Robust Estimation of Higher Order Derivatives of Solutions to Some Nonlinear Systems Under Uncertainties

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Abstract

Estimation of derivatives of a noisy signal is a practically important problem which frequently is approached via application of different filtering methodologies. Our work focuses on estimating derivatives of solutions to partly unknown nonlinear systems subject to bounded system and measurement noises. The robust observers are designed to approximate higher order derivatives for some nonlinear systems arising in various application domains and to estimate numerically the accuracy of such approximations. We also determine how the estimation errors are influenced by measurement and system noises and disclose amplification of the error norm is due to increased observer gains. Consequently, we use estimations of derivatives to forecast the system dynamics on relatively short time intervals.
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Contents

Abstract .......................................................... i
Acknowledgements ............................................... ii

1 Introduction ................................................... 1

2 Motivation of Identity Observer ......................... 4
  2.1 Estimation of Linear Time-Invariant Systems .......... 4
  2.2 Estimation for Linear System with Uncertainties ....... 14

3 Estimation of Derivatives for Nonlinear Systems ...... 17
  3.1 The Attempt of Identity Observer ...................... 17
  3.2 Luenberger Reduced Order Observer ................. 20
  3.3 Nonlinear System with Linear Uncertainties .......... 28

4 Forecasting Behaviors of Nonlinear Systems on Short-Time Intervals .................................. 34
  4.1 Double-Well Chaotic System ........................... 34
  4.2 Reduced Order Observer for Double-Well Chaotic System Under Linear Uncertainty ............... 42
  4.3 Short Forecast of Solutions to Process and Prediction of Double-Well Chaotic System .......... 44
  4.4 Lorenz System ............................................. 48
5 Estimation of Derivatives for Nonlinear System with Bounded Noise 56
  5.1 Identity Observer for Noised System 56
  5.2 Simulation of Performance of Reduced Order Observer for Nonlinear System with Noise 64

Appendices

A Section 2.1 72
B Section 2.2 74
C Section 3.1 75
D Section 3.2 76
E Section 4.1 78
F Section 4.3 81
G Section 4.4 83
H Section 5.1 86
I Section 5.2 89
Chapter 1

Introduction

Inherent uncertainties, instability and multiscale nature make the results of direct simulation of complex systems arising in various application domains sensitive to perturbations of their initial values, parameters and the simulation interval which might jeopardize the reliability and practical value of such elaborate simulations. Consequently, current researches in this area explores various approaches to assimilating a different kind and quality of observations in numerical simulations. Estimation of derivatives of solutions of partly unknown systems presents even more challenging problem since differentiation frequently amplifies sensitivity of the estimation to perturbations. Note that approximation of derivatives is a standard operation which is important to various fields of contemporary science and engineering. Estimation of derivatives of a known but noisy signal is a complex problem which is frequently approached through designing an appropriate filter. Respectively, approximation of derivatives to solutions of nonlinear systems under uncertainties presents a more challenging task which is addressed in this thesis.

In many cases a finite-dimensional control system can be written as

\[ \dot{x} = f(x, u, t), \]  

(1.0.1)
where

\[ x = [x_1, x_2, \ldots, x_m]^T, \]

\[ u = [u_1, u_2, \ldots, u_n]^T, \]

\[ f(x, u, t) = [f_1(x, u, t), f_2(x, u, t), \ldots, f_l(x, u, t)]^T. \]

Eq.(1.0.1) is called state equation. \( \dot{x}_i \) denotes the first derivative with respect to time variable \( t \) and \( u_j \) is the control/input variable. We call \( x_i \)'s the state variables. They are the memory that the dynamical system has of its past. We will assume that a vector-function \( f \) defining this system is partly unknown and to mitigate this uncertainty. We will also assume that the measurements of limited number of state variables are available and can be included in our analysis through an associated output equation

\[ y = g(x, u, t). \] (1.0.2)

\( y(t) \in \mathbb{R}^p \) is assumed to be measured or required to behave in a specific way. For linear time invariant systems with uncertain initial data Luenberger developed a concept of observer which exponentially fast recovers complete set of system states using incomplete but accurate measurements of linear combinations of these states.

Generalization of this concept on nonlinear systems derives essential problems. When the estimate lies into a neighborhood of the real state as it appears in Baras, Bensoussan, and James [1], Busvelle and Gauthier [2] and Song and Grizzle [3], the convergence of the estimated state to the real state can be guaranteed. The design of observers for nonlinear and discrete-time systems can be found in [4]-[10]. In physics literature, the terminology of observer design has recently been used for estimating of parameters and states of nonlinear and chaotic systems, see [11, 12].

High-gain observers for nonlinear systems amplify influence of a linear block on observer performance by enlarging the absolute values of negative real parts of cor-
responding close-loop systems. This kind of observer design was first studied by Esfandiari and Khalil [13, 14]. In [15] Pinsky has developed a predictor for time series which is based on designing of observer with adaptive control gains. His approach blends techniques of observer design and numerical approximation of solutions to ODEs.

The accuracy delivered by high-gain observers is sensitive to measurement noise. Observer theory reveals that there is a trade-off between the magnitude of the steady state observation errors and the speed of their reconstruction [16]. As the observer gain is increased, the bandwidth of the observer is extended which frequently amplify the influence of measurement noise. It is shown in [17] that the error component due to modeling uncertainty can be attenuated by gain amplification which, in turn, overweights the effect of noise. To mitigate this issue, Prasov and Khalil [17] designed an observer switching the gain magnitudes between two levels.

Measurement noise and model uncertainties essentially influence the accuracy of derivative estimation. Levant [18] proposed a bound of the error in derivative estimation which depends upon noise magnitude. In this thesis, we shall extend the observer theory on estimation of derivatives of nonlinear systems solutions. Next, using these estimations, we forecast the solution of dynamical systems on short time-intervals.

This thesis is organized as follows. Chapter 2 describes design of an identity Luenberger observer for linear systems with unknown initial conditions. In Chapter 3, this methodology is extended on estimating the states of nonlinear systems with unknown initial data. Chapter 4 develops forecast technique, utilizing a combination of Taylor expansion and Aitken delta-square process. In the reminder of this chapter, we will discuss the estimation of Lorenz system. In Chapter 5, we apply the concepts of two high-gain observers—the identity and the reduced order observers to estimate the derivatives of partly unknown nonlinear systems subject to model uncertainties and measurement noises.
Chapter 2

Motivation of Identity Observer

2.1 Estimation of Linear Time-Invariant Systems

The most obvious approach to estimate the state of a known system is called trivial observer. Suppose we have a linear system with the known input $u(t)$ as below,

$$\dot{x}(t) = Ax(t) + Bu(t).$$  \hfill (2.1.1)

Then we create a copy of it,

$$\dot{z}(t) = Az(t) + Bu(t)$$ \hfill (2.1.2)

whose state provides an estimate $z(t)$ of the state $x(t)$ of the original system. If the initial state of the copy matches that of the original system, i.e. $z(0) = x(0)$, the copy can provide the exact value of the state of the initial system. Otherwise, the error is

$$\dot{e}(t) = \dot{x} - \dot{z} = A[x(t) - z(t)] = Ae(t).$$  \hfill (2.1.3)
If the matrix $A$ is Hurwitz—all the eigenvalues have negative real parts, then the error will decay to zero. Otherwise, (if there is an eigenvalue of $A$ with positive real part) a nonzero initial error will be amplified and approach infinity when time increases.

Let’s consider a more general situation in which a fully-observable, linear, time-invariant system with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, and output $y(t) \in \mathbb{R}^p$:

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]

\[
y(t) = Cx(t).
\]

The more general case in which the output also depends on the input, $y(t) = Cx(t) + Du(t)$, can be addressed in the same manner as outlined here. $y(t)$ gives us information about the state of the system, even if somewhat indirectly. In particular, let’s augment the trivial observer above by incorporating a term depending on the difference between the observed or measured variables and expected output matrix $L$ in the following structure,

\[
\dot{z}(t) = Az(t) + L[y(t) - Cz(t)] + Bu(t),
\]

where $L$ is $n \times p$ matrix. Eq. (2.1.6) is known as the identity observer. The dimension of observer gain matrix $L$ is comprised to obtain a term of the correct dimension. The choice of the observer gain matrix will also affect the dynamic behavior of the state estimate and thus the state error.

The differential equation describing the observer error can be obtained by substituting for the output vector $y(t)$, that is

\[
\dot{e}(t) = \dot{x}(t) - \dot{z}(t) = (A - LC)[x(t) - z(t)] = (A - LC)e(t).
\]

If matrix $L$ is chosen such that matrix $A - LC$ is Hurwitz, then the norm of observation
error vanishes exponentially fast in forward time. Moreover, the speed of error decay is determined by the largest real part of eigenvalues of matrix $A - LC$.

Note that if a pair of matrices $(A, C)$ is observable, then the eigenvalues of matrix $A - LC$ can be placed in arbitrary complex conjugate locations of the complex plane. Mainly, Matlab function “place” serves this task. This function is based on application of Ackermanns formula. In particular, the initial observer state need not match the initial system state, as long as the observer gain (the largest real part of eigenvalue) is chosen such that all eigenvalues of the matrix $A - LC$ are in the left half plane. In the light of this, we can guarantee the desired self-conjugate closed-loop eigenvalues locations. In Matlab, “place”, the algorithm computes a gain matrix $K$ such that the state feedback

$$u = -Kx$$

places the closed-loop eigenvalues at the locations $p$. In other words, the eigenvalues of $A - BK$ match the entries of $p$. “Ackermanns formula” is used to prove the state-feedback eigenvalue placement problem: given $A$ and $B$ and a monic $n$th order polynomial $\Delta_d(s)$, find a matrix $K$ that makes

$$\det(sI - (A - BK)) = \Delta_d(s).$$

For the convenience of readers, we spell out this theorem as below [19].

**Theorem 2.1.1** (Eigenvalue Placement Theorem). *Suppose that $(A, B)$ is controllable. There exists a matrix $K$ such that

$$\det(sI - (A - BK)) = s^n + d_{n-1}s^{n-1} + \cdots + d_0,$$

*for an arbitrary $d_i$, $i = 0, 1, \ldots, n - 1$.***
Proof. If \((A, B)\) is in controllable canonical form:

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}.
\]  

(2.1.10)

The matrix \(K = [k_0 \ k_1 \ \cdots \ k_{n-1}]\) does the trick, if \(k_i = d_i - a_i\), for \(i = 0, 1, \ldots, n - 1\).

Suppose that the pair is not in controllable canonical form (CCF). We will show that there exists an invertible matrix \(T\) such that

\[
TAT^{-1} = \tilde{A}, TB = \tilde{B}
\]  

(2.1.11)

where \(\tilde{A}\) and \(\tilde{B}\) are in CCF.

Define \(C = [A^{n-1}B \ A^{n-2}B \ \cdots \ AB \ B]\) and \(\tilde{C} = [\tilde{A}^{n-1}\tilde{B} \ \tilde{A}^{n-2}\tilde{B} \ \cdots \ \tilde{A}\tilde{B} \ \tilde{B}]\).

Because both \((A, B)\) and \((\tilde{A}, \tilde{b})\) are controllable, the matrices \(C\) and \(\tilde{C}\) are invertible. \(\tilde{C}\) has the form

\[
\tilde{C} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
* & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
* & * & * & * & 1 \\
* & * & * & * & 1
\end{bmatrix},
\]  

(2.1.12)

where * corresponds to elements whose exact values are not needed.

The claim is that the matrix \(T = \tilde{C}C^{-1}\) puts the matrix in the correct form. To show this, it suffices to show that

\[
TB = \tilde{B} \iff \tilde{C}C^{-1}B = \tilde{B}
\]  

(2.1.13)
\[ TAT^{-1} = \tilde{A} \iff C^{-1}AC = \tilde{C}-1\tilde{A}\tilde{C}^{-1} \quad (2.1.14) \]

To show Eq.(2.1.13), we note that

\[ C^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \]

because \( C^{-1}C = I \) and \( B \) is the last column of \( C \); thus, the product \( C^{-1}B \) gives the last column of the identity.

Now,

\[ \tilde{C}C^{-1}B = \tilde{C} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \tilde{B}, \]

by the form of \( \tilde{C} \).

Next, we show Eq.(2.1.14) can be satisfied. First, note that

\[
AC = [A^nB \ A^{n-1}B \ \cdots \ A^2B \ AB]
= [-a_0B - a_1AB - \cdots - a_{n-1}A^{n-1}B \ A^{n-1}B \ \cdots \ A^2B \ AB],
\]

where the last line came from application of the Cayley-Hamilton theorem.

Since the matrix \( AB \) is the second last column of \( C \); that \( A^2B \) is the third last
column of $C$, etc. Thus,

$$C^{-1} [A^{n-1}B \, \cdots \, A^2B \, AB] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$ 

Also, $-a_0B - a_1AB - \cdots - a_{n-1}A^{n-1}B$ is

$$-a_0[nth \, col. \, of \, C] - a_1[(n-1)st \, col. \, of \, C] - \cdots - a_1[1st \, col. \, of \, C].$$

Therefore,

$$C^{-1}(-a_0B - a_1AB - \cdots - a_{n-1}A^{n-1}B) = \begin{bmatrix} -a_{n-1} \\ -a_{n-2} \\ \vdots \\ -a_0 \end{bmatrix}.$$ 

So the left side of Eq.(2.1.14) is

$$C^{-1}AC = \begin{bmatrix} -a_{n-1} & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & \vdots & \vdots & \ddots & \vdots \\ -a_0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$ 

However, since the characteristic polynomial of $A$ and $\tilde{A}$ are the same, following the
exact same procedure leads to

\[
\tilde{C}^{-1}A\tilde{C} = \begin{bmatrix}
-a_{n-1} & 0 & 0 & \cdots & 0 \\
-a_{n-2} & 1 & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \vdots \\
-a_1 & \vdots & \ddots & \ddots & \vdots \\
-a_0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

follows Eq.(2.1.14) holds.

If \( \tilde{K} \) is the state feedback matrix satisfying

\[
\det(sI - (\tilde{A} - \tilde{B}\tilde{K})) = s^n + d_{n-1}s^{n-1} + \cdots + d_0,
\]

then \( K = \tilde{K}T \) makes

\[
\det(sI - (A - BK)) = \det(sT^{-1}T - [T^{-1}\tilde{A}T + [T^{-1}\tilde{B}][\tilde{K}T])
\]

\[
= \det(T^{-1}[sI - \tilde{A} + \tilde{B}\tilde{K}]T)
\]

\[
= \det(T^{-1} \det(sI - \tilde{A} + \tilde{B}\tilde{K}) \det T
\]

\[
= \det(sI - \tilde{A} + \tilde{B}\tilde{K})
\]

\[
= s^n + d_{n-1}s^{n-1} + \cdots + d_0,
\]

as required. \(\blacksquare\)

In Matlab, the algorithm \( K = \text{place}(A, B, p) \) computes a feedback gain matrix \( K \) that achieves the desired closed-loop eigenvalues locations \( p \), assuming all the inputs of the plant are control inputs. The length of \( p \) must match the row size of \( A \). “place” works for multi-input systems and is based on the algorithm from [19]. One can also use “place” for estimator gain selection \( L \) by transposing the \( A \) matrix and substituting \( C' \) for \( B \), i.e. \( l = \text{place}(A', C', p)' \).
Next, an example is discussed. In this example, the identity observer and pole placement algorithm are applied.

**Example 2.1.1.** Consider an ODE:

\[
\ddot{x} - 0.1 \dot{x} + x = 0. \tag{2.1.15}
\]

In order to convert this ODE into first order system, set \( x_1 = x \) and \( x_2 = \dot{x} \).

\[
\begin{aligned}
\dot{x}_1 &= \dot{x} = x_2, \\
\dot{x}_2 &= \ddot{x} = -x + 0.1 \dot{x} = -x_1 + 0.1x_2.
\end{aligned} \tag{2.1.16}
\]

The Eq.(2.1.16) is a linear system without input \( u(t) \). It can be expressed by matrix

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0.1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}. \tag{2.1.17}
\]

So we have \( \dot{x} = Ax \), where

\[
A = \begin{bmatrix}
0 & 1 \\
-1 & 0.1
\end{bmatrix}. \tag{2.1.18}
\]

Since \( x = x_1 \), we can let \( C = [1, 0] \) and \( p = [-10.0, -11.0] \). Using function place (See Appendix A) we get that \( A - LC \) is defined as follows:

\[
\begin{bmatrix}
-21.1 & 1 \\
-112.11 & 0.1
\end{bmatrix}. \tag{2.1.19}
\]

The eigenvalues of this matrix are: -11.0 and -10.0.

Suppose we have a output \( y(t) = Cx(t) \), then the observer system has following structure

\[
\dot{z}(t) = Az(t) + L[y(t) - Cz(t)]. \tag{2.1.20}
\]
Let the initial values of initial system be $x_1(0) = 3$ and $x_2(0) = 2$. Note that one can pick arbitrary initial conditions for $z_1(0)$ and $z_2(0)$. Figures 2.1 and 2.2 show that the estimation: $z_1$ and $z_2$ converge to real solutions $x_1$ and $x_2$ while the error vector converge to zero in forward time.

![Figure 2.1](image1.png)  
**Figure 2.1:** shows convergence of observer and initial system solutions emanated from different initial values: $z_1(0) = 10$ and $z_2(0) = -3$.

![Figure 2.2](image2.png)  
**Figure 2.2:** displays time-histories of observation errors: $e_1(0) = 10$ and $e_2(0) = -8$.

The identity observer can also be used to estimate the derivatives of solutions by
applying cascade design. That is, estimating the first derivative of $x_1$ by assuming $x_1$ is known. After that, we can estimate the second derivative of $x_1$ by using the estimation of the first derivative of $x_1$. Repeating such step inductively, we can generate the higher derivatives of $x_1$. For convenience, this procedure can be coupled and solved simultaneously. For instance, we can include the second derivative of $x_1$ (or first derivative of $x_2$) in the observer system Eq.(2.1.20). Note that $\dot{x}_1 = x_2$ and $\ddot{x}_1 = \dot{x}_2$. Let $x_2 = x_3$ and $\dot{x}_3 = x_4$; and let $R = LC$. Then Eq.(2.1.20) can be rewritten as follows:

$$
\begin{align*}
\dot{z}_1(t) &= z_3(t) + R_{11}(z_1(t) - y_1(t)) + R_{12}(z_2(t) - y_2(t)), \\
\dot{z}_2(t) &= z_4(t) + R_{21}(z_1(t) - y_1(t)) + R_{22}(z_2(t) - y_2(t)), \\
\dot{z}_3(t) &= z_4(t) + R_{31}(z_1(t) - y_1(t)) + R_{32}(z_2(t) - y_2(t)), \\
\dot{z}_4(t) &= -z_3(t) + 0.1z_4(t) + R_{41}(z_1(t) - y_1(t)) + R_{42}(z_2(t) - y_2(t)).
\end{align*}
$$

(2.1.21)

The code that solves this equation can be found in Appendix A.

Applying the pole placement Matlab function, we can place the eigenvalues in the following locations $p = [-100 \ -110 \ -120 \ -130]$ in order to speed up the error convergence to be zero. See the convergence of initial system and estimator trajectories (Figure 2.3) and error trajectories (Figure 2.4).
Figure 2.3: shows convergence of observer and initial system solutions emanated from different initial values: $z_1(0) = 3; z_2(0) = 2; z_3(0) = 5; z_4(0) = -8$.

Figure 2.4: displays time-histories of observation errors with initial values: $e_1(0) = 6; e_2(0) = -3; e_3(0) = -1; e_4(0) = 1$.

### 2.2 Estimation for Linear System with Uncertainties

In this section, we shall discuss the utility of identity observer in estimation of the linear system with unknown linear perturbation.
Consider a linear system with unknown linear part $\Delta x(t)$ without input $u(t)$ as below,

$$\dot{x}(t) = Ax(t) + \Delta x(t)$$  \hspace{1cm} (2.2.1)

and

$$y(t) = Cx(t).$$  \hspace{1cm} (2.2.2)

In this case the observer system Eq.(2.1.20) remains intact and can be rewritten as follows:

$$\dot{z}(t) = Az(t) + LCe(t).$$

However, the error equations, Eq.(2.1.7) is modified as follow:

$$\dot{e}(t) = (A - LC)e(t) + \Delta(z(t) + e(t)).$$  \hspace{1cm} (2.2.3)

Note that the pair $(A, LC)$ is observable. An example presented below demonstrates that in this case the norm of the error can grow in time regardless of the location of eigenvalues of matrix $[A - LC]$. Let’s consider Example 2.2.1.

**Example 2.2.1.** We still use the system in Example 2.1 but add an unknown linear matrix as the perturbation

$$D = \begin{bmatrix} 0 & 0 \\ -d & r \end{bmatrix},$$  \hspace{1cm} (2.2.4)

where $d$ and $r$ can be considered as outside force and friction.

Then Eq.(2.1.17) can be rewritten into

$$\dot{x}(t) = Ax(t) + Dx(t) = \begin{bmatrix} 0 & 1 \\ -(1 + d) & (0.1 + r) \end{bmatrix} = \tilde{A}x.$$  \hspace{1cm} (2.2.5)

Let $d = 10$ and $r = 0.8$; let $x_1(0) = 3$; $x_2(0) = 2$. Note that the error increases as $t \to \infty$, see Figure 2.5.
Figure 2.5: shows the divergence of time-histories of observation errors with initial values: $e_1(0) = 10; e_2(0) = -8; z_1(0) = 6; z_2(0) = -3$.

Since there is one zero or two zero eigenvalues of $LC$ in this case, uncertainties can alter system stability and thus $e_2(t)$ is not decaying. This fact reveals that the identity observer fails to provide the estimation of solutions (derivatives) for linear system with uncertainties. Hence, we outline below the application of the reduced order observer to this problem.
Chapter 3

Estimation of Derivatives for Nonlinear Systems

3.1 The Attempt of Identity Observer

Before we start to introduce the reduced order observer, the efficiency of the identity observer should be noticed. Although the identity observer has already offered a satisfactory solution to estimation of states that are not directly measurable, its computational speed can be enhanced. Suppose that the length of our state vector is large, and that \( n - p \) is small. As pointed out by Luenberger, the computational load required to provide estimates of all \( n \) variables may be unacceptable.

First, apply the concept of identity observer to estimate the solutions to nonlinear system:

\[
\dot{x}(t) = Ax + f(x, t),
\]  

(3.1.1)

where \( f(x, t) \) is the nonlinear part.
Then using the output \( y(t) = Cx(t) \), the observer can be written as:

\[
\dot{z}(t) = Az(t) + L(y(t) - Cz(t)) + f(z(t), t).
\] (3.1.2)

Then the error equation takes the form

\[
\dot{e}(t) = (A - LC)e(t) + f(z(t) + e(t), t) - f(z(t), t).
\] (3.1.3)

Note that \( y(t) - Cz(t) = Cx(t) - Cz(t) = Ce(t) \). Thus the system coupling error and observer dynamics can be written as:

\[
\begin{cases}
\dot{e}(t) = (A - LC)e(t) + f(z(t) + e(t), t) - f(z(t), t), \\
\dot{z}(t) = Az(t) + LCe(t) + f(z(t), t).
\end{cases}
\] (3.1.4)

Let’s design the identity observer for the Van der Pol system.

**Example 3.1.1** (Van der Pol system). Write the differential equation for this system in the form:

\[
\ddot{x} = mx + q\dot{x} + q_3\dot{x}^3 + q_5\dot{x}^5 + m_3x^3 + m_5x^5,
\] (3.1.5)

where \( m, m_3, m_5, q_3, q_5 \) are constant.

Let \( x_1 = x \) and \( x_2 = \dot{x} \). Eq.(3.1.4) can be rewritten into

\[
\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = mx_1 + qx_2 + q_3x_2^3 + q_5x_2^5 + m_3x_1^3 + m_5x_1^5.
\end{cases}
\] (3.1.6)

For the convenience of computation, we write Eq.(3.1.5) into matrix form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\frac{m}{q} & -\frac{q}{m}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
q_3x_2^3 + q_5x_2^5 + m_3x_1^3 + m_5x_1^5
\end{bmatrix}.
\] (3.1.7)
Solve Eq.(3.1.4) for the parameters: \( m = -0.1, q = 0.1, m_3 = m_5 = q_5 = 0 \) and \( q_3 = -0.1 \), (see Appendix C); let \( t \in [0, 200] \). We find out that on the time interval \( t \in [0, 200] \) the error is oscillating in a very small range, see Figure 3.1.

Figure 3.1: displays the time-histories of observation errors with initial values: \( e_1(0) = 2; e_2(0) = -3; z_1(0) = 10; z_2(0) = -8 \).

However, if we add the linear perturbation \( D(x) \) for this Van der Pol system, like we did in Example 2.2.1, the error is not decaying. To check this, we set \( d = 10 \) and \( r = 0.5 \), see Figure 3.2.

Figure 3.2: displays the time-histories of observation errors with initial values: \( e_1(0) = 2; e_2(0) = -3; z_1(0) = 10; z_2(0) = -8 \) cannot decay to zero.
In the linear system without uncertainties (Example 2.1.1), the estimation errors are exponentially decaying to zero if we use small real part of the eigenvalues of the linear matrix $A - LC$. However, in the nonlinear system without uncertainties, the estimation errors do not converge to zero but oscillate in a relatively small interval. For the (linear or nonlinear) systems with uncertainties, the observation errors are not decaying or oscillating in a relatively small scale. But in the best case it would normally be bounded by a small value.

### 3.2 Luenberger Reduced Order Observer

To introduce the reduced order observer, we again consider the system

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t).
\end{aligned} \tag{3.2.1}$$

Apply a transformation matrix $P$ to the state $x(t)$ to reorder the states such that

$$\begin{bmatrix}
w(t) \\
y(t)
\end{bmatrix} = \tilde{x} = Px(t) = \begin{bmatrix} T \\ C \end{bmatrix} x(t). \tag{3.2.2}$$

The existence of the matrix $T$ that yields a nonsingular transformation matrix $P$ is guaranteed by the condition that $C$ has full row rank $p$.

Let us partition the matrices $A$ and $B$ in the obvious manner and write Eq.(3.2.2) into the form

$$\begin{bmatrix}
\dot{w}(t) \\
\dot{y}(t)
\end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} w(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t). \tag{3.2.3}$$

It is assumed below that $u(t) = 0$. Pretend that $L$ is a matrix gain which multiplies the output vector $y(t)$. Now, we derive the following equation for $w(t)$ and $y(t)$
separately, we construct the state equation of the difference $w(t) - Ly(t)$:

$$\dot{w}(t) - L\dot{y}(t) = (A_{11} - LA_{21})w(t) + (A_{12} - LA_{21})y(t).$$  \hspace{1cm} (3.2.4)

Rewrite Eq.(3.2.4) into the following form:

$$\dot{w}(t) - L\dot{y} = (A_{11} - LA_{21})[w(t) - Ly(t)] + (A_{11}L - LA_{21}L + A_{12} - LA_{22})y(t).$$  \hspace{1cm} (3.2.5)

Let $v(t) = w(t) - Ly(t)$, we have

$$\dot{v}(t) = (A_{11} - LA_{21})v(t) + (A_{11}L - LA_{21}L + A_{12} - LA_{22})y(t).$$  \hspace{1cm} (3.2.6)

The vector $v(t)$ represents the evolution of unmeasurable components of the initial state vector, $w(t)$ and $y(t)$. Now the observer equation can be written in the form:

$$\dot{z}(t) = (A_{11} - LA_{21})z(t) + A_{11}L - LA_{21}L + A_{12} - LA_{22})y(t).$$  \hspace{1cm} (3.2.7)

Next, the error equations are:

$$\dot{e}(t) = \dot{z}(t) - \dot{v}(t) = (A_{11} - LA_{21})[z(t) - v(t)] = (A_{11} - LA_{21})e(t).$$  \hspace{1cm} (3.2.8)

Now, if a pair of matrices $(A_{11}, A_{21})$ is observable, then the eigenvalues of matrix $A_{11} - A_{21}$ can be placed in arbitrary positions on the complex plane. This allows us to choose $L$ such that the error norm exponentially approaches zero in forward time.

Recalling that $v(t) = w(t) - Ly(t)$, we notice that the estimation for the original state-vector $\hat{x}(t)$ can be obtained as

$$\hat{x}(t) = \begin{bmatrix} \hat{w}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} z(t) + Ly(t) \\ y(t) \end{bmatrix}.$$  \hspace{1cm} (3.2.9)
Similarly, we can design a reduced order observer for nonlinear systems.

\[ \dot{x}(t) = Ax(t) + f(x, t), \quad \text{(3.2.10)} \]

(where \( f(x, t) \) is the nonlinear part) as follow. Partitioning matrix \( A \) into four blocks, that is

\[
\begin{bmatrix}
\dot{w} \\
\dot{y}
\end{bmatrix}
= \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
w \\
y
\end{bmatrix}
+ \begin{bmatrix}
f_1(w, y) \\
f_2(w, y)
\end{bmatrix}. \quad \text{(3.2.11)}
\]

Write the equation for the difference \( w(t) - Ly(t) \):

\[
\dot{w} - L\dot{y} = (A_{11} - LA_{21})(w - Ly) + (A_{11}L - LA_{21}L + A_{12} - LA_{22})y + f_1(w, y) - Lf_2(w, y). \quad \text{(3.2.12)}
\]

Defining as previously \( v(t) = w - Ly, w = v + Ly \), we obtain

\[
\dot{v} = (A_{11} - LA_{21})v + (A_{11}L - LA_{21}L + A_{12} - LA_{22})y + f_1(v + Ly, y) - f_1(z + Ly, y).
\quad \text{(3.2.13)}
\]

So the observer system is

\[
\dot{z} = (A_{11} - LA_{21})z + (A_{11}L - LA_{21}L + A_{12} - LA_{22})y + f_1(z + Ly, y) - Lf_2(z + Ly, y).
\quad \text{(3.2.14)}
\]

Note that \( e = v - z, v = e + z \), we write the error system as follows:

\[
\dot{e} = (A_{11} - LA_{21})e + f_1(e + z + Ly, y) - f_1(z + Ly, y)
- L(f_2(e + z + Ly) - f_2(z + Ly, y)). \quad \text{(3.2.15)}
\]
Example 3.2.1. Consider $2 \times 2$ nonlinear system (Van der-Pol) with,

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= mx_1 + qx_2 + q_3x_2^3 + q_5x_2^5 + m_3x_1^3 + m_5x_1^5.
\end{align*}
\] (3.2.16)

In order to estimate the second and third derivative of $x_1$, we can differentiate Eq. (3.2.16) as below. Let $\dot{x}_2 = x_3$. Denote $x_1^{(3)} = \ddot{x}_2 = \dot{x}_3$ as $x_4$, we obtain

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= mx_3 + qx_4 + 3q_3(2x_2x_2^3 + x_2^3x_4) + 5q_5(4x_3^3x_3 + x_2^5x_3) \\
&\quad + 3m_3(2x_1x_2^2 + x_1^2x_3) + 5m_5(4x_1^3x_2 + x_1^4x_3).
\end{align*}
\] (3.2.17)

Express the linear parts by matrices. Let $x_1 = y_1$, $\dot{x}_1 = x_2 = y_2$. Denote $x_3$ and $x_4$ as $w_1$ and $w_2$. Then the linear part of Eq. (3.2.17) can be rewritten in the form

\[
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
m & q & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
y_1 \\
y_2
\end{bmatrix}.
\] (3.2.18)

Partition the previous matrix in four $2 \times 2$ below blocks:

\[
A_{11} = \begin{bmatrix} 0 & 1 \\ m & q \end{bmatrix}, A_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]
Next, we note that

\[ f_1(w, y) = 3q_3(2y_2w_1^2 + y_2^2w_2) + 5q_5(4y_2^3w_1^2 + y_2^4w_2) + 3m_3(2y_1y_2^2 + y_1^2w_1) + 5m_5(4y_1^3y_2 + y_1^4w_1). \]  \hspace{1cm} (3.2.19)

\[ f_2(w, y) = 0. \]

For the code solving the coupled Eq.(3.2.14) and Eq.(3.2.15), see Appendix D.

If \( m = -0.1, \ q = 0.1, \ q_3 = -0.1 \) and \( m_3 = m_5 = q_5 = 0 \), we shall obtain a double-cycle system, see Figure 3.5. The observation errors are not decaying because \( e(t) = 0 \) is not a solution to the above error system. However, the errors can be reduced to a very small scale by adjusting gains, see Figure 3.4. Note that selection of the initial values is arbitrary in this case. In Figure 3.3, \( w_1 \) and \( w_2 \) represent the estimations of the first derivatives of \( y_1 \) and \( y_2 \) respectively.

![Figure 3.3](image-url)

Figure 3.3: shows the estimation derivatives of \( y_1 \) and \( y_2 \) in double-cycle system.

If \( m = -0.1, \ q = -0.1, \) and \( q_3 = q_5 = -0.1 \), then we shall have 1-cycle system,
Figure 3.4: displays the observation errors can be reduced to a very small scale.

Figure 3.5: Phase planes for cross sections $y_1 - y_2$ and $w_1 - w_2$.

see Figure 3.8. In this case, the error is also very small, see Figure 3.7. In Figure 3.6, $w_1$ and $w_2$ are the estimators of the first derivatives of $y_1$ and $y_2$ respectively. This case shows that the observation error can be further reduced to zero if the system is stable.

The above example shows that we can use two known states $y_1$ and $y_2$ to estimate
the derivatives of the solutions. Furthermore, we can apply cascade design to generate
the higher order derivatives by following steps. Using \( w_1 \) and \( w_2 \) as the known states
to estimate 5th and 6th derivatives \( w_3 \) and \( w_4 \) as follows:

\[
\begin{bmatrix}
\dot{w}_3 \\
\dot{w}_4 \\
\dot{w}_1 \\
\dot{w}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
m & q & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w_3 \\
w_4 \\
w_1 \\
w_2
\end{bmatrix}.
\] (3.2.20)

Repeating inductively on \( k \), we can obtain the \( k + 4 \)th and \( k + 5 \)th derivatives \( w_{k+2} \) and \( w_{k+3} \) by assuming that \( w_k \) and \( w_{k+1} \) are known.

\[
\begin{bmatrix}
\dot{w}_{k+2} \\
\dot{w}_{k+3} \\
\dot{w}_k \\
\dot{w}_{k+1}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
m & q & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w_{k+2} \\
w_{k+3} \\
w_k \\
w_{k+1}
\end{bmatrix}.
\] (3.2.21)
Figure 3.6: shows the estimation derivatives of $y_1$ and $y_2$ in 1-cycle system.

Figure 3.7: displays the observation errors can be reduced to a very small number.
3.3 Nonlinear System with Linear Uncertainties

In this section we design a reduced order observer to nonlinear systems under uncertainties which are considered in simulations as linear perturbations:

\[
\begin{align*}
    \dot{x} &= Ax + Dx + f(x, t), \\
    y &= Cx.
\end{align*}
\]  

(3.3.1)

where matrix \( D \) models unknown dynamics.

Partitioning the matrices \( A \) and \( D \) in the obvious manner to obtain

\[
\begin{bmatrix}
    \dot{w}(t) \\
    \dot{y}(t)
\end{bmatrix} =
\begin{bmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
    w(t) \\
    y(t)
\end{bmatrix} +
\begin{bmatrix}
    D_{11} & D_{12} \\
    D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
    w(t) \\
    y(t)
\end{bmatrix} +
\begin{bmatrix}
    f_1(w, y) \\
    f_2(w, y)
\end{bmatrix}.
\]

(3.3.2)
Write the equation of the difference $w(t) - Ly(t)$ in the form

$$
\dot{w}(t) - Ly(t) = (A_{11} - LA_{21})w(t) + (A_{12} - LA_{21})y(t) \\
+ (D_{11} - LD_{21})w(t) + (D_{12} - LD_{21})y(t) \\
+ f_1(v + Ly, y) - Lf_2(v + Ly, y).
$$

(3.3.3)

Rewrite Eq.(3.3.3) into

$$
\dot{w}(t) - Ly = (A_{11} - LA_{21} + D_{11} - LD_{21})(w(t) - Ly(t)) \\
+ (A_{11}L - LA_{21}L + A_{12} - LA_{22} + D_{11}L - LD_{21}L + D_{12} - LD_{22})y(t) \\
+ f_1(v + Ly, y) - Lf_2(v + Ly, y).
$$

(3.3.4)

Let $v(t) = w(t) - Ly(t)$, we have

$$
\dot{v}(t) = (A_{11} - LA_{21} + D_{11} - LD_{21})v(t) \\
+ (A_{11}L - LA_{21}L + A_{12} - LA_{22} + D_{11}L - LD_{21}L + D_{12} - LD_{22})y(t) \\
+ f_1(v + Ly, y) - Lf_2(v + Ly, y).
$$

(3.3.5)

Thus, the observer system takes the form:

$$
\dot{z}(t) = (A_{11} - LA_{21})z(t) + (A_{11}L - LA_{21}L + A_{12} - LA_{22})y(t) \\
+ f_1(z + Ly, y) - Lf_2(z + Ly, y).
$$

(3.3.6)

Consequently, the error system is

$$
\dot{e}(t) = \dot{z}(t) - \dot{v}(t) = (A_{11} - LA_{21})e(t) + (D_{11} - LD_{21})z(t) \\
+ (D_{11}L - LD_{21}L + D_{12} - LD_{22})y(t) + f_1(e + z + Ly, y) - f_1(z + Ly, y) \\
- L(f_2(e + z + Ly) - f_2(z + Ly, y)).
$$

(3.3.7)

Since $L$ is an arbitrary matrix, the task is to find $L$ such that the error norm either
asymptotically approach zero or a small steady state value. Below we consider designing of estimators for two nonlinear models which are frequently found in applications.

**Example 3.3.1.** Consider nonlinear system (Van der-Pol equation with linear perturbations) which are treated in simulations as uncertainties:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= (m + d)x_1 + (q + r)x_2 + q_3x_2^3 + q_5x_2^5 + m_3x_1^3 + m_5x_1^5.
\end{align*}
\]  

(3.3.8)

Here \(d\) and \(r\) simulate uncertain values. Write the linear part of these equations in matrix form: Let \(x_1 = y_1, \dot{x}_1 = x_2 = y_2\). Denote \(x_3\) and \(x_4\) as \(w_1\) and \(w_2\). Then Eq.(3.2.18) can be rewritten into

\[
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
\begin{bmatrix}
m & q \\
0 & 0
\end{bmatrix} & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
y_1 \\
y_2
\end{bmatrix} +
\begin{bmatrix}
0 & 0 & 0 & 0 \\
\begin{bmatrix}
d & r \\
0 & 0
\end{bmatrix} & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
y_1 \\
y_2
\end{bmatrix}.
\]  

(3.3.9)

Partition these matrices into four \(2 \times 2\) blocks:

\[
A_{11} = \begin{bmatrix} 0 & 1 \\ m & q \end{bmatrix}, A_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

and

\[
D_{11} = \begin{bmatrix} 0 & 0 \\ d & r \end{bmatrix}, D_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, D_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Note that the corresponding equations are similar to one derived for Example 3.2.1. Let \(m = -0.1, q = 0.1, q_3 = -0.1\) and \(m_3 = m_5 = q_5 = 0\); let \(d = -10\) and \(r = 0.5\). The initial data can be arbitrary chosen. Let the largest real part of the eigenvalues of \([A_{11} - LA_{21}]\) be sufficiently small such that the corresponding graph of the error
The further reduction of observation error can be achieved by further decreasing the largest real part of eigenvalues of the matrix \([A_{11} - LA_{21}]\). However, that will make the corresponding system very stiff which will greatly amplify the computation time. Next, let us show the observation errors inversely correlate with the largest real part of the eigenvalues of this matrix by selecting different eigenvalues as follows:

- \(p = [-100, 110]\), the range of the error is \([-2.5, 2.5]\).
- \(p = [-500, 510]\), the range of the error is \([-0.5, 0.5]\).
- \(p = [-1000, 1100]\), the range of the error is \([-0.25, 0.25]\).
- \(p = [-10000, -11000]\), the range of the error is \([-0.04, 0.04]\).

However, it should be noted that the errors will grow if the uncertainties become large enough.
For the stable nonlinear system, stabilization of the error equations can be a relatively easy problem. For instance, if the system is 1-cycle (see Figure 3.12), i.e. the solutions to the system are decaying (see Figure 3.10), the solutions to the error system will converge to zero, see Figure 3.11.

Figure 3.10: shows the derivatives of $y_1$ and $y_2$ in 1-cycle system with uncertainties.

In the next chapter, we shall estimate up to 6th derivative in order to produce the forecast of the solution, and all the designs of the observer systems will rely on the reduced order observer.
Figure 3.11: displays the time-histories observation errors are decaying in 1-cycle system with uncertainties.

Figure 3.12: Phase planes for cross sections $y_1$-$y_2$ and $w_1$-$w_2$. 
Chapter 4

Forecasting Behaviors of Nonlinear Systems on Short-Time Intervals

4.1 Double-Well Chaotic System

Chaos theory studies the behavior of dynamical systems that are highly sensitive to initial conditions. Small differences in initial conditions (such as those due to rounding errors in numerical computations) yield widely diverging outcomes for such dynamical systems, and rendering long-term prediction impossible in general. This theory is as summarized by Lorenz: When the present determines the future, but the approximate present does not approximately determine the future.

Since the sensitivity of solutions of chaotic system to small perturbation, its usually impossible to numerically forecast this system behavior on a long time interval. Furthermore, the forecast of solutions of the system, even in a short time interval, sounds like a challenge. Therefore, in this thesis, we test the accuracy of short-time forecast of solutions to chaotic system by using Taylor series expansion. We are going to generate this expansion using estimation of high order derivatives of states of partly unknown nonlinear systems as it has been outlined in the preceding chapters.
Consider a double-well system

\[ \ddot{x} + \delta \dot{x} - x + x^3 = F \cos \omega t. \]  

(4.1.1)

Let \( x_1 = x, x_2 = \dot{x} \). Then we can write above equation as a first order system

\[
\begin{cases}
    \dot{x}_1 = x_2, \\
    \dot{x}_2 = x_1 - \delta x_2 - x_1^3 + F \cos \omega t.
\end{cases}
\]

(4.1.2)

First, let’s find up to the 6th derivatives of \( x_1 \). This estimation comprises of the following steps.

(1) Let \( \dot{x}_2 = x_3 \). We just need \( x_1, x_2, x_3, \) since \( \dddot{x}_1 = \dot{x}_2 = x_3 \).

\[
\begin{cases}
    \dot{x}_1 = x_2, \\
    \dot{x}_2 = x_3, \\
    \dot{x}_3 = x_2 - \delta x_3 - 3x_1^2 x_2 - F \omega \sin \omega t.
\end{cases}
\]

(4.1.3)

Applying the reduced order observer. Let \( x_1 = y_1, \dot{y}_1 = y_2, \dot{y}_2 = w \).

\[
\begin{bmatrix}
\dot{w} \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} =
\begin{bmatrix}
-\delta & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w \\
y_1 \\
y_2
\end{bmatrix} +
\begin{bmatrix}
-3y_1^2 y_2 - F \omega \sin \omega t \\
0 \\
0
\end{bmatrix}.
\]

(4.1.4)

Partitioning the linear part matrix into blocks:

\[
A_{11} = -\delta, A_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

(2) We can also get an estimator of \( y_2^{(2)} \) or \( y_1^{(3)} \) by applying two known \( y_1 \) and \( y_2 \).
Let’s pretend we also don’t know \( \dot{y}_2 \). Extending Eq. (4.1.3), we have

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= x_3 - \delta x_4 - 6x_1 x_2^2 - 3x_1^2 x_3 - F \omega^2 \cos \omega t.
\end{align*}
\]

Let \( x_1 = y_1, \dot{y}_1 = y_2, \dot{y}_2 = w_1, \dot{w}_1 = \dot{y}_2 = w_2 \). Then we have

\[
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & -\delta & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
y_1 \\
y_2
\end{bmatrix} + \begin{bmatrix}
0 \\
-6y_1 y_2^2 - 3y_1^2 w_1 - F \omega^2 \cos \omega t \\
0 \\
0
\end{bmatrix}.
\]

(4.1.6)

Partitioning the linear part matrix into blocks:

\[
A_{11} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

Then the observer system will be

\[
\dot{z} = (A_{11} - L A_{21}) z + (A_{11} L - L A_{21} L + A_{12} - L A_{22}) y + f_1 (w, y),
\]

(4.1.7)

where

\[
w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} z_1 + L(1,1)y_1 + L(1,2)y_2 \\ z_2 + L(2,1)y_1 + L(2,2)y_2 \end{bmatrix}.
\]

(4.1.8)
The error system is \((e = v - z, v = e + z)\)

\[
\dot{e} = (A_{11} - LA_{21})e + f_1(e + z + Ly, y) - f_1(z + Ly, y).
\]

(4.1.9)

(3) Likewise, to get estimators of \(x^{(4)}, x^{(5)}\) and \(x^{(6)}\) with two known \(y_1\) and \(y_2\), we can differentiate Eq.(4.1.5) as follows,

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= x_5, \\
\dot{x}_5 &= x_6, \\
\dot{x}_6 &= x_7, \\
\dot{x}_7 &= x_6 - \delta x_7 - 90x_2x_3^2 - 60x_2^2x_4 - 60x_1x_3x_4 - 30x_1x_2x_5 - 3x_1^2x_6 - F\omega^5 \sin \omega t.
\end{align*}
\]

(4.1.10)

Let \(x_1 = y_1, \dot{y}_1 = y_2, \dot{y}_2 = w_1, \dot{w}_1 = \ddot{y}_2 = w_2, \dot{w}_2 = w_3\). Then we have

\[
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{w}_3 \\
\dot{w}_4 \\
\dot{w}_5 \\
\dot{y}_3 \\
\dot{y}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -\delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5 \\
y_1 \\
y_2
\end{bmatrix}.
\]

(4.1.11)
The nonlinear part is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-90y_2 w_1^2 - 60y_2^2 w_2 - 60y_1 w_1 w_2 - 30y_1 y_2 w_3 - 3y_1^2 w_4 - F \omega^5 \sin \omega t \\
0 \\
0
\end{bmatrix}.
\]  
(4.1.12)

Partitioning the linear part matrix into below blocks:

\[
A_{11} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -\delta
\end{bmatrix}, \\
A_{21} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}, \\
A_{12} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}, \\
A_{22} = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}.
\]

So the observer system is

\[
\dot{z} = (A_{11} - LA_{21}) z + (A_{11} L - L A_{21} L + A_{12} - L A_{22}) y \\
+ f_1(w, y),
\]  
(4.1.13)

where

\[
w = \begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5
\end{bmatrix} = \begin{bmatrix}
z_1 + L(1, 1)y_1 + L(1, 2)y_2 \\
z_2 + L(2, 1)y_1 + L(2, 2)y_2 \\
z_3 + L(3, 1)y_1 + L(3, 2)y_2 \\
z_4 + L(4, 1)y_1 + L(4, 2)y_2 \\
z_5 + L(5, 1)y_1 + L(5, 2)y_2
\end{bmatrix}.
\]  
(4.1.14)
The error system is

\[ \dot{e} = (A_{11} - L A_{21}) e + f_1(e + z + L y, y) - f_1(z + L y, y). \]  

(4.1.15)

Equations Eq.(4.1.13), Eq.(4.1.14) and Eq.(4.1.15) can be solved simultaneously, see Appendix E.

Let the parameters be \( \delta = 0.25, m = 1, F = 0.18 \) and \( \omega = 1 \). We plot the solutions of observer system (Figure 4.1); the error system (Figure 4.2); the phase planes (Figure 4.3).

Figure 4.1: shows the estimations of up to the 6th derivatives of \( y_1 \).

Let \( \delta = 0.25, m = 1, F = 0.4 \) and \( \omega = 1 \). We plot the graph of the chaotic system (Figure 4.4); error system (Figure 4.5); phase planes (Figure 4.6). For these parameter values, the system has one chaotic attractor, corresponding to forced oscillations
Figure 4.2: displays the time-histories observation errors are oscillating on a very small interval.

Figure 4.3: Phase planes for cross sections $y_1-y_2; w_1-w_2; w_2-w_3; w_3-w_4; w_4-w_5$. 
confined to the left or right well.

Figure 4.4: shows the up to 6th derivatives of $y_1$ in double-well chaotic system.

Figure 4.5: displays that the time-histories observation errors are oscillating on a very small interval.
Figure 4.6: Phase planes for cross sections $y_1-y_2$; $w_1-y_2$; $w_1-w_2$; $w_2-w_3$; $w_3-w_4$; $w_4-w_5$ in double-well chaotic system.

4.2 Reduced Order Observer for Double-Well Chaotic System Under Linear Uncertainty

In this section, we design estimators for the double-well chaotic system with uncertainties.

First, assume that the linear perturbations are modeled by adding an appropriate
term to equations (4.1.11) as it is done in section 3.3.

\[ D = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d & r & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]  

(4.2.1)

such that

\[ D_{11} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & r \\
\end{bmatrix}, \]

and other blocks \( D_{12}, D_{21} \) and \( D_{22} \) are zero.

Suppose that we use the same parameters as in previous sections. In the following simulations, we assume that the uncertain parameters are \( d = -10, r = -0.5 \) and the eigenvalues are \( p = [-100, -110, -120, -130, -140] \). For this purpose, one can readily adopt the same strategy presented in Example 3.3.1. Figure 4.7 shows time-histories of components of error vector corresponding to this estimation. Although the error does not converge to zero in this case, it remains bounded by small value which can be further decreased by adjusting the gain matrix. The estimation errors turn out to be relatively small in this case due to self-cancellation of \( z(t) + Ly(t) \). For instance,

- \( p = [-10, -15, -20, -25, -30] \), the bound of the error is \([-0.4, 0.4] \).
- \( p = [-50, -60, -70, -80, -90] \), the bound of the error is \([-0.05, 0.05] \).
4.3 Short Forecast of Solutions to Process and Prediction of Double-Well Chaotic System

In this section, we will use Taylors series and employ the derivatives we have estimated in the preceding section to forecast the behavior of the solutions in short time intervals. In addition, we shall use a special summation formula—Aitken $\delta^2$ process to increase the speed of convergence and the prediction accuracy.

The Aitken $\delta^2$ process is the most famous as well as the oldest nonlinear sequence transformation. It can accelerate linearly convergent sequences like Taylor series. The method can be concluded as below: Given a sequence $x = (x_n)_{n \in \mathbb{N}}$, one associates
this sequence to the new sequence

\[ A(x) = \left( \frac{x_{n+2}x_n - x_{n+1}^2}{x_{n+2} - 2x_{n+1} + x_n} \right)_{n \in \mathbb{N}}, \quad (4.3.1) \]

which normally will converge faster than the initial sequence. The above sequence can also be written as

\[ A(x_n) = x_n - \frac{(\delta x_n)^2}{\delta^2 x_n}, \quad (4.3.2) \]

where

\[ \delta x_n = x_{n+1} - x_n \quad (4.3.3) \]

and

\[ \delta^2 x_n = x_n - 2x_{n+1} + x_{n+2}. \quad (4.3.4) \]

Aitken’s algorithm [20] can be described as follows. \( x_n \) represents Aitken’s accelerated approximate solution, \( x_{n+1} \) and \( x_{n+2} \) are two iterations of the simulation start at \( x_1 \).

1. Choose an initial approximate solution \( x_n \), a tolerance \( \epsilon \) and a maximal number of iterations \( N \). Set \( n = 0 \).

2. Compute \( x_{n+1} = f(x_n) \) and \( x_{n+2} = f(x_{n+1}) \).

3. Update \( x_n \) with

\[ x_n = x_n - \frac{(\delta x_n)^2}{\delta^2 x_n}. \]

4. Increment \( n \).

5. If \( |f(x_n) - x_n| > \epsilon \) and \( n < N \) then go to step 2.

6. If \( |f(x_n) - x_n| > \epsilon \) and \( n > N \) then a warning that the maximal number of iterations has been reached without the desired convergence should be printed.
(7) Output the approximation $x_n$.

Suppose the estimators of the $(i + 1)$th derivative is $w_i$ (for $i = 1, \ldots, 4$). Consider $y_1$ and $y_2$ are the original function and 1st derivative. Then we can have the summands of Taylor series as below,

$$S_1 = y_1 + y_2(x - a) + \frac{w_1}{2!}(x - a)^2 + \frac{w_2}{3!}(x - a)^3 + \frac{w_3}{4!}(x - a)^4, \quad (4.3.5)$$

$$S_2 = S_1 + \frac{w_4}{5!}(x - a)^5, \quad (4.3.6)$$

$$S_3 = S_2 + \frac{w_5}{6!}(x - a)^6. \quad (4.3.7)$$

Apply the $\delta^2$ process on $S_1$, $S_2$ and $S_3$, we can generate a new function

$$A_1 = S_1 - \frac{(S_2 - S_1)^2}{S_1 - 2S_2 + S_3}. \quad (4.3.8)$$

Then $A_2$ can be produced by using $S_2$, $S_3$ and $A_1$,

$$A_2 = S_2 - \frac{(S_3 - S_2)^2}{S_2 - 2S_3 + A_1}. \quad (4.3.9)$$

Similarly, use $S_3$, $A_1$ and $A_2$, we have

$$A_3 = S_3 - \frac{(A_1 - S_3)^2}{S_3 - 2A_1 + A_2}. \quad (4.3.10)$$

Repeat above procedure inductively, the new sequence is

$$A_n = A_{n-3} - \frac{(A_{n-2} - A_{n-1})^2}{A_{n-3} - 2A_{n-2} + A_{n-1}}, \quad (4.3.11)$$

where $n \geq 4$.

So for sufficiently large $n$, $A_n$ frequently represents a better approximation of the original unknown solution function.
Let us test the prediction accuracy of this approach. For the corresponding Matlab code, see Appendix F. Thus, we can implement below codes into the program in section 4.1 (double-well chaotic system). For example, let’s examine the accuracy of double-well chaotic system in a short interval around \( t = 34 \), see Appendix F.

In this simulation, the system parameters are as follows: \( \delta = 0.25, m = 1, F = 0.4 \) and \( \omega = 1 \). In a short period around \( t = 34 \), the accuracy of prediction for double-well system and one with uncertainties are shown in the Figure 4.8. The blue prediction curve can shed some light on the behavior of the chaotic system.

![Figure 4.8](image.png)

Figure 4.8: The blue curve is the real solution function and red curve is the prediction. In a short time interval \([33, 35]\), the forecast is relatively accurate.

If the uncertainties \( d = -10 \) and \( r = -0.5 \) are added to the system, we can apply the same method in section 4.2. The prediction is as described in Figure 4.8. The relatively accurate prediction is possible for both systems in relatively short time interval of length 2 in both cases due to the chaotic nature of the system solutions.
4.4 Lorenz System

One of the most well-known chaotic dynamic system is Lorenz system [21], which is derived by Lorenz in 1963. Lorenz discovered that this simple-looking deterministic system could have chaotic dynamics over a wide range of parameters, the solutions oscillate irregularly, never exactly repeating themselves but always remaining in a bounded region of phase space. When he plotted the trajectories in three dimensions, he discovered that they settled onto a strange attractor. Unlike stable fixed points and limit cycles, the strange attractor is not a point or a curve or even a surface—it’s a fractal, with a fractional dimension around 2.06. The estimation of solutions to Lorenz system especially the derivatives look like a challenging problem due to the chaotic nature of the underlying system.
Consider the Lorenz equations

\[
\begin{align*}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - bz,
\end{align*}
\]  

(4.4.1)

where \(\sigma, r, b > 0\) are parameters. \(\sigma\) is the Prandtl number, \(r\) is the Rayleigh number, and \(b\) has no name. The system (4.3.1) has two nonlinearities, the quadratic terms \(xy\) and \(xz\). Just as we did in section 4.1, we are going to estimate \(z\) by assuming \(x\) and \(y\) are known.

Let \(z = w, x = y_1\) and \(y = y_2\). Rewrite Eq.(4.3.1) into

\[
\begin{bmatrix}
\dot{w} \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} =
\begin{bmatrix}
-b & 0 & 0 \\
0 & -\sigma & \sigma \\
0 & r & -1
\end{bmatrix}
\begin{bmatrix}
w \\
y_1 \\
y_2
\end{bmatrix} +
\begin{bmatrix}
y_1 y_2 \\
0 \\
-y_1 w
\end{bmatrix}.
\]  

(4.4.2)

Partitioning the matrix into 4 blocks, we get

\[
A_{11} = -b, A_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}, A_{22} = \begin{bmatrix} -\sigma & \sigma \\ r & -1 \end{bmatrix}.
\]

Solve the coupled system Eq.(3.2.15)-(3.2.16). See Appendix G.

Let \(\sigma = 10, b = 8/3\) and \(r = 28\). Numerical simulations show that for chaotic Lorenz system the estimation errors decay to zero regardless of initial data assumed for the observer. Consequently the estimator \(w\) converges to the real solution \(z\), see Figure 4.10.

Similarly, we can estimate the derivatives of \(x, y\) and \(z\) by using the procedure outlined in section 4.1. First, we have to estimate \(z\) under assumption that \(x\) and \(y\) are measurable. Then, to estimate the derivatives of \(x, y\) and \(z\), we need to differentiate
Figure 4.10: shows convergence of observer and initial system solutions emanated from different initial values.

the system

\[ \dot{x} = \sigma(y - x), \]
\[ \dot{y} = rx - y - xz, \quad (4.4.3) \]
\[ \dot{z} = -bz + xy. \]

This defines the following set of equations:

\[ \ddot{x} = \sigma(\dot{y} - \dot{x}), \]
\[ \ddot{y} = r\dot{x} - \dot{y} - \dot{x}z - x\dot{z}, \quad (4.4.4) \]
\[ \ddot{z} = -b\dot{z} + \dot{x}y + x\dot{y}. \]

We recast our strategy in a cascade form which can be partitioned into the following steps: (1) Use known variables \( y_1, y_2 \) to estimate \( z(w_1) \); (2) Assume that the estimator of \( z(w_1) \) is another known variable \( y_3 \). Use \( y_1, y_2, y_3 \) to estimate the derivatives.
Recusing this procedure leads to generating of observer equations which estimate higher order derivatives for states of Lorenz system states. This would require \( n \)-subsequent application of reduced order observer to estimate matrix to \( 3n \times 3n \) matrix \((n \geq 2)\) times to get the \( n \)-th derivatives.

To avoid such repetition, we can comprise all these steps into one equation. Recalling the standard Lorenz system Eq.(4.4.3), and replacing \( z \) by the estimation \( w_1 \). Then we have new system as follows:

\[
\dot{x} = \sigma(y_2 - y_1),
\]
\[
\dot{y} = ry_1 - y_2 - y_1 w_1,
\]
\[
\dot{z} = -bw_1 + y_1 y_2. 
\]

Rewrite Eq. (4.4.4) by letting \( \dot{x} = \sigma(y_2 - y_1) \), \( \dot{y} = ry_1 - y_2 - y_1 w_1 \) and \( \dot{z} = -bw_1 + y_1 y_2 \).

We have

\[
\begin{align*}
\dot{w}_2 &= \sigma(ry_1 - y_2 - y_1 w_1 - w_2), \\
\dot{w}_3 &= rw_2 - w_3 - w_2 w_1 - y_1(-bw_1 + y_1 y_2), \\
\dot{w}_4 &= -bw_4 + w_2 y_2 + y_1 w_3,
\end{align*}
\]

where \( w_2, w_3 \) and \( w_4 \) are estimation derivatives of \( x, y \) and \( z \) respectively. Coupling Eq.(4.4.2) and Eq.(4.4.6) together, we obtain

\[
\begin{bmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{w}_3 \\
\dot{w}_4 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} =
\begin{bmatrix}
-b & 0 & 0 & 0 & 0 & 0 \\
0 & -\sigma & 0 & 0 & \sigma r & -\sigma \\
0 & r & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -b & 0 & 0 \\
0 & 0 & 0 & 0 & -\sigma & \sigma \\
0 & 0 & 0 & 0 & r & -1
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
y_1 \\
y_2
\end{bmatrix} +
\begin{bmatrix}
y_1 y_2 \\
-\sigma y_1 w_1 \\
-w_2 w_1 + by_1 w_1 - y_1^2 y_2 \\
w_2 y_2 + y_1 w_3 \\
y_1 \\
y_2
\end{bmatrix}.
\]

Finally, the last system is simulated by using the code listed in Appendix G for
standard Lorenz parameters $\sigma = 10, b = 8/3$ and $r = 28$. The simulation results are presented in Figures 4.11-4.16. The estimated trajectories in Figures 4.11-4.12 show the convergence of observer and initial system solutions emanated from different initial values for Lorenz system. Simulation errors are relatively small, see Figure 4.13.

Figure 4.11: shows convergence of observer and initial system solutions emanated from different initial values.
Figure 4.12: shows convergence of observer and initial system solutions emanated from different initial values.

Figure 4.13: displays time-histories observation errors are oscillating in a relatively small interval.
Figure 4.14: Phase plane for cross sections $x$-$z$ and $x$-$w_1$ (which is estimator of $z$).

Figure 4.15: Phase planes for cross sections derivative of $y$ versus derivative of $z$ and $w_3$-$w_4$ (which are estimators of derivatives of $y$ and $z$).
Figure 4.16: Phase planes for cross sections derivative of $y$ versus derivative of $x$ and $w_3-w_2$ (which are estimators of derivatives of $y$ and $x$).
Chapter 5

Estimation of Derivatives for Nonlinear System with Bounded Noise

5.1 Identity Observer for Noised System

In this section, we study influence of measurement and system noises on the estimation error of high-gain observers for some nonlinear systems such as Van der-Pol and double-well systems.

Suppose we have a nonlinear system with unknown but bounded noise $\phi(t)$

$$\dot{x}(t) = Ax(t) + f(x(t), t) + \phi(t)$$

$$y = C(x + \psi(t)),$$

where $\psi(t)$ is the measurement noise.
Design the identity observer as follows:

\[
\dot{z}(t) = A z(t) + f(z(t), t) + L(y(t) - C z(t)) \\
= A z(t) + f(z(t), t) + LC(x(t) - z(t)) + LC\psi(t).
\]  

(5.1.2)

Then the error system is

\[
\dot{e}(t) = \dot{x}(t) - \dot{z}(t) \\
= (A - LC)e(t) + f(z(t) + e(t), t) - f(z(t), t) - LC\psi(t) + \phi(t).
\]  

(5.1.3)

**Example 5.1.1 (Van der-Pol).** Consider the Van der-Pol system with noise

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= m x_1 + q x_2 + q_3 x_2^3 + q_5 x_2^5 + m_3 x_1^3 + m_5 x_1^5.
\end{align*}
\]  

(5.1.4)

Recall Eq. (3.2.18)

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= m x_3 + q x_4 + 3 q_3 (2 x_2 x_3^2 + x_2^2 x_4) + 5 q_5 (4 x_2^3 x_3^2 + x_2^4 x_4) \\
&+ 3 m_3 (2 x_1 x_2^2 + x_1^2 x_3) + 5 m_5 (4 x_1^3 x_2 + x_1^4 x_3).
\end{align*}
\]  

(5.1.5)

We readily obtain the observer for estimating system derivatives in the presence of
measurement noise as follows:

\[
\begin{bmatrix}
\dot{z}_1(t) \\
\dot{z}_2(t) \\
\dot{z}_3(t) \\
\dot{z}_4(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & m & q
\end{bmatrix}
\begin{bmatrix}
z_1(t) \\
z_2(t) \\
z_3(t) \\
z_4(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
f(z_1(t), z_2(t), t) \\
0 \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
x_1(t) - z_1(t) \\
x_2(t) - z_2(t) \\
x_3(t) - z_3(t) \\
x_4(t) - z_4(t)
\end{bmatrix} + LC
\begin{bmatrix}
\phi(t) \\
\psi(t)
\end{bmatrix},
\]

(5.1.6)

where

\[
f(z_1(t), z_2(t), t) = 3q_3(2z_2z_3^2 + z_2^2z_4 + z_2^2z_3) + 5q_5(4z_3^3z_2^2 + z_2^4z_4) + 3m_3(2z_1z_2^2 + z_1^2z_3) + 5m_5(4z_1^3z_2 + z_1^4z_3).
\]

Let \( m = -0.1, q = 0.1 \). The parameters \( m_3 = m_5 = q_5 = 0 \) and \( q_3 = -0.1 \). Suppose the noise functions are defined as: \( \phi(t) = 0.01 \sin(20t), \psi(t) = 0.01 \sin(10t) \). Note that the magnitude of measurement noise is scaled by matrix \( L \). Thus, the norm of steady-state error in this case does not inversely correlate with the magnitude of the largest real part of eigenvalues of matrix \( L \), but instead, reaches minimum for some choice of \( L \). By using numerical simulations, we find out that the following set of eigenvalues of matrix \([A_{11} - LA_{12}] \) (\( p = [-0.1, -0.2, -0.3, -0.4] \)) minimizes the norm of steady-state error vector such that the observation errors are oscillating within the interval: \([-1.5 \times 10^{-3}, 1.5 \times 10^{-3}] \), see Figure 5.1. We also find out that these eigenvalues minimize the bound of observation errors. For instance, a different set of eigenvalues \([-10, -20, -30, -40]\) amplifies the corresponding error, the errors will oscillate between \(-0.1\) and \(0.1\), see Figure 5.2.

For the corresponding eigenvalues, the errors practically do not change if we set \( \phi(t) = 0, \psi(t) = 0.01 \sin(10t) \) and \( p = [-10, -20, -30, -40] \), see Figures 5.1 and 5.2.
Figure 5.1: shows the observation errors oscillate within interval $[-1.5 \times 10^{-3}, 1.5 \times 10^{-3}]$ for the eigenvalues $[-0.1, -0.2, -0.3, -0.4]$.

Figure 5.2: shows the observation errors oscillate within interval $[-0.1, 0.1]$ for the eigenvalues $[-10, -20, -30, -40]$.

However, higher gain can reduce the errors when the measurement noise is equal to zero, i.e. $\phi(t) = 0.01 \sin(20t), \psi(t) = 0$ and $p = [-100, -200, -300, -400]$, see Figure 5.3. Thus we conclude that high gain observers can effectively attenuate the
error caused by system noise but effect of measurement noise can be reduced only up to a certain degree.

Figure 5.3: shows that the error can be reduced if the measurement noise is zero by using higher gain which correspond to the following eigenvalues are $[-100, -200, -300, -400]$.

Moreover, we also observe that the higher frequency of $\phi(t)$ and $\psi(t)$ inversely correlate with the magnitude of steady-state error, see Figures 5.1 and 5.4 for comparison. In the latter case: $\phi(t) = 0.01 \sin(100t)$ and $\psi(t) = 0.01 \sin(80t)$, which result in significant error reduction.

**Example 5.1.2** (Double-Well Chaotic System). Let’s recall Eq.(4.1.5)

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= x_3 - \delta x_4 - 6x_1x_2^2 - 3x_1^2x_3 - F\omega^2 \cos \omega t.
\end{align*}
\]  

(5.1.7)

Combining Eq.(5.1.2) and Eq.(5.1.3), one can readily comprise the codes for observer
Figure 5.4: The eigenvalues are as the same as those in Figure 5.1. Compared with Figure 5.1, the errors are smaller because of the increase of the frequencies of the noises.

and error systems, see Appendix H. The simulations of observer for double-well chaotic system with parameters $F = 0.4, \omega = 1, q = 0.25, m = -0.1$ yield the following inferences:

- $\phi(t) = 0.01 \sin(100t), \psi(t) = 0.01 \sin(80t)$ and $p = [-10, -20, -30, -40]$, the error will be larger if we use the higher gains, i.e. $p = [-100, -200, -300, -400]$, see Figure 5.5 and Figure 5.6. The error will be reduced for smaller frequency of the noises such as $\phi(t) = 0.01 \sin(10t), \psi(t) = 0.01 \sin(8t)$ and $p = [-10, -20, -30, -40]$, see Figure 5.7. But even for smaller noise frequency, the higher gains $p = [-100, -200, -300, -400]$ will amplify the error, see Figure 5.8.

- $\phi(t) = 0.01 \sin(10t), \psi(t) = 0$, the error will be amplified if one uses higher gains. Also, a higher frequency of the noise will increase the error.

- $\phi(t) = 0, \psi(t) = 0.01 \sin(8t)$, the error will be amplified if we use higher gains.
In this case as well, a higher frequency of the noise will increase the error.

Figure 5.5: shows the observation errors with noises: \( \phi(t) = 0.01 \sin(100t) \), \( \psi(t) = 0.01 \sin(80t) \) and the eigenvalues of the linear part of the error equations are \([-10, -20, -30, -40]\).

Different from the double-cycle system, the lower frequencies of the noises can reduce the observation errors in double-well chaotic system. The higher gain always amplify the observation errors in double-well chaotic system. However, for double-cycle system without measurement noise, the errors can be reduced by using higher gain.
Figure 5.6: shows the observation errors with noises: $\phi(t) = 0.01 \sin(100t), \psi(t) = 0.01 \sin(80t)$ and the eigenvalues of the linear part of error equations are $[-100, -200, -300, -400]$.  

Figure 5.7: displays the observation errors with noises: $\phi(t) = 0.01 \sin(10t), \psi(t) = 0.01 \sin(8t)$ and the eigenvalues of the linear part of error equations are $[-10, -20, -30, -40]$. In comparison with Figure 5.5, the errors are smaller in this case.
Figure 5.8: displays the observation errors with noises: \( \phi(t) = 0.01 \sin(10t) \), \( \psi(t) = 0.01 \sin(8t) \) and the eigenvalues of the linear part of the error equations are \([-100, -200, -300, -400]\).

## 5.2 Simulation of Performance of Reduced Order Observer for Nonlinear System with Noise

In this section, we analyze the impact of noise on the estimation error when reduced order observers are applied.

Consider the nonlinear system with noise \( \phi(t) \) and \( \psi(t) \) as follows:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + f(x, t) + \phi(t) \\
y(t) &= \hat{y}(t) + \psi(t).
\end{align*}
\]  

(5.2.1)

Partitioning the matrix \( A \) as we did in Eq.(3.2.3), we obtain

\[
\begin{bmatrix}
\dot{w}(t) \\
\dot{\hat{y}}(t)
\end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} w(t) \\
\hat{y}(t) + \psi(t) \end{bmatrix} + \begin{bmatrix} f_1(w, \hat{y} + \psi) \\
f_2(w, \hat{y} + \psi) \end{bmatrix} + \begin{bmatrix} \phi_1(t) \\
\phi_2(t) \end{bmatrix}.
\]

(5.2.2)
Write an equation of the difference $w(t) - Ly(t)$:

$$
\dot{w}(t) - L\dot{y}(t) = (A_{11} - LA_{21})w(t) + (A_{12} - LA_{21})(\dot{y}(t) + \psi(t)) + \phi_1(t) - L\phi_2(t).
$$

(5.2.3)

Rewrite Eq.(5.2.2) in the form

$$
\dot{w}(t) - L\dot{y} = (A_{11} - LA_{21})[w(t) - Ly(t)] + (A_{11}L - LA_{21}L + A_{12} - LA_{22})\dot{y}(t)
$$

$$
+ (A_{11}L - LA_{21}L + A_{12} - LA_{22})\psi(t) + f_1(w, \ddot{y} + \psi) - Lf_2(w, \ddot{y} + \psi)
$$

$$
+ \phi_1(t) - L\phi_2(t).
$$

(5.2.4)

Let $v(t) = w(t) - Ly(t)$, we have

$$
\dot{v}(t) = (A_{11} - LA_{21})v(t) + (A_{11}L - LA_{21}L + A_{12} - LA_{22})\dot{y}(t)
$$

$$
+ (A_{11}L - LA_{21}L + A_{12} - LA_{22})\psi(t) + f_1(v + Ly, y) - Lf_2(v + Ly, y)
$$

$$
+ \phi_1(t) - L\phi_2(t).
$$

(5.2.5)

Thus the observer system is:

$$
\dot{z} = (A_{11} - LA_{21})z + (A_{11}L - LA_{21}L + A_{12} - LA_{22})(\ddot{y} + \psi(t))
$$

$$
+ f_1(z + Ly, y) - Lf_2(z + Ly, y).
$$

(5.2.6)

Subtract Eq.(5.2.5) and Eq.(5.2.6) we get the error equation for a reduced order estimator observer:

$$
\dot{e}(t) = (A_{11} - LA_{21})e + f_1(e + z + Ly, y) - f_1(z + Ly, y)
$$

$$
- L(f_2(e + z + Ly) - f_2(z + Ly, y)) + \phi_1(t) - L\phi_2(t)
$$

(5.2.7)
Then we derive an inequality for the norm of $w(t)$:

$$
||w(t)|| \leq ||z(t) + e(t) + L(\dot{y}(t) + \psi(t))|| \\
\leq ||z(t) + Ly(t)|| + ||e(t)||.
$$

(5.2.8)

The last inequality shows that the accuracy of estimation is determined by the norm of estimation error as well as the norm of $z(t) + Ly(t)$.

**Example 5.2.1 (Van der-Pol Noised System).** Recall Eq.(3.2.18), Eq.(3.2.19) and Eq.(3.2.20), one can immediately derive the codes for Van der-Pol noised system, see Appendix I. Let $m = -0.1, q = -0.1$. Because of the disappearance of $\psi(t)$, we only discuss the system with noise $\phi(t) = 0.01 \sin(\omega_1 t)$. The parameters $m_3 = m_5 = q_3 = 0$ and $q_3 = -0.1$. Here are some observations.

- $\phi(t) = 0.01 \sin(100t), p = [-1, -3]$, the error oscillates in $[-0.1, 0.1]$, see Figure 5.9. Higher gain $[-100, -200]$ will reduce the error, see Figure 5.10.

- $\phi(t) = 0.01 \sin(10t), p = [-1, -3]$, see Figure 5.11. This shows that variation of noise frequency does not affect the observation error but this error will be slightly amplified in case of higher gains $[-100, -200]$, see Figure 5.12.

**Example 5.2.2 (Double-Well Chaotic Noised System).** Recall Eq.(4.1.10), Eq.(4.1.11) and Eq.(4.1.12). Combine Eq.(5.1.2) and Eq.(5.1.3), the codes for double-well noised system are presented in Appendix I. Set the parameters as: $F = 0.4, \omega = 1, q = 0.25, m = -0.1$, which comprises a chaotic behavior of such system. The observations can be concluded as below:

- $\phi(t) = 0.01 \sin(100t), p = [-10, -11, -12, -13, -14]$. The observation errors oscillating in $[-20, 20]$, see Figure 5.13. The higher gain, i.e. $p = [-50, -60, -70, -80, -90]$, amplifies the errors, see Figure 5.14.
Figure 5.9: shows the observation errors with noise: $\phi(t) = 0.01 \sin(100t)$ and the eigenvalues of the linear part of error equations are $[-1, -3]$.

Figure 5.10: displays the observation errors with noise: $\phi(t) = 0.01 \sin(100t)$ and the eigenvalues of the linear part of error equations are $[-100, -200]$.

- $\phi(t) = 0.01 \sin(10t), p = [-10, -11, -12, -13, -14]$. This shows that variation of noise frequency does not affect the observation error, see Figure 5.15. The observation errors $e_1, e_2$ and $e_3$ will be reduced if we use higher gain, i.e. $p =$
Figure 5.11: shows the observation errors with noise: $\phi(t) = 0.01 \sin(10t)$ and the eigenvalues of the linear part of error equations are $[-1, -3]$.

Figure 5.12: displays the observation errors with noise: $\phi(t) = 0.01 \sin(10t)$ and the eigenvalues of the linear part of error equations are $[-100, -200]$. In comparison with Figure 5.12, the error will be slightly amplified in case of higher gains.

$[-50, -60, -70, -80, -90]$. However, the error of higher derivatives, i.e., $e_4$ and $e_5$ will be amplified, see Figure 5.16.
Figure 5.13: displays the observation errors with noise: $\phi(t) = 0.01 \sin(100t)$ and the eigenvalues of the linear part of error equations are $[-10, -11, -12, -13, -14]$.

Figure 5.14: displays the observation errors with noise: $\phi(t) = 0.01 \sin(100t)$ and the eigenvalues of the linear part of error equations are $[-50, -60, -70, -80, -90]$.
Figure 5.15: shows the observation errors with noise: $\phi(t) = 0.01\sin(10t)$ and the eigenvalues of the linear part of error equations are $[-10, -11, -12, -13, -14]$.

Figure 5.16: shows the observation errors with noise: $\phi(t) = 0.01\sin(10t)$ and the eigenvalues of the linear part of error equations are $[-50, -60, -70, -80, -90]$. 
Appendices
Appendix A

Section 2.1

\[
p=\begin{bmatrix} -10.0 & -11.0 \end{bmatrix}; \quad \% \text{targeted eigenvalues}
\]
\[
A=\begin{bmatrix} 0 & 1; -1 & 0.1 \end{bmatrix}; \quad \% \text{matrix A}
\]
\[
C=\begin{bmatrix} 1 & 0 \end{bmatrix}; \quad \% \text{Substitute C' instead of B}
\]
\[
L=\text{place}(a',c',p)' \quad \% \text{print gain matrix L}
\]
\[
K=A-L*C
\]
\[
d=\text{eig}(K) \quad \% \text{check if the eigenvalues are on the left plane}
\]

**Figure 2.1-2.2**

% \text{A(i,j) represents the ith-row, jth-col. element of A}
% \text{K denotes the matrix A - LC}
% \text{R denotes the matrix LC}
% \text{z(1)=x1 and z(2)=x2; z(3)=e1 and z(4)=e2; z(5)=z1 and z(6)=z2}
\[
dz = \text{zeros}(6,1);
\]
\[
dz(1)=A(1,1)*z(1)+A(1,2)*z(2);
\]
\[
dz(2)=A(2,1)*z(1)+A(2,2)*z(2);
\]
\[
dz(3)=K(1,1)*z(3)+K(1,2)*z(4);
\]
\[
dz(4)=K(2,1)*z(3)+K(2,2)*z(4);
\]
\[
dz(5)=A(1,1)*z(5)+A(1,2)*z(6)+R(1,1)*(z(1)-z(5))+R(1,2)*(z(2)-z(6));
\]
\[
dz(6)=A(2,1)*z(5)+A(2,2)*z(6)+R(2,1)*(z(1)-z(5))+R(2,2)*(z(2)-z(6));
\]
Figure 2.3-2.4.

% z(1)=x1; z(2)=x2; z(3)=x3; z(4)=x4; z(5)=e1; z(6)=e2; z(7)=e3; z(8)=e4;  
% z(9)=z1; z(10)=z2; z(11)=z3; z(12)=z4

dz = zeros(12,1); % column vector

dz(1)=A(1,1)*z(1)+A(1,2)*z(2)+A(1,3)*z(3)+A(1,4)*z(4);  
dz(2)=A(2,1)*z(1)+A(2,2)*z(2)+A(2,3)*z(3)+A(2,4)*z(4);  
dz(3)=A(3,1)*z(1)+A(3,2)*z(2)+A(3,3)*z(3)+A(3,4)*z(4);  
dz(4)=A(4,1)*z(1)+A(4,2)*z(2)+A(4,3)*z(3)+A(4,4)*z(4);  
dz(5)=K(1,1)*z(5)+K(1,2)*z(6)+K(1,3)*z(7)+K(1,4)*z(8);  
dz(6)=K(2,1)*z(5)+K(2,2)*z(6)+K(2,3)*z(7)+K(2,4)*z(8);  
dz(7)=K(3,1)*z(5)+K(3,2)*z(6)+K(3,3)*z(7)+K(3,4)*z(8);  
dz(8)=K(4,1)*z(5)+K(4,2)*z(6)+K(4,3)*z(7)+K(4,4)*z(8);  
dz(9)=A(1,1)*z(9)+A(1,2)*z(10)+A(1,3)*z(11)+A(1,4)*z(12)  
+R(1,1)*(z(1)−z(9)) +R(1,2)*(z(2)−z(10));  
dz(10)=A(2,1)*z(9)+A(2,2)*z(10)+A(2,3)*z(11)+A(2,4)*z(12)  
+R(2,1)*(z(1)−z(9)) +R(2,2)*(z(2)−z(10));  
dz(11)=A(3,1)*z(9)+A(3,2)*z(10)+A(3,3)*z(11)+A(3,4)*z(12)  
+R(3,1)*(z(1)−z(9)) +R(3,2)*(z(2)−z(10));  
dz(12)=A(4,1)*z(9)+A(4,2)*z(10)+A(4,3)*z(11)+A(4,4)*z(12)  
+R(4,1)*(z(1)−z(9)) +R(4,2)*(z(2)−z(10));
Appendix B

Section 2.2

Figure 2.5

\[ p = \begin{bmatrix} -100 & -110 \end{bmatrix}; \quad \% \text{place the eigenvalues on the left plane} \]
\[ a = \begin{bmatrix} 0 & 1 \\ -1+d & (0.1+r) \end{bmatrix}; \quad \% \text{matrix } \tilde{A} \]
\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0.1 \end{bmatrix}; \quad \% \text{initial matrix } A \]
\[ D = \begin{bmatrix} 0 & 0 \\ -d & r \end{bmatrix}; \quad \% \text{matrix } A \]
\[ C = [1 \ 0]; \quad \% \text{Substitute } C' \text{ instead of } B \]
\[ L = \text{place}(A',C',p)' \quad \% \text{print gain matrix } L \]
\[ R = L \times C \quad \% \text{Denote } LC \text{ as } R \]
\[ K = A - R \]
\[ d = \text{eig}(K) \quad \% \text{check if the eigenvalues are on the left plane} \]

\[ dz = \text{zeros}(6,1); \quad \% \text{column vector} \]
\[ dz(1) = a(1,1) \times z(1) + a(1,2) \times z(2); \quad \% z(1) = x_1; \ z(2) = x_2; \ z(3) = e_1; \ z(4) = e_2 \]
\[ dz(2) = a(2,1) \times z(1) + a(2,2) \times z(2); \quad \% z(5) = z_1; \ z(6) = z_2 \]
\[ dz(3) = K(1,1) \times z(3) + K(1,2) \times z(4) + D(1,1) \times (z(5) + z(3)) + D(1,2) \times (z(6) + z(4)); \]
\[ dz(4) = K(2,1) \times z(3) + K(2,2) \times z(4) + D(2,1) \times (z(5) + z(3)) + D(2,2) \times (z(6) + z(4)); \]
\[ dz(5) = A(1,1) \times z(5) + A(1,2) \times z(6) + R(1,1) \times (z(1) - z(5)) + R(1,2) \times (z(2) - z(6)); \]
\[ dz(6) = A(2,1) \times z(5) + A(2,2) \times z(6) + R(2,1) \times (z(1) - z(5)) + R(2,2) \times (z(2) - z(6)); \]
Appendix C

Section 3.1

Figure 3.1-3.2

\[ \begin{align*}
A &= \begin{bmatrix} 0 & 1 \\ m & q \end{bmatrix}; \quad \text{\%matrix \( A \)} \\
C &= \begin{bmatrix} 1 & 0 \end{bmatrix}; \\
p &= \begin{bmatrix} -100 & -110 \end{bmatrix}; \quad \text{\%place the eigenvalues on the left plane} \\
L &= \text{place}(a',c',p)' \\
R &= L*c; \quad \text{\%denote matrix \( LC \) as \( R \)} \\
K &= A - R; \\
d &= \text{eig}(K) \quad \text{\%check if the eigenvalues are on the left plane} \\
\end{align*} \]

\[ \begin{align*}
\%z(1) &= e1; \quad z(2) = e2; \quad z(3) = e3; \quad z(4) = e4 \\
dz &= \text{zeros}(4,1); \\
dz(1) &= K(1,1) * z(1) + K(1,2) * z(2); \\
dz(2) &= K(2,1) * z(1) + K(2,2) * z(2) + (q3 \ast (z(4) + z(2))^3 + q5 \ast (z(4) + z(2))^5) \\
& \quad + m3 \ast (z(3) + z(1))^3 + m5 \ast (z(3) + z(1))^5 - (q3 \ast (z(4))^3 \\
& \quad + q5 \ast (z(4))^5 + m3 \ast (z(3))^3 + m5 \ast (z(3))^5); \\
dz(3) &= A(1,1) * z(3) + A(1,2) * z(4) + R(1,1) * z(1) + R(1,2) * z(2); \\
dz(4) &= A(2,1) * z(3) + A(2,2) * z(4) + R(2,1) * z(1) + R(2,2) * z(2) + q3 \ast (z(4))^3 \\
& \quad + q5 \ast (z(4))^5 + m3 \ast (z(3))^3 + m5 \ast (z(3))^5. \\
\end{align*} \]
Appendix D

Section 3.2

Figure 3.3-3.8

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-m & -q & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

\[
A11=a(1:2,1:2); \quad \text{%Block A11}
\]
\[
A21=a(3:4,1:2); \quad \text{%Block A21}
\]
\[
A12=a(1:2,3:4); \quad \text{%Block A12}
\]
\[
A22=a(3:4,3:4); \quad \text{%Block A22}
\]
\[
p=[-3000 \quad -3100]; \quad \text{%target eigenvalues on the left plane}
\]
\[
L=place(A11',A21',p)' \quad \text{%Obtain gain matrix}
\]
\[
Q=A11-L*A21;
\]
\[
E=eig(Q) \quad \text{%place the eigenvalues on the left plane}
\]

For the convenience of programming,

\[
C=A11-L*A21; \quad \text{%denote A11-LA21 as C}
\]
\[
D=A11*L-L*A21*L+A12-L*A22; \quad \text{%denote A11*L-LA21*L+A12-LA22 as D}
\]

\[
%z(1)=y1; \quad z(2)=y2; \quad z(3)=e1; \quad z(4)=e2; \quad z5=v1; \quad z6=v2
\]
\[
dz=zeros(6,1);
\]
\[
dz(1)=z(2);
\]
\[
dz(2)=m*z(1)+q*z(2)+q3*z(2)^3+q5*z(2)^5+m3*z(1)^3+m5*z(1)^5;
\]
\[ dz(3) = C(1,1) \cdot z(3) + C(1,2) \cdot z(4); \]
\[ dz(4) = C(2,1) \cdot z(3) + C(2,2) \cdot z(4) + (3 \cdot m3 \cdot z(1)^2 + 5 \cdot m5 \cdot z(1)^4 + 20 \cdot q5 \cdot z(2)^3 \cdot (z(3) + z(5) + L(1,1) \cdot z(1) + L(1,2) \cdot z(2))) \]
\[ + 5 \cdot q5 \cdot z(2)^4 \cdot (z(4) + z(6) + L(2,1) \cdot z(1) + L(2,2) \cdot z(2)) \]
\[ \cdot (z(3) + z(5) + L(1,1) \cdot z(1) + L(1,2) \cdot z(2)) \]
\[ - (3 \cdot m3 \cdot z(1)^2 + 5 \cdot m5 \cdot z(1)^4 + 20 \cdot q5 \cdot z(2)^3 \cdot (z(5) + L(1,1) \cdot z(1) + L(1,2) \cdot z(2))) \]
\[ + L(1,2) \cdot z(2)) \cdot (z(5) + L(1,1) \cdot z(1) + L(1,2) \cdot z(2)) \]
\[ + (3 \cdot q3 \cdot z(2)^2 + 5 \cdot q5 \cdot z(2)^4) \cdot (z(6) + L(2,1) \cdot z(1) + L(2,2) \cdot z(2)); \]
\[ dz(5) = C(1,1) \cdot z(5) + C(1,2) \cdot z(6) + D(1,1) \cdot z(1) + D(1,2) \cdot z(2); \]
\[ dz(6) = C(2,1) \cdot z(5) + C(2,2) \cdot z(6) + D(2,1) \cdot z(1) + D(2,2) \cdot z(2) \]
\[ + (3 \cdot m3 \cdot z(1)^2 + 5 \cdot m5 \cdot z(1)^4 + 20 \cdot q5 \cdot z(2)^3 \cdot (z(5) + L(1,1) \cdot z(1) + L(1,2) \cdot z(2))) \]
\[ + L(1,2) \cdot z(2)) \cdot (z(5) + L(1,1) \cdot z(1) + L(1,2) \cdot z(2)) \]
\[ + 5 \cdot q5 \cdot z(2)^4 \cdot (z(6) + L(2,1) \cdot z(1) + L(2,2) \cdot z(2)) + 6 \cdot m3 \cdot z(2)^2 \cdot z(1) \]
\[ + (6 \cdot q3 \cdot z(5) + L(1,1) \cdot z(1) + L(1,2) \cdot z(2)) \cdot (2 + 20 \cdot m5 \cdot z(1)^3) \cdot z(2); \]

% After obtaining z1 and z2, we need use z+Ly to get w
% z1o, z2o, z3o, z4o, z5o, z6o are the initial conditions
% dt is the step size
[t, z] = ode15s(@fun_z, [t1:dt:t2], [z1o z2o z3o z4o z5o z6o]);
Y = z(:, 1:2); % Y is the vector [y1 y2]
W = (z(:, 5:6)'+ L*Y)'; % W is the estimator of x
Appendix E

Section 4.1

Figure 4.1-4.6

% q denotes the coefficient of \( \delta \) in Eq.(4.1.2)
% v denotes the frequency \( \omega \) in Eq.(4.1.2)
% F denotes the force in Eq.(4.1.2)
% a is the linear matrix in Eq.(4.1.2)
% aij is the block after partitioning
% XX is the eigenvalues of the block all
% p is the pole vector
% L is the gain matrix derived by place function
% Q denote A11–LA21
% E is the eigenvalues of (A11–LA21) which are placed on the left plane

q=0.25;
m=1;
F=0.18;
v=1;

a=[0 1 0 0 0 0;0 0 1 0 0 0;0 0 0 1 0 0;
0 0 0 0 1 0;0 0 0 -1 -q 0;0 0 0 0 0 1;1 0 0 0 0 0];
\[ \text{a11} = \text{a}(1:5,1:5); \]
\[ \text{a21} = \text{a}(6:7,1:5); \]
\[ \text{a12} = \text{a}(1:5,6:7); \]
\[ \text{a22} = \text{a}(6:7,6:7); \]
\[ \text{XX} = \text{eig} (\text{a11}) \]
\[ \text{p} = [-100 \ -110 \ -120 \ -130 \ -140]; \]
\[ \text{L} = \text{place} (\text{a11}',\text{a21}',\text{p})' \]
\[ \text{Q} = \text{a11} - \text{L} \text{a21}; \]
\[ \text{E} = \text{eig} (\text{Q}) \]

% for the convenience of computation, let
\[ \text{C} = \text{a11} - \text{L} \text{a21}; \]
\[ \text{D} = (\text{a11} \text{L} - \text{L} \text{a21} \text{L} + \text{a12} - \text{L} \text{a22}); \]

% \text{z}(1) = \text{y}1; \text{z}(2) = \text{y}2; \text{z}(i) = \text{e}i, \text{ for } i = 3, \ldots, 7
\[ \% \text{z}(i) = \text{z}i, \text{ for } i = 8, \ldots, 12 \]

\[ \text{dz} = \text{zeros}(12,1); \]
\[ \text{dz}(1) = \text{z}(2); \]
\[ \text{dz}(2) = \text{z}(1) - \text{q} \text{z}(2) - \text{z}(1)^3 + \text{F} \text{cos}(\text{v}\text{t}); \]
\[ \text{dz}(3) = \text{C}(1,1) \text{z}(3) + \text{C}(1,2) \text{z}(4) + \text{C}(1,3) \text{z}(5) + \text{C}(1,4) \text{z}(6) + \text{C}(1,5) \text{z}(7); \]
\[ \text{dz}(4) = \text{C}(2,1) \text{z}(3) + \text{C}(2,2) \text{z}(4) + \text{C}(2,3) \text{z}(5) + \text{C}(2,4) \text{z}(6) + \text{C}(2,5) \text{z}(7); \]
\[ \text{dz}(5) = \text{C}(3,1) \text{z}(3) + \text{C}(3,2) \text{z}(4) + \text{C}(3,3) \text{z}(5) + \text{C}(3,4) \text{z}(6) + \text{C}(3,5) \text{z}(7); \]
\[ \text{dz}(6) = \text{C}(4,1) \text{z}(3) + \text{C}(4,2) \text{z}(4) + \text{C}(4,3) \text{z}(5) + \text{C}(4,4) \text{z}(6) + \text{C}(4,5) \text{z}(7); \]
\[ \text{dz}(7) = \text{C}(5,1) \text{z}(3) + \text{C}(5,2) \text{z}(4) + \text{C}(5,3) \text{z}(5) + \text{C}(5,4) \text{z}(6) + \text{C}(5,5) \text{z}(7) \]
\[ - (90 \text{z}(2) * (\text{z}(3) + \text{z}(8) + \text{L}(1,1) * \text{z}(1) + \text{L}(1,2) * \text{z}(2)) + 2 + 60 \text{z}(2)^2 \]
\[ * (\text{z}(4) + \text{z}(9) + \text{L}(2,1) * \text{z}(1) + \text{L}(2,2) * \text{z}(2)) + 60 \text{z}(1)^2 + (\text{z}(3) + \text{z}(8) \]
\[ + \text{L}(1,1) * \text{z}(1) + \text{L}(1,2) * \text{z}(2)) * (\text{z}(4) + \text{z}(9) + \text{L}(2,1) * \text{z}(1) + \text{L}(2,2) * \text{z}(2)) \]
\[ + 30 \text{z}(1) * \text{z}(2) + (\text{z}(5) + \text{z}(10) + \text{L}(3,1) * \text{z}(1) + \text{L}(3,2) * \text{z}(2)) + 3 \text{z}(1)^2 \]
\[ * (\text{z}(6) + \text{z}(11) + \text{L}(4,1) * \text{z}(1) + \text{L}(4,2) * \text{z}(2)) + (90 \text{z}(2) * (\text{z}(8) \]
\[ + \text{L}(1,1) * \text{z}(1) + \text{L}(1,2) * \text{z}(2)) + 2 + 60 \text{z}(2)^2 + 2 * (\text{z}(9) + \text{L}(2,1) * \text{z}(1) \]
\[ + \text{L}(2,2) * \text{z}(2)) + 60 \text{z}(1) * (\text{z}(8) + \text{L}(1,1) * \text{z}(1) + \text{L}(1,2) * \text{z}(2)) * (z(9) \]
\[+L(2,1)\cdot z(1)+L(2,2)\cdot z(2))+30\cdot z(1)\cdot z(2)\cdot \left( z(1) + L(3,1)\cdot z(1) + L(3,2)\cdot z(2) \right)\];

dz(8)=C(1,1)\cdot z(8)+C(1,2)\cdot z(9)+C(1,3)\cdot z(10)+C(1,4)\cdot z(11)+C(1,5)\cdot z(12)
+D(1,1)\cdot z(1)+D(1,2)\cdot z(2);

dz(9)=C(2,1)\cdot z(8)+C(2,2)\cdot z(9)+C(2,3)\cdot z(10)+C(2,4)\cdot z(11)+C(2,5)\cdot z(12)
+D(2,1)\cdot z(1)+D(2,2)\cdot z(2);

dz(10)=C(3,1)\cdot z(8)+C(3,2)\cdot z(9)+C(3,3)\cdot z(10)+C(3,4)\cdot z(11)+C(3,5)\cdot z(12)
+D(3,1)\cdot z(1)+D(3,2)\cdot z(2);

dz(11)=C(4,1)\cdot z(8)+C(4,2)\cdot z(9)+C(4,3)\cdot z(10)+C(4,4)\cdot z(11)+C(4,5)\cdot z(12)
+D(4,1)\cdot z(1)+D(4,2)\cdot z(2);

dz(12)=C(5,1)\cdot z(8)+C(5,2)\cdot z(9)+C(5,3)\cdot z(10)+C(5,4)\cdot z(11)+C(5,5)\cdot z(12)
+D(5,1)\cdot z(1)+D(5,2)\cdot z(2)-(90\cdot z(2)\cdot \left( z(8) + L(1,1)\cdot z(1) + L(1,2)\cdot z(2) \right)\);
Appendix F

Section 4.3

Figure 4.8-4.9

\[ [t,z]=\text{ode15s}(@fun_z,[t_{1:d}:t_{2}],[z_{10} z_{20} z_{30} z_{40} z_{50} z_{60} z_{70} z_{80} z_{90} z_{100} z_{110} z_{120}]); \text{ % } z_{i0}, \text{ i}=1,\ldots,12 \text{ is the arbitrary initial values}\]

\[ Y=[z(:,1:2)]; \text{ % } Y \text{ is } y_{1} \text{ and } y_{2}\]

\[ w=(z(:,8:12)'+L*Y')'; \text{ % } w \text{ is vector of } w_{i}, \text{ i}=1,\ldots,5\]

\[ X=[Y \ w]; \]

% S1, S2 and S3 are summands described in Eq.(4.2.5)-(4.2.7)
% c0,c1,...,c6 denotes y1,y2,w1,...,w5 at t=34

\[ S1=\text{inline('c0+c1*(x-34)+c2/factorial(2)*(x-34)^2+c3/factorial(3)\}
\]
\[ *(x-34)^3+c4/factorial(4)*(x-34)^4,'x','c0','c1','c2','c3','c4');\]

\[ S2=\text{inline('c0+c1*(x-34)+c2/factorial(2)*(x-34)^2+c3/factorial(3)\}
\]
\[ *(x-34)^3+c4/factorial(4)*(x-34)^4+c5/factorial(5)\]
\[ *(x-34)^5,'x','c0','c1','c2','c3','c4','c5');\]

\[ S3=\text{inline('c0+c1*(x-34)+c2/factorial(2)*(x-34)^2+c3/factorial(3)\}
\]
\[ *(x-34)^3+c4/factorial(4)*(x-34)^4+c5/factorial(5)*(x-34)^5\]
\[ +c6/factorial(6)*(x-34)^6,'x','c0','c1','c2','c3','c4','c5','c6');\]

\[ c0=X(7000,1); \]
82

c1=X(7000,2);
c2=X(7000,3);
c3=X(7000,4);
c4=X(7000,5);
c5=X(7000,6);
c6=X(7000,7);

%below iteration is Eq.(4.2.8)-(4.2.11)
s=zeros(11,N+1);
for i=3:10
    for j=1:N+1;
        x1(j)=S1(t(j),c0,c1,c2,c3,c4);
        x2(j)=S2(t(j),c0,c1,c2,c3,c4,c5);
        x3(j)=S3(t(j),c0,c1,c2,c3,c4,c5,c6);
        A(1,j)=x1(j)-((x2(j)-x1(j))^2/(x1(j)-2*x2(j)+x3(j)));
        A(2,j)=x2(j)-((x3(j)-x2(j))^2/(x2(j)-2*x3(j)+s(1,j)));
        A(3,j)=x3(j)-((s(1,j)-x3(j))^2/(x3(j)-2*s(1,j)+s(2,j)));
        A(i+1,j)=A(i-2,j)-(A(i-1,j)-A(i-2,j))^2/(A(i-2,j)
                    -A*s(i-1,j)+A(i,j));
    end
end
P=s;
Pred=s(10,:); %we can use 10 iterations to get the prediction
Appendix G

Section 4.4

Figure 4.10-4.11

% denote \sigma as m
% a is matrix as in Eq.(4.3.2)
% Aij is the block of A
a=[-b 0 0;0 -m 0 r -1];
A11=a(1:1,1:1);
A21=a(2:3,1:1);
A12=a(1:1,2:3);
A22=a(2:3,2:3);
c=[1 0]';
p=[-100];
L=place(a11',c',p)'
Q= a11-L*a21;
E=eig(Q)

A=A11-L*A21;
B=(A11*L-L*A21*L+A12-L*A22);

%z(1)=y1, z(2)=y2, z(3)=z, z(4)=e, z(5)=w
dz=zeros(5,1);
dz(1) = m \times (z(2) - z(1));
dz(2) = r \times z(1) - z(2) - z(1) \times z(3);
dz(3) = -b \times z(3) + z(1) \times z(2);
dz(4) = A(1) \times z(4);
dz(5) = A(1) \times z(5) + B(1, 1) \times z(1) + B(1, 2) \times z(2) + z(1) \times z(2) - L(2) \times (-z(1) \times z(3));

Figure 4.12-4.17

A = [-b \ 0 \ 0 \ 0 \ 0 \ 0; -m \ 0 \ 0 \ m \times r \ -m; 0 \ r \ -1 \ 0 \ 0 \ 0; 0 \ 0 \ 0 \ -b \ 0 \ 0; 0 \ 0 \ 0 \ 0 \ -m \ m; 0 \ 0 \ 0 \ 0 \ r \ -1];

A11 = A(1:4, 1:4);
A21 = A(5:6, 1:4);
A12 = A(1:4, 5:6);
A22 = A(5:6, 5:6);

C = A11 - L \times A21;
D = (A11 \times L - L \times A21 + A12 - L \times A22);

% z(1) = x, z(2) = y, z(3) = z
% z(4) = e1, z(5) = e2, z(6) = e3, z(7) = e4
% z(8) = z1, z(9) = z2, z(10) = z3, z(11) = z4
dz = zeros(11, 1);
dz(1) = m \times (z(2) - z(1));
dz(2) = r \times z(1) - z(2) - z(1) \times z(3);
dz(3) = -b \times z(3) + z(1) \times z(2);
dz(4) = C(1, 1) \times z(4) + C(1, 2) \times z(5) + C(1, 3) \times z(6) + C(1, 4) \times z(7) - L(1, 2) \times (-1)
\times z(1) \times (z(4) + z(8) + L(1, 1) \times z(1) + L(1, 2) \times z(2)) + z(1) \times (z(8) + L(1, 1) \times z(1) + L(1, 2) \times z(2));
dz(5) = C(2, 1) \times z(4) + C(2, 2) \times z(5) + C(2, 3) \times z(6) + C(2, 4) \times z(7) - L(2, 2)
\times ((-1) \times z(1) \times (z(4) + z(8) + L(1, 1) \times z(1) + L(1, 2) \times z(2)) + z(1) \times (z(8)
+ L(1, 1) \times z(1) + L(1, 2) \times z(2))) - m \times z(1) \times (z(4) + z(8) + L(1, 1) \times z(1)
+ L(1, 2) \times z(2)) + m \times z(1) \times (z(8) + L(1, 1) \times z(1) + L(1, 2) \times z(2));
\[dz(6) = C(3,1)z(4) + C(3,2)z(5) + C(3,3)z(6) + C(3,4)z(7) - L(3,2) \\
\times ((-1)z(1) * (z(4) + z(8) + L(1,1)z(1) + L(1,2)z(2)) + z(1) * (z(8) \\
+ L(1,1)z(1) + L(1,2)z(2)) - ((z(4) + z(8) + L(1,1)z(1) + L(1,2) \\
z(2)) * z(2)) + ((z(8) + L(1,1)z(1) + L(1,2)z(2)) * z(2)) + ((z(8) + L(1,1) \\
z(1) + L(1,2)z(2)) * z(2)) + b * (z(4) + z(8) + L(1,1)z(1) \\
+ L(1,2) + z(2)) * z(1) - b * (z(8) + L(1,1)z(1) + L(1,2)z(2)) * z(1); \\
\]

\[dz(7) = C(4,1)z(4) + C(4,2)z(5) + C(4,3)z(6) + C(4,4)z(7) - L(4,2) \\
\times ((-1)z(1) * (z(4) + z(8) + L(1,1)z(1) + L(1,2)z(2)) + z(1) * (z(8) \\
+ L(1,1)z(1) + L(1,2)z(2)) - ((z(4) + z(8) + L(1,1)z(1) + L(1,2) \\
z(2)) * z(2)) + ((z(8) + L(1,1)z(1) + L(1,2)z(2)) * z(2)) + ((z(8) + L(1,1) \\
z(1) + L(1,2)z(2)) * z(2)) - (z(10) + L(3,1)z(1) + L(3,2)z(2)) * z(1); \\
\]

\[dz(8) = C(1,1)z(8) + C(1,2)z(9) + C(1,3)z(10) + C(1,4)z(11) + D(1,1)z(1) \\
+ D(1,2) * z(2) + z(1) * z(2) + L(1,2) * z(1) * (z(8) + L(1,1)z(1) \\
+ L(1,2) + z(2)); \\
\]

\[dz(9) = C(2,1)z(8) + C(2,2)z(9) + C(2,3)z(10) + C(2,4)z(11) + D(2,1)z(1) \\
+ D(2,2) * z(2) - m * z(1) * (z(8) + L(1,1)z(1) + L(1,2) + z(2)) + L(1,2) \\
* (z(1) * (z(8) + L(1,1)z(1) + L(1,2) + z(2))); \\
\]

\[dz(10) = C(3,1)z(8) + C(3,2)z(9) + C(3,3)z(10) + C(3,4)z(11) + D(3,1)z(1) \\
+ D(3,2) * z(2) - (z(8) + L(1,1)z(1) + L(1,2) + z(2)) * z(1) + L(1,2) \\
* (z(8) + L(1,1)z(1) + L(1,2) + z(2)); \\
\]

\[dz(11) = C(4,1)z(8) + C(4,2)z(9) + C(4,3)z(10) + C(4,4)z(11) + D(4,1)z(1) \\
+ D(4,2) * z(2) + (z(8) + L(2,1)z(1) + L(2,2) + z(2)) * z(2) + (z(10) \\
+ L(3,1)z(1) + L(3,2)z(2)) * z(1) + L(4,2) * (z(1) * (z(8) + L(1,1)z(1) \\
+ L(1,2) + z(2))); \\
\]
Appendix H

Section 5.1

Figure 5.5-5.8

% i denotes \omega_1
% j denotes \omega_2
% v denotes the frequency of \psi(t)
% a denotes the matrix A
% dt denotes the step-size
% z1,...,z4 denotes e1,...,e2
% z5,...,z8 denotes estimator of y1,...,y4
% zio represents the initial value, i=1,...,8

q=0.25;
m=-0.1;
v=80;
r=0.01;
F=0.4;
i=10;
j=1;

a=[0 1 0 0; 0 0 1 0; 0 0 0 1; 0 0 -m -q];
c= [1 0 0 0; 0 1 0 0];
p=[-100 -200 -300 -400];
L=place(a',c',p)'
K=L*c;
Q=a−K
E=eig(Q)

N=10000;
t1=0;
t2=500;
dt=t2/N;

z1o=2;
z2o=−3;
z3o=10;
z4o=−8;
z5o=2;
z6o=−3;
z7o=10;
z8o=−8;

options = odeset('RelTol',1e−13,'AbsTol',1e−13);
[t,z] = ode15s(@fun_z,[t1:0.1: t2],[z1o z2o z3o z4o z5o z6o z7o z8o]);

dz = zeros(8,1); % column vector

dz(1)=Q(1,1)*z(1)+Q(1,2)*z(2)+Q(1,3)*z(3)+Q(1,4)*z(4)−K(1,1)*r*sin(v*t)−K(1,2)*r*sin(v*t);
dz(2)=Q(2,1)*z(1)+Q(2,2)*z(2)+Q(2,3)*z(3)+Q(2,4)*z(4)−K(2,1)*r*sin(v*t)−K(2,2)*r*sin(v*t);
dz(3)=Q(3,1)*z(1)+Q(3,2)*z(2)+Q(3,3)*z(3)+Q(3,4)*z(4)−K(3,1)*r*sin(v*t)−K(3,2)*r*sin(v*t);
dz(4)=Q(4,1)*z(1)+Q(4,2)*z(2)+Q(4,3)*z(3)+Q(4,4)*z(4)−K(4,1)*r*sin(v*t)−K(4,2)*r*sin(v*t)+0.01*sin(i*t)−6*(z(5)+z(1))*(z(6)+z(2))^2
\[-3*(z(5)+z(1))^2*(z(7)+z(3))+6*z(5)*z(6)^2+3*z(5)^2*z(7);
\]
\[dz(5)=a(1,1)*z(5)+a(1,2)*z(6)+a(1,3)*z(7)+a(1,4)*z(8)+K(1,1)*z(1)
+K(1,2)*r*sin(v*t);
\]
\[dz(6)=a(2,1)*z(5)+a(2,2)*z(6)+a(2,3)*z(7)+a(2,4)*z(8)+K(2,1)*z(1)
+K(2,2)*r*sin(v*t);
\]
\[dz(7)=a(3,1)*z(5)+a(3,2)*z(6)+a(3,3)*z(7)+a(3,4)*z(8)+K(3,1)*z(1)
+K(3,2)*r*sin(v*t);
\]
\[dz(8)=a(4,1)*z(5)+a(4,2)*z(6)+a(4,3)*z(7)+a(4,4)*z(8)+K(4,1)*z(1)
+K(4,2)*r*sin(v*t)-6*(z(5)+z(1))*(z(6)+z(2))^2
-3*(z(5)+z(1))^2*(z(7)+z(3))-F*j^2*cos(j*t);\]
Appendix I

Section 5.2

Figure 5.9-5.14

% i denotes $\omega_1$
% v denotes $\omega_2$
% a denotes the matrix A
% dt denotes the step-size
% z1,z2 denotes y1,y2
% z3,z4 denotes e1,e2
% z5,z6 denotes estimator of v1,v2
% zio is initial value
%2-cycle
i=100;
v=80;
q=-0.1;
q3=-0.1;
q5=0;
m=0.1;
m3=0;
m5=0;
d=0;
r=0;
\[ A = \begin{bmatrix} 0 & 1 & 0 & 0; & -m & -q & 0 & 0; & 0 & 0 & 1; & 0 & 0 & 0 \end{bmatrix}; \]
\[ A_{11} = A(1:2,1:2); \]
\[ A_{21} = A(3:4,1:2); \]
\[ A_{12} = A(1:2,3:4); \]
\[ A_{22} = A(3:4,3:4); \]
\[ p = [-1 \quad -3]; \]

\[ L = \text{place}(a11',a21',p)' \]
\[ Q = a11 - L * a21; \]
\[ E = \text{eig}(Q) \]

\[ N = 1e + 3; \]
\[ t1 = 0; \]
\[ t2 = 500; \]
\[ dt = t2 / N; \]

\[ z1o = 10; \]
\[ z2o = -5; \]
\[ z3o = 2; \]
\[ z4o = -3; \]
\[ z5o = 0; \]
\[ z6o = 0; \]

\[ \text{options} = \text{odeset}('RelTol',1e-13,'AbsTol',1e-15); \]
\[ [t, z] = \text{ode15s}(@fun_z,[t1:dt:t2],[z1o \ z2o \ z3o \ z4o \ z5o \ z6o]); \]

\[ \text{delta} = \begin{bmatrix} 0 & 0; & -d & -r \end{bmatrix}; \]
\[ \text{uncertainties} \]
\[ H = \text{delta} \times L; \]

\[ C = A_{11} - L \times A_{21}; \]
\[ B = (A_{11} \times L - L \times A_{21} + L \times A_{12} - L \times A_{22}); \]
% i denotes \( \omega_1 \)
% j denotes \( \omega_2 \)
% v denotes \( \omega \)
% a denotes the matrix A
% dt denotes the step-size
% z1, z2 denotes y1, y2
% z3,z4,z5,z6,z7 denotes e1,e2,e3,e4,e5
% z8,z9,z10,z11,z12 denotes estimator of v1,v2,v3,v4,v5
% zio is initial value
q=0.25;
q3=0;
q5=0;
m=-1;
m3=0;
m5=0;
d=0;
r=0;
F=0.4;
v=1;
i=100;
j=80;

\[
a = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -m & q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

A11=a(1:5,1:5);
A21=a(6:7,1:5);
A12=a(1:5,6:7);
A22=a(6:7,6:7);
c=[1 0 0 0 0; 0 0 0 1 1];
p=[-10 -11 -12 -13 -14];
L=place(a11',a21',p)'
Q=a11-L*c
E=eig(Q)

N=1e+4;
t1=0;
t2=100;
\[ dt = \frac{t_2}{N}; \]
\[ t = (0:N) \cdot dt; \]

\[ z_{1o} = 0; \]
\[ z_{2o} = 0; \]
\[ z_{3o} = 2; \]
\[ z_{4o} = -3; \]
\[ z_{5o} = 3; \]
\[ z_{6o} = -2; \]
\[ z_{7o} = -5; \]
\[ z_{8o} = 6; \]
\[ z_{9o} = 1; \]
\[ z_{10o} = -1; \]
\[ z_{11o} = 1.5; \]
\[ z_{12o} = -0.5; \]

options = odeset('RelTol', 1e-13, 'AbsTol', 1e-15);
\[
[t, z] = \text{ode15s}(\@fun, z, [t_1: dt: t_2], [z_{1o} \ z_{2o} \ z_{3o} \ z_{4o} \ z_{5o} \ z_{6o} \ z_{7o} \ z_{8o} \ z_{9o} \ z_{10o} \ z_{11o} \ z_{12o}]);
\]

\[ Y = [z(:, 1:2)]; \quad \% \text{vector } y_1, y_2 \]
\[ w = (z(:, 8:12)'+ L \cdot Y')'; \quad \% z+Ly \]
\[ X = [Y \ w]; \]

\[ \delta = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d -r \end{bmatrix}; \]
\[ H = \delta \cdot L; \]

\[ C = A_{11} - L \cdot A_{21}; \]
\[ D = (A_{11} \cdot L - L \cdot A_{21} + A_{12} \cdot L - A_{22}); \]

\[ dz = \text{zeros}(12, 1); \]
\[ dz(1) = z(2); \]
\[ dz(2) = -(m+d)z(1) - (q+r)z(2) - z(1)^3 + F \cos(v*t) + 0.01 \sin(i*t); \]

\[ dz(3) = C(1,1)z(3) + C(1,2)z(4) + C(1,3)z(5) + C(1,4)z(6) + C(1,5)z(7) + \delta(1,1)z(8) + \delta(1,2)z(9) + \delta(1,3)z(10) \]

\[ + (D(1,1) + D(1,2)) \times 0.01 \sin(j*t) + 0.01 \sin(i*t) - (L(1,1) \times 0.01 \sin(i*t) + L(1,2) \times 0.01 \sin(i*t)); \]

\[ dz(4) = C(2,1)z(3) + C(2,2)z(4) + C(2,3)z(5) + C(2,4)z(6) + C(2,5)z(7) + \delta(2,1)z(8) + \delta(2,2)z(9) + \delta(2,3)z(10) + \delta(2,4)z(11) + H(2,1)z(1) + H(2,2)z(2) \]

\[ + (D(2,1) + D(2,2)) \times 0.01 \sin(j*t) + 0.01 \sin(i*t) - (L(2,1) \times 0.01 \sin(i*t) + L(2,2) \times 0.01 \sin(i*t)); \]

\[ dz(5) = C(3,1)z(3) + C(3,2)z(4) + C(3,3)z(5) + C(3,4)z(6) + C(3,5)z(7) + \delta(3,1)z(8) + \delta(3,2)z(9) + \delta(3,3)z(10) + \delta(3,4)z(11) + H(3,1)z(1) + H(3,2)z(2) \]

\[ + (D(3,1) + D(3,2)) \times 0.01 \sin(j*t) + 0.01 \sin(i*t) - (L(3,1) \times 0.01 \sin(i*t) + L(3,2) \times 0.01 \sin(i*t)); \]

\[ dz(6) = C(4,1)z(3) + C(4,2)z(4) + C(4,3)z(5) + C(4,4)z(6) + C(4,5)z(7) + \delta(4,1)z(8) + \delta(4,2)z(9) + \delta(4,3)z(10) + \delta(4,4)z(11) + H(4,1)z(1) + H(4,2)z(2) \]

\[ + (D(4,1) + D(4,2)) \times 0.01 \sin(j*t) + 0.01 \sin(i*t) - (L(4,1) \times 0.01 \sin(i*t) + L(4,2) \times 0.01 \sin(i*t)); \]

\[ dz(7) = C(5,1)z(3) + C(5,2)z(4) + C(5,3)z(5) + C(5,4)z(6) + C(5,5)z(7) + \delta(5,1)z(8) + \delta(5,2)z(9) + \delta(5,3)z(10) + \delta(5,4)z(11) + H(5,1)z(1) + H(5,2)z(2) \]

\[ + (D(5,1) + D(5,2)) \times 0.01 \sin(j*t) + 0.01 \sin(i*t) - (L(5,1) \times 0.01 \sin(i*t) + L(5,2) \times 0.01 \sin(i*t)); \]
\[ \begin{align*}
+ & L(4,1) \cdot z(1) + L(4,2) \cdot z(2)) ; \\
\text{dz}(8) = & C(1,1) \cdot z(8) + C(1,2) \cdot z(9) + C(1,3) \cdot z(10) + C(1,4) \cdot z(11) + C(1,5) \cdot z(12) \\
& + D(1,1) \cdot z(1) + D(1,2) \cdot z(2) ; \\
\text{dz}(9) = & C(2,1) \cdot z(8) + C(2,2) \cdot z(9) + C(2,3) \cdot z(10) + C(2,4) \cdot z(11) + C(2,5) \cdot z(12) \\
& + D(2,1) \cdot z(1) + D(2,2) \cdot z(2) ; \\
\text{dz}(10) = & C(3,1) \cdot z(8) + C(3,2) \cdot z(9) + C(3,3) \cdot z(10) + C(3,4) \cdot z(11) + C(3,5) \cdot z(12) \\
& + D(3,1) \cdot z(1) + D(3,2) \cdot z(2) ; \\
\text{dz}(11) = & C(4,1) \cdot z(8) + C(4,2) \cdot z(9) + C(4,3) \cdot z(10) + C(4,4) \cdot z(11) + C(4,5) \cdot z(12) \\
& + D(4,1) \cdot z(1) + D(4,2) \cdot z(2) ; \\
\text{dz}(12) = & C(5,1) \cdot z(8) + C(5,2) \cdot z(9) + C(5,3) \cdot z(10) + C(5,4) \cdot z(11) + C(5,5) \cdot z(12) \\
& + D(5,1) \cdot z(1) + D(5,2) \cdot z(2) - (90 \cdot z(2) \cdot z(8) + L(1,1) \cdot z(1) + L(1,2) \cdot z(2)) ^ 2 \\
& + 60 \cdot z(1) ^ 2 \cdot z(9) + L(2,1) \cdot z(1) + L(2,2) \cdot z(2)) + 60 \cdot z(1) \cdot z(8) + L(1,1) \cdot z(1) \\
& + L(1,2) \cdot z(2)) + (z(9) + L(2,1) \cdot z(1) + L(2,2) \cdot z(2)) + 30 \cdot z(1) \cdot z(2) \cdot z(10) \\
& + L(3,1) \cdot z(1) + L(3,2) \cdot z(2)) + 3 \cdot z(1) ^ 2 \cdot z(11) + L(4,1) \cdot z(1) + L(4,2) \cdot z(2) \\
& - F \cdot v^5 \cdot \sin(v \cdot t) ;
\end{align*} \]
Bibliography


