligned} V_0 &= \{(0,1)\}, \\ V_1 &= \{(1,1),(1,2)\}, \\ V_2 &= \{(2,1),(2,2)\}, \\ &\vdots \end{aligned}$$

As well as the sets of edges,

$$E_1 = \{(1,1,1,0),(1,1,2,0)\},$$

$$E_2 = \{(2,1,1,10),(2,1,1,8),(22,1,1,6),(2,1,1,3),(2,1,1,1),\ \ldots$$

$$\ldots\ ,(2,1,2,3),(2,1,2,1),(2,2,1,5),(2,2,1,0),(2,2,2,0)\},$$

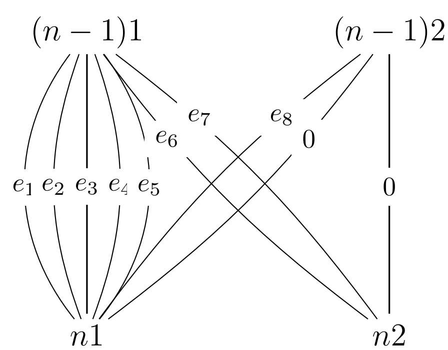
$$\vdots$$

$$E_n = \{(n,1,1,e_1),(n,1,1,e_2),(n,1,1,e_3),(n,1,1,e_4),(n,1,1,e_5),\ \ldots$$

$$\ldots\ ,(n,1,2,e_6),(n,1,2,e_7),(n,1,2,e_8),(n,2,1,0),(n,2,2,0)\},$$

$$\vdots$$

Where $e_1 = p_{2n-2} - p_{2n-3}$, $e_2 = p_{2n-3} + p_{2n-4} + p_{2n-5}$, $e_3 = p_{2n-3} + p_{2n-5}$, $e_4 = e_6 = p_{2n-4} + p_{2n-5}$, $e_5 = e_7 = p_{2n-5}$ and $e_8 = p_{2n-3}$. This produces the stationary properly ordered Bratteli diagram, noting that in this presentation, $n > 1$,



Chapter 4

Symbolic Dynamical Systems

4.1 Introduction to Symbolic Dynamics

At this point we would like to introduce a particular subset of topological dynamical systems, namely symbolic dynamical systems. The primary references for the work in this chapter are [4] and [12]. Our goal here is to eventually define what is meant by a *substitution dynamical system*, which is an instance of a symbolic dynamical system. The area of symbolic dynamics seeks to study discrete dynamical systems in the form of shift spaces. In order to define the concept of a shift space, we begin by considering a finite *alphabet* $\mathcal{A} = \{a_1, a_2, \ ... \ , a_n\}$, which is simply a finite set of symbols. We will call the elements of $\mathcal{A}$ letters. From hereon forward, the finiteness of an alphabet will be assumed unless otherwise stated. We define a *word* on the alphabet $\mathcal{A}$ to be a finite sequence of letters in $\mathcal{A}$. We denote $\mathcal{A}^+$ to be the set of nonempty words on $\mathcal{A}$, and $\mathcal{A}^*$ to be $\mathcal{A}^+ \cup \emptyset$ where $\emptyset$ is the empty word.

For $\omega = \omega_1\omega_2\cdots\omega_n \in \mathcal{A}^+$, $|\omega| = n$ is the *length* of ω. Furthermore, given $a \in \mathcal{A}$, $|\omega|_a$ is the number of occurrences of the letter a in the word ω. Note that $|\emptyset| = 0$ and that $|\emptyset|_a = 0$ for all $a \in \mathcal{A}$. A word v is said to be a *factor* of a word ω if $\omega = u_1vu_2$ for some $u_1, u_2 \in \mathcal{A}^*$. In this case, we write $v \prec \omega$

We denote $\mathcal{A}^{\mathbb{N}}$ to be the set of infinite sequences of symbols from $\mathcal{A}$ and $\mathcal{A}^{\mathbb{Z}}$ to be the set of bi-infinite sequences of symbols from $\mathcal{A}$. Elements of $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ are called *sequences* over $\mathcal{A}$. From hereon forward, we will work primarily with bi-infinite sequences. Consequently, a sequence will be assumed bi-infinite unless otherwise specified. The term *factor* is used for sequences in a manner very similar to that used for words. That is, if x is a sequence over $\mathcal{A}$ and $\omega \in \mathcal{A}^*$, then $\omega \prec x$ if $x = y_1\omega y_2$ for some infinite sequences y_1 and y_2 over $\mathcal{A}$ which are not bi-infinite. The language, $\mathcal{L}(x)$ is the set of words which are factors of x.

Note that we may view a sequence x over $\mathcal{A}$ in the form $x = (\ \ldots\ x_{-2}x_{-1}x_0x_1x_2\ \ldots\ )$, where each $x_j \in \mathcal{A}$. If we endow $\mathcal{A}^{\mathbb{Z}}$ with the metric $d(x,y) = 1/2^N$ where $N = \max\{m : x_j = y_j \text{ for } |j| < m\}$, we can see that we have a basis of clopen cylinder sets, $C_{(n,t)} = [c_0,\ \ldots\ , c_n] = \{x : x_t = c_0,\ \ldots\ , x_{t+n} = c_n\}$. An argument very similar to that used for the **p**-adic Integers shows that this space is a Cantor space. With this in mind, we can begin to define the notion of a *substitution* on an alphabet $\mathcal{A}$. We will begin by looking at *subshifts* on $\mathcal{A}$. In particular, we define the shift T on $\mathcal{A}^{\mathbb{Z}}$ as

$$(Tx)_n = x_{n+1} \text{ for every } x \in \mathcal{A}^{\mathbb{Z}} \text{ and every } n \in \mathbb{Z}.$$

It is immediate that T is a homeomorphism of $\mathcal{A}^{\mathbb{Z}}$. With this in mind, we may define what we mean by a subshift on $\mathcal{A}$. Formally,

Definition 4.1.1. A *subshift*, (X, T) on the alphabet $\mathcal{A}$ is a closed, T-invariant subset X of $\mathcal{A}^{\mathbb{Z}}$ along with a restriction of T to X (also denoted as T). Furthermore, given two words $u, v \in \mathcal{A}^*$, we denote

$$[u] = \{x \in X : \ x_{[0,|u|)} = u\} \text{ and } [u.v] = \{x \in X : \ x_{[-|u|,|v|)} = uv\},$$

Note that these are simply cylinder sets, and that they again form a basis of clopen subsets of X.

4.2 Substitution Dynamical Systems

Using the ideas of subshifts, we are in a position to approach the definition of a substitution dynamical system. We begin with the idea of a substitution. In particular, a *substitution* on the alphabet $\mathcal{A}$ is a map $\sigma : \mathcal{A} \to \mathcal{A}^+$. Note that for each $n \in \mathbb{N}$, it follows, using the extension to words by concatenation, that $\sigma^n : \mathcal{A} \to \mathcal{A}^+$ is also a substitution.

In order to complete this picture, however, we must first complete our definition of what is meant by a substitution dynamical system. Now, in a natural manner, we may speak of the *language of the substitution*, σ, which we will denote $\mathcal{L}(\sigma)$. That is, we define $\mathcal{L}(\sigma)$ to be the set of words in $\mathcal{A}^*$ which are factors of $\sigma^n(a)$ for some $a \in \mathcal{A}$ and $n \in \mathbb{N}$.

In this manner we define the subshift X_σ associated to $\mathcal{L}(\sigma)$ to be the set of $x \in \mathcal{A}^{\mathbb{Z}}$ whose every finite factor is a word in $\mathcal{L}(\sigma)$. It follows that X_σ is closed in $\mathcal{A}^{\mathbb{Z}}$ and invariant under T.

Definition 4.2.1. Given an alphabet, $\mathcal{A}$, and a substitution, σ, on $\mathcal{A}$, we define (X_σ, T_σ), where T_σ is the restriction of T to X_σ, to be the *substitution dynamical system* on $\mathcal{A}$ associated to σ.

In his lecture notes, [15], M. Queffelec showed that every substitution dynamical system is minimal. Now, in order that we can study orbit equivalence through the results which we will introduce, we would like to restrict our attention to certain types of substitutions. We begin by defining a substitution, σ, to be periodic if X_σ is finite. We would like to work primarily with dynamical systems which are not finite, so we will henceforth assume that σ is aperiodic. There are various algorithms for deciding periodicity of a substitution, but this is outside of the scope of this work. Consequently, we will proceed by further defining the types of substitution dynamical systems with which we would like to work.

Definition 4.2.2. We say that a substitution is *primitive* if there exists some $n \in \mathbb{N}$ such that, for all $a, b \in \mathcal{A}$, it is the case that $b \prec \sigma^n(a)$, and if there exists some $c \in \mathcal{A}$ such that $\lim_{n\to\infty} |\sigma^n(c)| = \infty$.

The importance of this distinction will become clear shortly through the notion of fixed points, the properties and definition of which we will now present. Note that for every $n \in \mathbb{Z}$, σ^n defines the same language, and consequently the same dynamical system, as σ. Therefore we may assume that there exist some $r, l \in \mathcal{A}$ such that, for some $n, m \in \mathbb{Z}$,

1. r is the last letter of $\sigma^n(r)$,
2. l is the first letter of $\sigma^m(l)$,
3. $rl \in \mathcal{L}(\sigma)$.

In order to see this, consider the second condition above. That is, we are assuming that there exists some $l \in \mathcal{A}$ such that l is the first letter of $\sigma^m(l)$ for some $m \in \mathbb{N}$. Now, let $\mathcal{A} = \{a_1, a_2, \ \dots \ , a_j\}$ for some $j \in \mathbb{N}$. Suppose that there is no $a_k \in \mathcal{A}$ such that a_k is the first letter of $\sigma(a_k)$. Since σ acts as a permutation on the first letter of $\sigma(a_k)$ for each k, it follows that repeated iterations of σ will 'shuffle' around the first letters of each word until we obtain the desired equality. In fact, the only class of substitutions which does not permute the first letter of each word is the class of substitutions in which each substitution acts as the identity on the first letter of each word; and this class of substitutions satisfies the second condition trivially on the first iteration! In a similar manner, one can see that the first condition will always be satisfied.

Thus, given first and second conditions, it takes a little more work to see that the third condition will always hold. Suppose that l only occurs as the first letter of any word in $\mathcal{L}(\sigma)$. Then it follows that $\sigma(l) = l$, in which case we may set $r = l$; thus, this condition is trivially satisfied. Otherwise, we may choose a word $u \in \mathcal{L}(\sigma)$ such that $l \prec u$ but l is not the first letter. Denote, $a_l \prec u$ to be the letter that directly precedes l in u. If $a_l = r$, we done. Otherwise, note that σ will shuffle a_l through $\mathcal{A}$ until $a_l = r$ since l will always 'retain' its position in u due to the second condition. Consequently, it follows that condition 3 is again satisfied. With this in mind we may proceed to the actual definition of a fixed point.

Definition 4.2.3. We say that $\omega \in \mathcal{A}^{\mathbb{Z}}$ is a *fixed point* of σ if $\omega_{-1} = r$, $\omega_0 = l$, and $\sigma(\omega) = \omega$ if r and l satisfy conditions 1 and 2 above. If r and l also satisfy 3 above, then we say that ω is an *admissible fixed point* of σ.

It can be shown that if ω is an admissible fixed point of σ, then X_σ is the closure of the orbit of ω under T. Specifically, we have that if ω is a fixed point of a primitive substitution, σ, then the following conditions are equivalent,

1. ω is admissible,
2. $\omega \in X_\sigma$,
3. X_σ is the subshift spanned by ω.

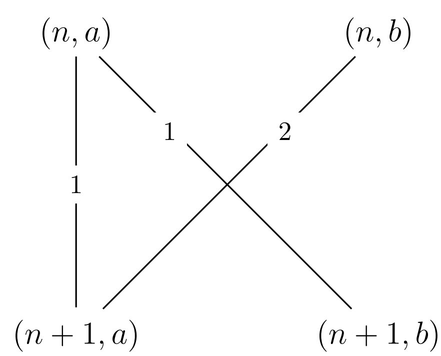
Note from above that the statement that ω is a fixed point of σ should be taken to mean that ω is a fixed point of σ^n for some n. With this convention, every substitution dynamical system has at least one admissible fixed point. It is natural then to differentiate the class of substitution dynamical systems which have exactly one admissible fixed point.

Definition 4.2.4. A substitution σ on an alphabet $\mathcal{A}$ is *proper* if there exist some $n \in \mathbb{N}$ and $r, l \in \mathcal{A}$ such that

1. For every $a \in \mathcal{A}$, r is the last letter of $\sigma^n(a)$,
2. For every $a \in \mathcal{A}$, l is the first letter of $\sigma^n(a)$.

In other words, a proper substitution admits only one admissible fixed point. It should noted that the substitution read from a stationary, properly ordered Bratteli diagram is necessarily primitive and proper. That being stated, we ought to discuss the manner in which we may associate an ordered Bratteli diagram and a given substitution over an alphabet $\mathcal{A}$.

Now, using the notation for Bratteli diagrams as established in Chapter 1, we label the vertices, V_n for $n > 0$ using the alphabet $\mathcal{A}$. In other words, we have that $V_n = \{V(n, a_j) : a_j \in \mathcal{A}\}$ for each $n \in \mathbb{N}$. In order to determine the set of edges E_{n+1} connecting V_n and V_{n+1}, we take a vertex, $V(n+1, a_j)$ and we connect this vertex with each $V(n, a_k)$ such that $a_k \prec \sigma(a_j)$. Note that the number of edges between $V(n, a_k)$ and $V(n+1, a_j)$ will be exactly $|\sigma(a_j)|_{a_k}$. Finally, we get the ordering on the edges from the natural left-to-right ordering given by the substitution. As an example, it follows that the substitution σ on $\mathcal{A} = \{a, b\}$ given by $\sigma(a) = ab$ and $\sigma(b) = a$ is given by



We now find ourselves at the point at which we can present the result of [4]. Note that odometer is a term used equivalently for an adding machine, and that an odometer with a stationary base is simply an adding machine for which there exists an $n \in \mathbb{N}$ such that $p_j/p_{j+1} = p_{j+1}/p_{j+2}$ for all $j \geq n$.

Theorem 4.2.1. Let σ be a proper primitive substitution on the alphabet $\mathcal{A}$. Let

$\mathcal{B} = (V, E)$ be the stationary ordered Bratteli diagram associated to σ. Then

1. If σ is aperiodic, the substitution dynamical system (X_σ, T_σ) associated to σ is isomorphic to the Bratteli-Vershik system $(X_\mathcal{B}, V_\mathcal{B})$.

2. If σ is periodic, the Bratteli-Vershik system $(X_\mathcal{B}, V_\mathcal{B})$ is isomorphic to an odometer with a stationary base.

In order to appreciate the power of this theorem we need one more result also given in the work [4]. Namely, we cite the result that every substitution dynamical system is isomorphic to the system associated to some proper substitution. Note that this is trivial in the case of a periodic substitution since all periodic substitutions are proper to begin with. In order to examine the case of an aperiodic substitution, we need to build up a bit more structure. In so doing, we will present the method for the actual construction of such a proper substitution. For what follows, we will assume (X, T) to be a minimal subshift on the alphabet $\mathcal{A}$. Furthermore, for $x \in X$, note that $\mathcal{L}(x)$ is the language of x as defined in the usual manner.

Definition 4.2.5. We say that $x \in X$ is *uniformly recurrent* if, for all $u \in \mathcal{L}(x)$, there is some $n \geq 1$ such that for all $v \in \mathcal{L}(x)$, $|v| \geq n$ implies that $u \prec v$.

Now, suppose that $x \in X$ is uniformly recurrent with $u, v \in \mathcal{L}(x)$ such that u is a suffix of $x_{(-\infty,-1]}$ and v is a prefix of $x_{[0,\infty)}$ with at least one of u and v not equal to the empty word.

Definition 4.2.6. We say that $n \in \mathbb{Z}$ is an *occurrence* of $u.v$ in x if $x_{[n-|u|,n+|v|)} = uv$. Furthermore, we say that a word $\omega \in \mathcal{L}(x)$ is a *return* word to $u.v$ in x if there exist two consecutive occurrences j, k of $u.v$ in x such that $\omega = x_{[j,k)}$.

Since x is uniformly recurrent, one can see that $|\omega| < \infty$, and that the set of return words to $u.v$, which we will denote $\mathcal{R}_{u.v}$, is finite. Furthermore, it is immediate that a word $\omega \in \mathcal{A}^+$ is a return word if and only if

1. $u\omega v \in \mathcal{L}(x)$,

2. v is a prefix of ωv and u is a suffix of $u\omega$,

3. $u\omega v$ contains exactly two occurrences of uv.

Now, define $R_{u.v} = \{1, \ \dots \ , \mathrm{Card}(\mathcal{R}_{u.v})\}$. Then we may define $\phi_{u.v} : R_{u.v} \to \mathcal{R}_{u.v}$ in the following manner. Taking the order on $\mathcal{R}_{u.v}$ given by the rank of the first occurrence in $x_{[0,\infty)}$, then we define $\phi_{u.v}(k)$ to be the k^{th} ordered element of $\mathcal{R}_{u.v}$. If we consider $R_{u.v}$ to be an alphabet, then it follows that $\phi_{u.v}$ is a map from $R_{u.v}$ to $\mathcal{A}^+$.

Definition 4.2.7. The *u.v-derivative* of x is the unique sequence, denoted $\mathcal{D}_{u.v}(x)$, on the alphabet $R_{u.v}$ such that

$$\phi_{u.v}(\mathcal{D}_{u.v}(x)) = x.$$

Now that we have the requisite structure, we may consider a primitive, aperiodic substitution σ on the alphabet $\mathcal{A}$. Let x be a fixed point of $\mathcal{A}$ with $r = x_{-1}$ and $l = x_0$. Thus, r is the last letter of $\sigma(r)$ and l is the first letter of $\sigma(l)$. We will from this point forward consider return words to $r.l$. Consequently, $\mathcal{R}, \ R, \ \phi,$ and $\mathcal{D}(x)$ will refer to $\mathcal{R}_{r.l}, \ R_{r.l}, \ \phi_{r.l},$ and $\mathcal{D}(x)_{r.l}$, respectively.

If we let $j \in R$ and $\omega = \omega_1 \cdots \omega_k = \phi(j) \in \mathcal{R}$, then from above, we have that $r\omega l \in \mathcal{L}(\sigma)$, $\omega_1 = l$, and $\omega_k = r$. Since $\sigma(x) = x$, it follows that $\sigma(r\omega l) = \sigma(r)\sigma(\omega)\sigma(l)$ is a word in $\mathcal{L}(\sigma)$. Furthermore, since, by construction, r is the last letter of $\sigma(r)$ and l is the first letter of $\sigma(l)$, it follows that $r\sigma(\omega)l \in \mathcal{L}(\sigma)$. We note that rl is both a prefix and a suffix of $r\sigma(\omega)l$ since the first letter of $\sigma(\omega$ is l and the last letter of $\sigma(\omega)$ is r. Consequently, $\sigma(\omega) \in \phi(R^+)$ as it is a concatenation of return words, in which case it is a result from [4] that there exists a unique $u \in R^+$ such that $\sigma(\omega) = \phi(u)$.

We may therefore define a substitution τ on R to be $\tau(j) = u$. It follows that

$$\phi \circ \tau^n = \sigma^n \circ \phi \text{ for each } n \geq 0.$$

Furthermore, it can be shown that τ is proper, primitive, aperiodic, and that $\mathcal{D}(x)$ is its fixed point. At this point, it might be tempting to assert that we have identified the proper substitution which yields the substitution dynamical system which is conjugate to that system induced by σ. Unfortunately, this is not necessarily the case. The substitution τ is used to find another proper substitution, which we will denote ζ, that does yield the substitution dynamical system conjugate to σ. Furthermore, τ will become important later on when we want to calculate the dimension group of a substitution dynamical system.

Now, we would like to define ζ. We will take the following as yet another result of [4], looking primarily to the explicit construction of ζ. We will begin by defining the alphabet, B, on which ζ is a substitution. Thus, we may begin by assuming that $|\tau(j)| \geq |\phi(j)|$ for all $j \in R$, substituting τ for a power of τ if need be. Consequently, for each $j \in R$, we may write $m_j = |\tau(j)|$ and $n_j = |\phi(j)|$, in which case we may define B as follows,

$$B = \{(j,p) \ : \ j \in R, \ 1 \leq p \leq n_j\}.$$

We may also define the maps $\theta : B \to A$ and $\psi : R \to B^+$ as,

$$\theta(j,p) = (\phi(j))_p \quad \text{and} \quad \psi(j) = (j,1)(j,2)\cdots(j,n_j).$$

It is immediate that $\theta \circ \psi = \phi$. Now, it can be shown that θ is an isomorphism between (X_σ, T_σ) and (X_ζ, T_ζ), and we will use ψ in order to define the substitution ζ. Specifically, we defin ζ on B for $j \in R$ and $1 \leq p \leq n_j$ by,

$$\zeta(j,p) = \begin{cases} \psi\left((\tau(j))_p\right) & \text{if } 1 \leq p < n_j, \\ \psi\left((\tau(j))_{[n_j,m_j]}\right) & \text{if } p = n_j. \end{cases}$$

At this point, we have developed all of the necessary tools in order to move fluidly between substitution dynamical systems and Bratteli-Vershik systems. In other words, given a stationary, properly ordered Bratteli diagram, we may identify a substitution which will yield a dynamical system conjugate to the associated Bratteli-Vershik system; and conversely, given an aperiodic substitution, we may identify the Bratteli-Vershik system conjugate to the associated substitution dynamical system.

Before we close this chapter, we would like to give an example of the explicit construction of a proper substitution from an arbitrary substitution. Specifically, we would like to look at the Fibonacci substitution on the alphabet $\mathcal{A} = \{a, b\}$ given by,

$$\sigma(a) = b,$$
$$\sigma(b) = ab.$$

Now, clearly σ is primitive, and it is well known that σ is aperiodic. We will begin by constructing τ. To this extent, let x be the fixed point of σ such that $x_{-1} = b$ and $x_0 = a$,

$$x = \cdots babab ba bab ba bba bab bab.abbabbababbab \cdots$$

Notice that we must use σ^2 since σ alternates between x and another fixed point. Furthermore, note that x is admissible since $ba \prec \sigma^2(a)$ and $ba \prec \sigma^2(b)$. We may now identify the first return word to $b.a$. In order to do this, we begin reading x from $b.a$ looking for the next occurrence of ba. Consequently, we have that the first return word is $\omega_1 = abb$ since we have $b.\boldsymbol{abb}a$. Thus, we may set $\phi(1) = \omega_1$. In order to define τ and identify the other return words to $b.a$, we look at $\sigma^2(\omega_1)$. That is,

$$\sigma^2(\omega_1) = \sigma^2(abb) = \sigma(babab) = abbabbab = \omega_1\omega_1\omega_2,$$

Where $\omega_2 = ab$. We may then set $\phi(2) = \omega_2$, and we get $\tau(1) = 112$. We repeat this process with ω_2,

$$\sigma^2(\omega_2) = \sigma^2(ab) = \sigma(bab) = abbab = \omega_1\omega_2,$$

Which yields $\tau(2) = 12$. Since τ is primitive, it follows that the alphabet R cannot contain any other letter, in which case $\mathcal{R}$ cannot contain any other return word. Consequently, we have that

$$R = \{1, 2\}, \quad \mathcal{R} = \{abb, ab\}, \quad \phi(1) = abb, \quad \phi(2) = ab, \quad \tau(1) = 112, \quad \tau(2) = 12.$$

It follows that τ is primitive, proper, aperiodic, and has as its fixed point,

$$y \;=\; \cdots 112112121121121211212.1121121211212 \cdots$$

Now, we have that $|\tau(j)| = |\phi(j)|$ for $j = 1, 2$ satisfying the requirement that $|\tau(j)| \geq |\phi(j)|$ for each j. We have that $n_1 = m_1 = 3$ and $n_2 = m_2 = 2$. Consequently, we may define B,

$$B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2)\},$$

In which case, we have that $\psi(1) = (1, 1)(1, 2)(1, 3)$ and $\psi(2) = (2, 1)(2, 2)$. In order to ease notation we denote

$$\alpha = (1, 1), \quad \beta = (1, 2), \quad \gamma = (1, 3), \quad \delta = (2, 1), \quad \epsilon = (2, 2).$$

This allows us to then calculate the substitution ζ on B,

$$\begin{aligned}
\zeta(\alpha) &= \psi(\tau(1)_1) &= \psi(1) &= \alpha\beta\gamma, \\
\zeta(\beta) &= \psi(\tau(1)_2) &= \psi(1) &= \alpha\beta\gamma, \\
\zeta(\gamma) &= \psi(\tau(1)_{[3,3]}) &= \psi(2) &= \delta\epsilon, \\
\zeta(\delta) &= \psi(\tau(2)_1) &= \psi(1) &= \alpha\beta\gamma, \\
\zeta(\epsilon) &= \psi(\tau(2)_{[2,2]}) &= \psi(2) &= \delta\epsilon.
\end{aligned}$$

In which case, ζ is proper with fixed point $[\epsilon.\alpha]$.

Chapter 5

The Dimension Group of a Cantor Minimal System

5.1 Introduction to Dimension Groups

The motivation for this chapter stems from the results of [7], [9], [11] and [14], where it is shown that to each Cantor minimal system we may associate a dimension group. The significance of this is that these dimension groups are invariant under orbit equivalence. In order to explain this in more detail, we must begin by building up the structure necessary in order to define what is meant by a dimension group.

The following is taken from [10]. We start with some basic definitions in order to set terminology and notation. We say that a *partial order*, $\leq$, on a set X is a reflexive, antisymmetric, transitive relation on X. Naturally, we say that a partially ordered set, or *poset*, is a set X equipped with a specified partial order. Note that this definition does not specify that any two elements in a poset may be comparable. We say that a poset $(X, \leq)$ is *totally ordered* if every pair of elements, $x, y \in X$ are comparable. That is, either $x \leq y$ or $y \leq x$. We say that the *dual ordering* of a poset $(X, \leq)$ is the partial order $\geq$. The dual ordering is often denoted $\leq^*$. In this manner, the *dual* of $(X, \leq)$ is the poset $(X, \leq^*)$. A poset $(X, \leq)$ is said to be *upward (downward) directed* if every finite subset of X has an upper (lower) bound in X.

From the above foundation, we say that a *partially ordered abelian group*, $(G, \leq)$, is an abelian group, G, equipped with a specified translation-invariant partial order, $\leq$. By translation invariant, it is meant that for $x, y, z \in G$, $x \leq y$ implies that $x + z \leq y + z$. In the usual sense, a *positive element* in a partially ordered abelian group $(G, \leq)$ is any element $x \in G$ which satisfies $0 \leq x$. A *strictly positive element* in $(G, \leq)$ is any element $x \in G$ which satisfies $0 < x$. Note that the idea of a nonnegative element cannot be used in place of a positive element except in the case that G is totally ordered. Thus, we may define a *cone* in an abelian group G as any subset, C, of G such that $0 \in C$ and C is closed under addition. A cone $C \subseteq G$ is a *strict cone* if 0 is the only element $x \in G$ for which both $x \in C$ and $-x \in C$. Consequently, given a cone $C \subseteq G$, we may define a relation $\leq_C$ on G such that given $x, y \in G$, $x \leq_C y$ if and only if $x - y \in C$. Furthermore, we can see that $\leq_C$ is a translation-invariant reflexive transitive relation on G which is a partial order if and only if C is a strict cone.

Definition 5.1.1. The *positive cone* of a partially ordered abelian group G is the set G^+ of all positive elements of G.

In this line of thought, we also define an *order-unit* in a partially ordered abelian group G to be any positive element $u \in G^+$ such that given any $x \in G$, there is some positive integer n such that $x \leq n \cdot u$. With these definitions, we may cite a common result from [10].

Proposition 5.1.1. Let H be a subgroup of a partially ordered abelian group G equipped with the induced partial order from G. Then $H^+ = H \cap G^+$ and we have that the following conditions are equivalent:

1. H is upward directed.

2. H is downward directed.

3. H is generated (as a subgroup of G) by a subset of G^+.

4. All elements of H have the form $x - y$ for $x, y \in H^+$.

From this result we may write the following definition,

Definition 5.1.2. A *directed subgroup* of a partially ordered abelian group G is any subgroup H of G that satisfies the conditions of the above proposition. We say that a group G is *directed* if it is a directed subgroup of itself. A *directed abelian group* is any directed partially ordered abelian group.

As an example, it is easy to see that any totally ordered abelian group is directed. Specifically, $\mathbb{Z}$ equipped with the usual ordering is a directed group with positive cone $\mathbb{N} \cup \{0\}$. For an example which is slightly less immediate, one can verify that $\mathbb{Z}^n$ equipped with the lexicographic ordering is a directed group with positive cone $(Z^n)^+ = \{(x_1, x_2, \ \dots \ , x_n) \in \mathbb{Z}^n : x_j \geq 0 \ \forall \ 1 \leq j \leq n\}$.

Definition 5.1.3. Let G be a partially ordered abelian group, and let n be a positive integer. We say that G is *n-perforated* if there exists an element $x \in G$ such that $n \cdot x \geq 0$ but $x \ngeq 0$. Otherwise, we say that G is *n-unperforated.* If G is n-perforated for some positive integer n, then we say that G is *perforated.* Consequently, if G is n-unperforated for all positive integers n, then we say that G is *unperforated.*

For example, if we equip $\mathbb{Z}$ with the positive cone $2\mathbb{N} \cup \{0\}$ and the associated partial ordering, then it follows that $\mathbb{Z}$ is 2-perforated since $2 \cdot (2n+1) \geq 0$ for $n \geq 1$ but $2n + 1 \ngeq 0$ for $n \geq 1$ since $2n + 1$ is not in the defined positive cone. On the other hand, it is a quick verification that $\mathbb{Z}$ equipped with the usual ordering is unperforated.

From here, we need only establish one more definition before we can define what we mean by a dimension group. Namely, we need to define that which is meant by an interpolation group. To this end we cite a result from [10].

Proposition 5.1.2. For a partially ordered abelian group G, the following conditions are equivalent:

1. Given $x_1, x_2, y_1, y_2 \in G$ such that $x_j \leq y_k$ for each j, k, there exists $z \in G$ such that $x_j \leq z \leq y_k$ for each j, k.

2. Given $x, y_1, y_2 \in G^+$ such that $x \leq y_1 + y_2$, there exist $x_1, x_2 \in G^+$ such that $x = x_1 + x_2$ and $x_j \leq y_j$ for each j.

3. Given $x_1, x_2, y_1, y_2 \in G^+$ such that $x_1+x_2 = y_1+y_2$, there exist $z_{11}, z_{12}, z_{21}, z_{22} \in G^+$ such that $x_j = z_{j1} + z_{j2}$ for each j and $y_k = z_{1k} + z_{2k}$ for each k.

With this result, we may cite a definition.

Definition 5.1.4. A poset $(X, \leq)$ is said to satisfy the *Riesz interpolation property* provided X satisfies condition (1) in the above proposition. A partially ordered abelian group G is said to satisfy the *Riesz decomposition properties* provided G satisfies the conditions (2) and (3) in above proposition. Consequently, we say that a partially ordered abelian group G is an *interpolation group* if it satisfies the Riesz interpolation property as well as the Riesz decomposition properties.

It is a quick verification, for example that $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ each equipped with the usual ordering are interpolation groups. For an example of a partially ordered abelian group which is not an interpolation group, consider $\mathbb{Z}^2$ with the positive cone $(Z^2)^+ = \{(x, y) \in \mathbb{Z}^2 : 2x \geq y \geq 0\}$. Setting $x_1 = (0, 0)$, $x_2 = (0, 1)$, $y_1 = (1, 1)$ and $y_2 = (1, 2)$, we can see that $x_j \leq y_k$ for each j, k, but there does not exist an element $z \in G$ such that $x_j \leq z \leq y_k$ for each j, k. We are now able to define a dimension group.

Definition 5.1.5. We say that a *dimension group* is any directed, unperforated, interpolation group.

One should note immediately that $\mathbb{Z}^n$ equipped with the usual ordering is a dimension group for any $n \in \mathbb{N}$. In fact, most of the dimension groups we will work with will be $\mathbb{Z}^n$ equipped with some appropriate positive cone. This being stated, we may now consider the problem of associating a dimension group to a given Cantor minimal system.

5.2 The K^0 Group of a Cantor Minimal System

As it turns out, there are multiple approaches to the problem of associating a dimension group to a given Cantor minimal system. We will begin with an examination of the K^0 group of a Cantor minimal system. In particular, we will follow the work of Y.T. Poon, [14].

In [14], Poon studies the quotient group $G(X, \varphi)$ for a zero-dimensional topological dynamical system (X, φ). Specifically, we define $G(X, \varphi) = C(X, \mathbb{Z})/B(X, \varphi)$, where $C(X, \mathbb{Z})$ is the group of continuous functions from X to $\mathbb{Z}$ and $B(X, \varphi)$ is the subgroup of functions of the form $f - f \circ \varphi$. The motivation for the study of this group comes from [9], where this group is denoted $K^0(X, \varphi)$, in which the authors show that two Cantor minimal systems (X_1, φ_1) and (X_2, φ_2) are Kakutani strong orbit equivalent if and only if $K^0(X_1, \varphi_1)$ is order isomorphic to $K^0(X_2, \varphi_2)$ by a map which does not necessarily preserve the distinguished order units. Furthermore, if we strengthen the condition on the isomorphism so that the map must preserve the distinguished order units, then we may replace Kakutani strong orbit equivalence with strong orbit equivalence.

Now, let (X, φ) be a Cantor minimal system. In order to construct $K^0(X, \varphi)$ as shown in [14], we begin by noting that for each $f \in C(X, \mathbb{Z})$, there exists a partition $\{O_j : 1 \leq j \leq n\}$ of X and a sequence of integers $\{a_j : 1 \leq j \leq n\}$ such that $f = \sum_{j=1}^{n} a_j \chi_{O_j}$. Here we use χ_O to denote the characteristic function of O. Thus, one can see that we may work with partitions of X in lieu of functions from $C(X, \mathbb{Z})$.

Furthermore, Poon goes on to show that for a given partition $\{O_j : j = 1 \leq j \leq n\}$ of X, we may define a directed graph on the set of vertices $V = \{1, \ldots, n\}$ by letting $j \to k$ if $O_j \cap \varphi^{-1}(O_k) \neq \emptyset$. The reason for doing this lies in the fact that we can construct from this directed graph an ordered group.

Specifically, let Γ be a directed graph on a finite set V. Let $C_\Gamma = \{f : V \to \mathbb{Z}\}$ and $B_\Gamma = \{f \in C_\Gamma : \sum_s f = 0,$ for every cycle s of $\Gamma\}$. It follows that C_Γ is a group under the usual addition, and that B_Γ is a subgroup of C_Γ. We may then define $G_\Gamma = C_\Gamma / B_\Gamma$ with

$$G_\Gamma^+ = \left\{ [f] : \sum_s f \geq 0 \text{ for every cycle } s \text{ of } \Gamma \right\}.$$

Thus, we come back to the consideration of a Cantor minimal system, (X, φ). Since X is separable, we may define a sequence of finite partitions $P_1 < P_2 < \cdots$, such that $\bigcup_{n=1}^{\infty}$ is a basis for the topology of X. Suppose that $P_n = \{O_v : v \in V_n\}$. Then we may let Γ_n be the directed graph on V_n associated with the partition P_n and $G_n = G_{\Gamma_n}$. Since $P_n < P_{n+1}$, it follows that there exists a unique map, ϕ_n of V_{n+1} onto V_n such that, for every v' of V_n, $O_{v'} = \bigcup\{O_v : v \in \phi_n^{-1}(v')\}$. Therefore, we have a map $\Phi_n : G_n \to G_{n+1}$ by $\Phi_n([f]) = ([f \circ \phi_n])$. It can be checked that Φ_n is a well-defined unital order homomorphism for each n, and that consequently, we can define the direct limit $\varinjlim(G_n, \Phi_n)$. With this mind, we may cite the main result of [14].

Theorem 5.2.1. Suppose (X, φ) is topologically transitive. Let $P_1 < P_2 < \cdots$ be a sequence of partitions of X such that $\bigcup_{n=1}^{\infty} P_n$ is a basis for the topology of X. Then $K^0(X, \varphi)$ is order isomorphic to $\varinjlim(G_n, \Phi_n)$.

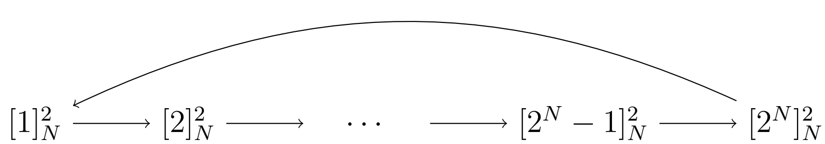
With this background, we can move on to actually calculate the K^0 group for several Cantor minimal systems. Note that in [14], the directed graphs given for the set of examples to follow were not properly directed. We correct this mistake here. We will begin with the dyadic adding machine, $(\mathbb{Z}_{(2)}, \varphi)$. In order to simplify notation, we define

$$[N]_n^2 = 2^n \mathbb{Z}_{(2)} + N,$$

For $n, N \in \mathbb{N}$ with $1 \leq N \leq 2^N$. Thus, we may define the sequence of partitions:

$$\begin{aligned}
P_1 &= \{[1]_1^2,\ [2]_1^2\}, \\
P_2 &= \{[1]_2^2,\ [2]_2^2,\ [3]_2^2,\ [4]_2^2\}, \\
P_3 &= \{[1]_3^2,\ [2]_3^2,\ [3]_3^2,\ [4]_3^2,\ [5]_3^2,\ [6]_3^2,\ [7]_3^2,\ [8]_3^2\}, \\
&\vdots
\end{aligned}$$

From which φ yields the following directed graph for P_N:



So that, for $f \in C(P_N, \mathbb{Z})$, we have that

$$f = \sum_{j=1}^{2^N} a_j \chi_{[j]_N^2}.$$

We can identify $[f] \in C(P_N, \mathbb{Z})/B(P_N, \mathbb{Z})$ with $\sum_{j=1}^{2^N} a_j \in \mathbb{Z}$. This identifies the isomorphism $G_N \cong \mathbb{Z}$ under which the positive cone is mapped to $\mathbb{Z}^+$. Furthermore, we have that $[1] = 2^N$, and the map $\Phi_N : G_N \to G_{N+1}$ is given by

$$\Phi_N(a) = 2 \cdot a.$$

Thus, it follows that $K^0(\mathbb{Z}_{(2)}, \varphi)$ is order isomorphic to the direct limit

$$\mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\psi} \cdots$$

Where $\psi : \mathbb{Z} \to \mathbb{Z}$ is the map $\psi(x) = 2x$. Now, in order to calculate this direct limit, we set

$$L = \{q \in \mathbb{Q} : \exists\, n > 0 \text{ such that } q \cdot \psi^n(1) \in \mathbb{Z}\},$$
$$H = \{q \in \mathbb{Q} : \exists\, n > 0 \text{ such that } q \cdot \psi^n(1) = 0\}.$$

Note then that $K^0(\mathbb{Z}_{(2)}, \varphi)$ is order isomorphic to L/H, and that the positive cone of $K^0(\mathbb{Z}_{(2)}, \varphi)$ is the image of the positive cone of L in the quotient. Since $\psi^n(1) = 2^n$, it follows that $q \cdot \psi^n(1) \in \mathbb{Z}$ whenever $q \in 2^{-n} \cdot \mathbb{Z}$. It follows that

$$L = \bigcup_{n=0}^{\infty} 2^{-n} \cdot \mathbb{Z} = \mathbb{Z}\left[\tfrac{1}{2}\right],$$

With positive cone

$$L^+ = \bigcup_{n=0}^{\infty} 2^{-n} \cdot \mathbb{Z}^+ = \mathbb{Z}\left[\tfrac{1}{2}\right]^+.$$

Now, since $q \cdot \psi^n(1) = 0$ only when $q = 0$, it follows that $K^0(\mathbb{Z}_{(2)}, \varphi)$ is order isomorphic to L with positive cone L^+ and distinguished order unit 1.

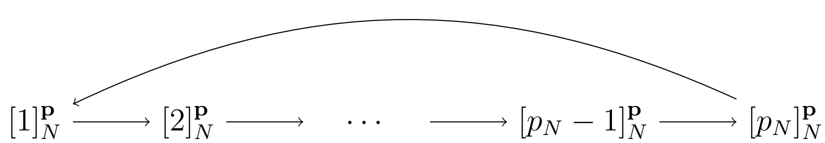
In a similar manner, we can approach the more general **p**-adic adding machine. In this case, define

$$[N]_n^{\mathbf{p}} = p_{n-1}\mathbb{Z}_{(\mathbf{p})} + N,$$

For $n, N \in \mathbb{N}$ with $1 \leq N \leq p_{n-1}$. As in the section on the Kakutani-Rohlin partitions, we set $q_n = p_n/p_{n-1}$. Then it follows that we may define the sequence of partitions:

$$\begin{aligned} P_1 &= \{[1]_1^{\mathbf{p}}, [2]_1^{\mathbf{p}},\ \dots\ , [p_0]_1^{\mathbf{p}}\}, \\ P_2 &= \{[1]_2^{\mathbf{p}}, [2]_2^{\mathbf{p}},\ \dots\ , [p_1]_2^{\mathbf{p}}\}, \\ P_3 &= \{[1]_3^{\mathbf{p}}, [2]_3^{\mathbf{p}},\ \dots\ , [p_2]_3^{\mathbf{p}}\}, \\ &\vdots \end{aligned}$$

From which φ yields the following directed graph for P_N:



Similar to the case for the dyadic adding machine, for $f \in C(P_N, \mathbb{Z})$ we have that

$$f = \sum_{j=1}^{p_N} a_j \chi_{[j]_N^{\mathbf{p}}}.$$

We can consequently identify $[f] \in C(P_N, \mathbb{Z})/B(P_N, \mathbb{Z})$ with $\sum_{j=1}^{p_N} a_j \in \mathbb{Z}$. This identifies the isomorphism $G_N \cong \mathbb{Z}$ under which the positive cone is mapped to $\mathbb{Z}^+$. Furthermore, we have that $[1] = p_N$, and the map $\Phi_N : G_N \to G_{N+1}$ is given by

$$\Phi_N(a) = q_{N+1} \cdot a.$$

Using the same approach as in the dyadic adding machine above, we can see that $K^0(\mathbb{Z}_{(\mathbf{p})}, \varphi)$ is order isomorphic to the group

$$L = \bigcup_{n=0}^{\infty} p_n^{-1} \cdot \mathbb{Z},$$

With positive cone

$$L^+ = \bigcup_{n=0}^{\infty} p_n^{-1} \cdot \mathbb{Z}^+,$$

In which the distinguished order unit is 1. Having calculated the K^0 group for the adding machine examples, we now turn our attention to the irrational rotation examples. We will start with the system $(\mathbb{T}_C, \varphi_\alpha)$ for $\alpha = \frac{1+\sqrt{5}}{2}$. In what follows, we will use the notation introduced in the Kakutani-Rohlin partition calculation. Thus, in a similar manner, we construct the partition P_n from the set

$$Q_n = \{\varphi_\alpha^n(0) \;:\; -q_n \leq n < q_{n-1}\}.$$

Note that the bounds in Q_n defined here are different from those used in the Kakutani-Rohlin partition calculations. Note also that the notation used here differs from that used in [14]. In this case, it is easy to see that $|P_n| = q_n + q_{n-1}$. Consequently, we have the following directed graph derived from P_n and φ_α,

$$\begin{tikzcd}
{[\alpha^r_{-q_n}, \alpha^l_0]} \arrow[r] & {[\alpha^r_{1-q_n}, \alpha^l_1]} \arrow[r] & \cdots \arrow[r] & {[\alpha^r_{q_{n-1}-q_n-1}, \alpha^l_{q_{n-1}-1}]} \arrow[d] \\
{[\alpha^r_{q_{n-1}-1}, \alpha^l_{-1}]} \arrow[u] \arrow[rrr, bend left] & {[\alpha^r_{q_{n-1}-2}, \alpha^l_{-2}]} \arrow[l] & \cdots \arrow[l] & {[\alpha^r_{q_{n-1}-q_n}, \alpha^l_{-q_n}]} \arrow[l]
\end{tikzcd}$$

When n is even and

$$\begin{tikzcd}
{[\alpha^r_0, \alpha^l_{-q_n}]} \arrow[r] & {[\alpha^r_1, \alpha^l_{1-q_n}]} \arrow[r] & \cdots \arrow[r] & {[\alpha^r_{q_{n-1}-1}, \alpha^l_{q_{n-1}-q_n-1}]} \arrow[d] \\
{[\alpha^r_{-1}, \alpha^l_{q_{n-1}-1}]} \arrow[u] \arrow[rrr, bend left] & {[\alpha^r_{-2}, \alpha^l_{q_{n-1}-2}]} \arrow[l] & \cdots \arrow[l] & {[\alpha^r_{-q_n}, \alpha^l_{q_{n-1}-q_n}]} \arrow[l]
\end{tikzcd}$$

When n is odd. Denote $[q_n]_j^N$ to be the intervals in P_N of size $\varphi_\alpha^{-q_n}(0)$. Denote also $[N]_j$ to be the j^{th} interval in P_N. Then, for $f \in C(P_n, \mathbb{Z})$, we have that

$$f = \sum_{j=1}^{q_n+q_{n-1}} a_j \chi_{[n]_j} + \sum_{j=1}^{q_n} b_j \chi_{[q_{n-1}]_j^n}.$$

We can identify $[f] \in C(P_n, \mathbb{Z})/B(P_n, \mathbb{Z})$ with $\left(\sum_{j=1}^{q_n+q_{n-1}} a_j,\ \sum_{j=1}^{q_n} b_j\right) \in \mathbb{Z}^2$. This identifies the isomorphism $G_n \cong \mathbb{Z}^2$ under which the positive cone is mapped to $(\mathbb{Z}^2)^+$.

Furthermore, we have that $[1] = (q_n + q_{n-1}, q_n)$, and the map $\Phi_n : G_n \to G_{n+1}$ is given by

$$\Phi_n(a, b) = (a \cdot a_{n+1} + b, a).$$

Where, as usual, a_{n+1} is the $(n+1)^{\text{st}}$ partial quotient of the continued fraction decomposition of α. Note that the map Φ_n is defined by the fact that P_{n+1} is obtained from P_n by splitting each interval of length $\varphi_\alpha^{(-1)^{n-1}q_{n-1}}(0)$ into a_{n+1} intervals of length $\varphi_\alpha^{(-1)^n q_n}(0)$ and one interval of length $\varphi_\alpha^{(-1)^{n+1}q_{n+1}}(0)$. Thus, it follows that $K^0(\mathbb{T}_C, \varphi_\alpha)$ is order isomorphic to the direct limit

$$\mathbb{Z}^2 \xrightarrow{\psi_1} \mathbb{Z}^2 \xrightarrow{\psi_2} \mathbb{Z}^2 \xrightarrow{\psi_3} \cdots$$

Where ψ_n is the linear map

$$\psi_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Since, in this case, $a_n = 1$ for each n, it follows that

$$\psi_n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

In order to calculate this direct limit, we will follow the work of [6]. Let P_α denote the positive cone for $\mathbb{Z}^2$ defined by

$$P_\alpha = \{(n, m) \in \mathbb{Z}^2 : \alpha \cdot n + m \geq 0\}.$$

Now, consider the following diagram,

$$\begin{array}{ccc}
\mathbb{Z}^2 & \xrightarrow{\theta_0} & (\mathbb{Z}^2, P_\alpha) \\
\downarrow{\scriptstyle \psi^2} & & \downarrow{\scriptstyle id} \\
\mathbb{Z}^2 & \xrightarrow{\theta_1} & (\mathbb{Z}^2, P_\alpha) \\
\downarrow{\scriptstyle \psi^2} & & \downarrow{\scriptstyle id} \\
\mathbb{Z}^2 & \xrightarrow{\theta_2} & (\mathbb{Z}^2, P_\alpha) \\
\downarrow{\scriptstyle \psi^2} & & \downarrow{\scriptstyle id}
\end{array}$$

Where $\theta_0 = id$ and $\theta_n = (\psi^{2n})^{-1}$. It is shown in [6] that (θ_n) determines an order isomorphism, in which case we have that $K^0(\mathbb{T}_C, \varphi_\alpha)$ is isomorphic to $(\mathbb{Z}^2, P_\alpha)$. Note that the above process holds for any irrational α. Hence we get a similar result for any choice of α. For example, if we set $\alpha = [2, 2, 2 \ldots]$, then we have that

$$\psi_n = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},$$

In which case, the result proceeds as illustrated above.

5.3 The K_0 Group of a Cantor Minimal System

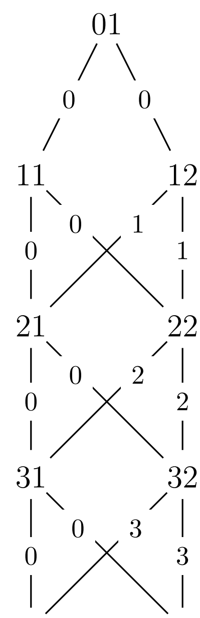
It is shown in [11] that there is an alternative method for construction the dimension group of a Cantor minimal system. In particular, the authors show that the K_0 group of an essentially simple ordered Bratteli diagram is order isomorphic to the K^0 group of the associated Vershik system. Consequently, Theorem 3.5.1 implies that the K^0 group of a Cantor minimal system, (X, φ) is order isomorphic to the K_0 group of the ordered Bratteli diagram constructed from (X, φ) using Kakutani-Rohlin partitions.

The K_0 group of a Bratteli diagram originates in the study of AF-algebras where the work of [2] proved to be very useful in that a Bratteli diagram provides a simple exposition of the structure of its associated AF-algebra. Furthermore, this structure is reflected in the K_0 group of the Bratteli diagram allowing work in AF-algebras to be relegated to the level of dimension groups.

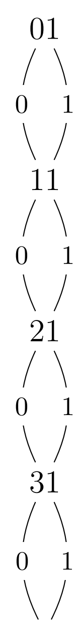
In the context of Cantor minimal systems, we can leverage this work as explained in the first paragraph of this section. In particular, if we represent each vertex in the set V_n of the ordered Bratteli diagram $(V, E, \geq)$ as a copy of $\mathbb{Z}$, then we can define the inductive limit on $(V, E, \geq)$ as

$$\mathbb{Z}^{n_0} \xrightarrow{A_1} \mathbb{Z}^{n_1} \xrightarrow{A_2} \mathbb{Z}^{n_2} \xrightarrow{A_3} \cdots ,$$

Where $n_j = |V_{n_j}|$ and A_j is the $n_j \times n_{j-1}$ incidence matrix. In the case of $(\mathbb{Z}_{(2)}, \varphi)$, we calculated in section 3.6.1 that the Bratteli diagram associated with this system is given by



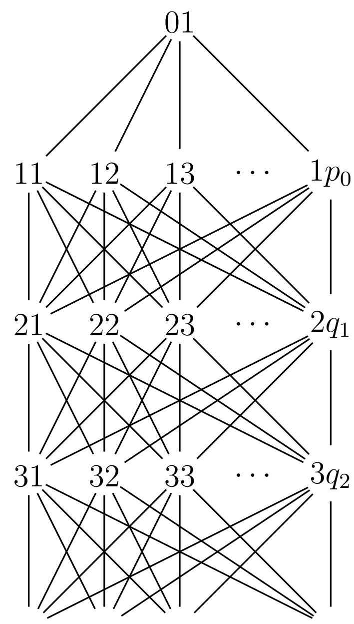
It is not difficult to see that this ordered Bratteli diagram is isomorphic to the ordered Bratteli diagram



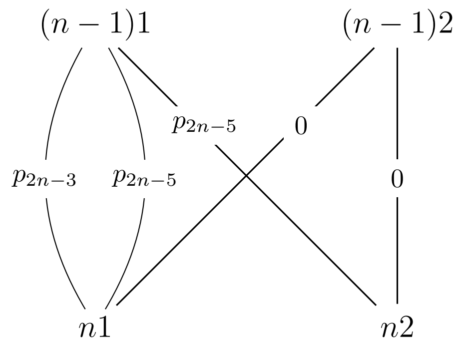
Which yields the inductive limit

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots .$$

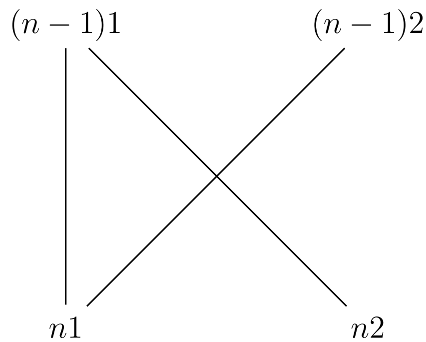
As we saw in the previous section, this direct limit is order isomorphic to the group $\mathbb{Z}[\frac{1}{2}]$ with positive cone $\mathbb{Z}[\frac{1}{2}]^+$ and distinguished order unit 1. Thus, we can see directly that $K^0(\mathbb{Z}_{(2)}, \varphi) \cong K_0(\mathbb{Z}_{(2)}, \varphi)$. Similarly, in the case of the Cantor minimal system $(\mathbb{Z}_{(\mathbf{p})}, \varphi)$, we have its associated Bratteli diagram from section 3.6.1,



From which we can derive an isomorphic ordered Bratteli diagram with one vertex at each level yielding an inductive limit which is order isomorphic to that which is calculated in section 5.2. Moving on to the irrational rotation examples, we look at the system $(\mathbb{T}_C, \varphi_\alpha)$ where $\alpha = [1, 1, 1, \ ...]$. In section 3.6.2, we calculated the associated ordered Bratteli diagram



Which is isomorphic to the Bratteli diagram



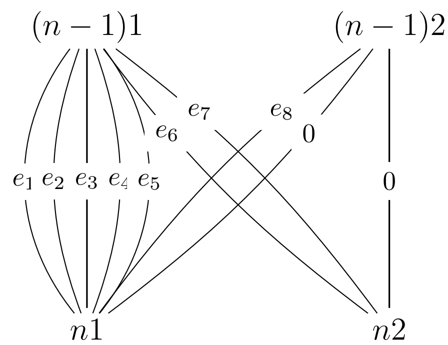
Note that we have left the ordering out of the above Bratteli diagram since it is not reflected in the inductive limit,

$$\mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{A} \cdots ,$$

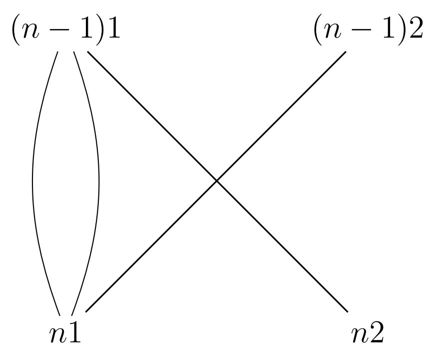
Where the incidence matrix, A, at each level is given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Again, this inductive is equivalent to the inductive limit calculated in section 5.2 giving the desired isomorphism. Similarly, for $\alpha = [2, 2, 2, \ ...]$, we have the ordered Bratteli diagram



Which we can see is isomorphic to the Bratteli diagram



From which we obtain the inductive limit

$$\mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{A} \cdots ,$$

Where the incidence matrix, A, at each level is given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},$$

Yielding the desired isomorphism. The importance of this result lies in the fact that we are now in the position to easily calculate the dimension group of a substitution minimal system. More importantly, we now have the ability to easily traverse between substitution minimal systems and their associated Cantor minimal systems through the lens of ordered Bratteli diagrams and their associated K_0 groups.

For a specific example, we simply take the Fibonacci substitution described in section 4.2. We can easily see that this substitution is orbit equivalent to the Cantor minimal system $(\mathbb{T}_C, \varphi_\alpha)$ with $\alpha = [1, 1, 1, \ ...]$ since they both yield isomorphic Bratteli diagrams and consequently have isomorphic dimension groups. Similarly, we may use substitutions in order to study Cantor minimal systems without ever having to explicitly work with a given Cantor minimal system. This becomes potentially useful, for example, in the study of $\mathbb{Z}^d$-actions on a Cantor set when $d > 1$.

Chapter 6

Conclusion

In this work, we have built up the structure necessary to study several classes of topological dynamical systems and to trace several examples through these various perspectives. We have looked at topological dynamical systems in general, with a focus on Cantor minimal systems, substitution minimal systems, and Bratteli-Vershik systems. The idea of a dimension group associated to a Cantor minimal system was developed and examined as a tool to bring these seemingly disparate ideas together.

The original goal of explicitly calculating the range of the dimension group for $\mathbb{Z}^d$-actions on a Cantor set for some $d > 1$ was not accomplished. On the other hand, we were able to explicitly calculate the Kakutani-Rohlin partitions for several adding machine and irrational rotation systems. Furthermore, we were able to make corrections to the work of [14]. Specifically, the proper directed graphs for the given systems were calculated.

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