Dynamical Systems: Chaotic Attractors and Synchronization using Time-Averaged Partial Observations of the Phase Space

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Abstract

We study the synchronization of chaotic systems when the coupling between them contains both time averages and stochastic noise. Our model dynamics are given by the Lorenz equations which are a system of three ordinary differential equations in the variables $X$, $Y$ and $Z$. Our theoretical results show that coupling two copies of the Lorenz equations using a feedback control which consists of time averages of the $X$ variable leads to exact synchronization provided the time-averaging window is known and sufficiently small. In the presence of noise the convergence is to within a factor of the variance of the noise. The novelty of our investigation hinges on the analysis of the time averages. We also consider the case when the time-averaging window is not known and show that it is possible to tune the feedback control to recover the size of the time-averaging window. Further numerical computations show that synchronization is more accurate and occurs under much less stringent conditions than our theory requires.
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Chapter 1

Introduction

We produce synchronization results when coupling is applied on two chaotic systems when the coupling between them contains both time averages and stochastic noise. Our model dynamics are given by the Lorenz equations which are a system of three ordinary differential equations in the variables $X$, $Y$ and $Z$. We build off of investigations of the Lorenz model in [12, 14], where space and continuous time observations [2] by use of a control theory perspective similar to that which arises from the derivation of continuous time limits of discrete time filters [3]. We seek to determine relationships between the underlying dynamical system and the observation operator to ensure that synchronization can be obtained. For a coupled pair of chaotic dynamical systems, where the driven system’s initialization is not known precisely, the extent of the use of observed data over time is a topic of interest in meteorology, weather forecasting and any attempt to predict the future using incomplete information about the current state of a dynamical model.

In this thesis, we extend the approach of [2] to use time-averaged continuous-in-time partial observations of the phase space. We examine both the noise-free case and the case where the observations include stochastic noise. We are able to show that the use of time-averaged observations results in synchronization. The techniques investigated here are applicable to different dissipative dynamical systems, and provide many points of continued study, which are detailed in a chapter dedicated to future work.

The outline of the thesis is as follows. In Chapter 1 we introduce the concept of synchronization on the global attractor. We also provide a detailed description of our model problem, a simplified model of atmospheric convection, the Lorenz system, and present an analytical bound on the trajectories which is used through-
out the thesis. We also introduce the coupling method that is the focus of the thesis and introduce the notation that will be used to describe our experiments. In Chapter 2 we present a proof of the existence and uniqueness of our solution. In addition, we present a generalized Gronwall integration proof that will be used throughout the thesis and is applicable to the convergence of our model. In chapter three we study the convergence of the approximating solution in a noise-free setting, ending with an investigation into the sensitivity of the time-averaging window. Chapter 4 presents an analysis when the observations contain stochastic noise. Chapters 5 and 6 present our numerical methods and conclusions. Finally, Chapter 7 provides points of future work.

1.1 Synchronization

Synchronization means that the difference between two solution trajectories converge to zero as time tends to infinity. The sensitivity to initial conditions of dynamical systems means that two identical systems with even a slight perturbation of initial conditions quickly become uncorrelated even though they are bounded and converging to the same attractor. On the theoretical side, we would like to determine when coupling between two identical copies of a dynamical system leads to synchronization and to study the effects that realistic conditions, such as a moving time-averages and noisy observations, have on the synchronization. On the computational side, we would like to simulate the same things and test the sharpness of our theory.

Consider a dynamical system given by the ordinary differential equations

$$\frac{dU}{dt} = \mathcal{F}(U)$$

(1.1)

where $U(0) = U_0 \in \mathbb{R}^n$ and $\mathcal{F}: \mathbb{R}^n \to \mathbb{R}^n$. We use partial information about $U$ to
drive a second solution \( u \) as follows:

\[
\frac{du}{dt} = F(u) + \mu\left(\tilde{R}(U) - R(u)\right)
\]  

(1.2)

with arbitrary initial condition \( u(0) = u_0 \). Here \( \tilde{R}(U) \) denotes an interpolation of partial observations of \( U \) back into the phase space \( \mathbb{R}^n \) that may contain noise as well moving time averages, and \( R(u) \) denotes a feedback control used to nudge \( u \) towards \( U \) over time. Note that the relaxation parameter \( \mu > 0 \) determines the rate at which \( u \) is nudged towards \( U \) as it evolves forward in time and represents a tunable parameter in the system.

Note, we assume the forcing function \( F \) is known exactly and leads to a well posed dynamical system that governs the freely evolving solution. More realistically, \( F \) might only be known up to an approximation \( \tilde{F} \). Ideas for work along these lines is discussed in Section 7.5 in the chapter on future work. We assume the dynamics given by \( F \) are dissipative. However, our analysis relies not just on dissipativity, but on a squeezing property with respect to the observable part of the phase space.

If \( \tilde{R} = R \) and \( u_0 = U_0 \), then \( \mu((\tilde{R}(U_0) - R(u_0)) = 0 \). It follows that both \( u \) and \( U \) evolve according to the same dynamics, which in turn implies \( u(t) = U(t) \) for all \( t \geq 0 \). When \( u_0 \neq U_0 \) it may still happen that \( \|u(t) - U(t)\| \to 0 \) as \( t \to \infty \). When this happens we say that \( u \) synchronizes with \( U \). Note, however, when \( \tilde{R} \neq R \) it may happen that \( \mu((\tilde{R}(U_0) - R(u_0)) \neq 0 \) even if \( U_0 = u_0 \). It immediately follows that \( U(t) \neq u(t) \) for \( t > 0 \). In this case, we look for bounds on \( \|U - u\| \) that depend on the quality of the approximation \( \tilde{R} \approx R \).

If the observations \( \tilde{R} \) have been contaminated using a moving time average with a known window size, then \( R \) can also include an identical moving time average so that \( \tilde{R} = R \). However, if the size of the averaging window is unknown, then we can only guess its size. In this case \( R \) is only an approximation of \( \tilde{R} \). Similarly, if the observations \( \tilde{R} \) contain a unknown noise term, this this noise can not be
including in the feedback term and again $\mathcal{R}$ will differ from $\tilde{\mathcal{R}}$, this time by the noise. The main result of this thesis are theoretical and numerical bounds on the expected value of $\|U - u\|^2$ in the case when $\tilde{\mathcal{R}}$ contains noisy observations that have been averaged with respect to an unknown window size.

1.2 Global Attractors

For general continuous time dynamical systems the number of degrees of freedom is often associated with the number of first order ordinary differential equations (ODEs) in the model. If
\[
\sum_{i=1}^{n} \frac{\partial \mathcal{F}_i}{\partial U_i} < 0,
\]
then the system is said to be dissipative system. This indicates that volumes transported by the dynamics of the system are contracting. In this case, there exists a compact set to which all bounded sets converge as they are transported by the flow. In particular, all trajectories eventually converge that set. Moreover, if this set is invariant, that is, if every point on the attractor is the image under the flow of the dynamics over a fixed period of time of another point on the attractor, then the set is unique and called the global attractor. Since the attractor is bounded and invariant, then any trajectory on the attractor is bounded backwards in time. This leads to another characterization of the attractor as the set of all solutions which are bounded backwards in time. Note that the set of fixed points such that $\{x : \mathcal{F}(x) = 0\}$ are included in the attractor, as is the unstable manifold of the fixed point. This observation can be used to find lower bounds on the size of the attractor.

Due to the presence of an attractor, the sum of the Lyapunov exponents is expected to be negative; an indication that the system is dissipative. The Lyapunov exponents control the exponential growth or contraction of volume elements in
phase space [6]. They quantify the exponential divergence of infinitesimally close state-space trajectories and estimate the amount of chaos in a system. A positive exponent is consistent with diagnosing chaos and represents local instability in a particular direction.

The Lyapunov spectrum is represented geometrically as a small $n$-dimensional sphere of initial conditions. In a chaotic attractor, the sphere evolves into an ellipsoid whose principal axes expand (or contract) at rates given by the Lyapunov exponents. Calculating the largest Lyapunov exponent can yield an estimate of the “level of chaos” using the slope of the divergence as was calculated by Rosenstein and Collins [21]. This determines how two trajectories with nearby initial conditions diverge over time. In fact, we expect that two randomly chosen initial conditions in arbitrary phase space will diverge exponentially at the rate given by the largest Lyapunov exponent. In the context of the synchronization studies in this thesis, it is this tendency for the trajectories to diverge that needs to be overcome by the feedback controller represented by $\mu(\hat{R}(U) - R(u))$ in equation (1.2).

1.3 The Lorenz ’63 Model

The Lorenz ’63 Model is a system of three ordinary differential equations that represent a simplified model of atmospheric convection [16]. The model is dissipative with a quadratic energy-conserving nonlinearity. This system of three coupled non-linear ordinary differential equations satisfies equation (1.1) where $U = (X, Y, Z)$ and

$$F(U) = \begin{bmatrix}
\sigma(Y - X) \\
-\sigma X - Y - XZ \\
-bZ + XY - b(r + \sigma)
\end{bmatrix}.$$ (1.3)
Here $\sigma$ is the Prandtl number, $r$ is the Rayleigh number, and $b$ is a geometric factor. Note that we have employed a coordinate system where the center is shifted to the point $(0, 0, 0)$. We will use the standard parameter values $\sigma = 10$, $b = 8/3$ and $r = 28$ in our study. At these values, the system is known to have a chaotic global attractor $\mathcal{A}$, Tucker [26].

Note that
\[ \sum_{i=1}^{3} \frac{\partial F_i}{\partial U_i} = -\sigma - 1 - b \]
which shows the system is dissipative. Moreover, linearization about the fixed point
\[ (\sqrt{b(r - 1)}, \sqrt{b(r - 1)}, -\sigma - 1) \]
gives the matrix
\[
\begin{bmatrix}
-\sigma & \sigma & 0 \\
1 & -1 & -\sqrt{b(r - 1)} \\
\sqrt{b(r - 1)} & \sqrt{b(r - 1)} & -b
\end{bmatrix},
\]
which has eigenvalues
\[
\lambda_1 \approx -13.85457791 \\
\lambda_2 \approx 0.093955624 - 10.19450522i \\
\lambda_3 \approx 0.093955624 + 10.19450522i.
\]
Since the real parts of two of these eigenvalues is positive, nearby solutions will spiral away onto a two-dimensional unstable manifold. We conclude that the unstable manifold at the fixed point is two dimensional. Therefore, the global attractor contains a two dimensional set and is itself at least two dimensional.

The Lorenz attractor has been shown to be a global attractor contained in a volume bounded by a sphere, a cylinder, the volume between two parabolic sheets, an ellipsoid and a cone [6], see also [5]. A smaller sphere which also contains the attractor was discovered while attempting to improve estimates used in the proof.
of Proposition 3.3. While we did not make use of that bound in this thesis, it has been included as Appendix A because it is of independent interest and may be useful in future work, see Section 7.1. Throughout this thesis we assume that the free running solution lies on the global attractor. Thus, $U_*$ reflects the long term behavior of the Lorenz systems and obeys the \textit{a priori} bounds discussed above. Of those bounds we make exclusive use of the spherical bound given as Theorem 1.1 stated below.

The following proof is essentially the same as the one given in [6]; however, it is simpler because we have already translated the $Z$ variable in the formulation of the Lorenz system.

**Theorem 1.1.** Let $U$ be a trajectory with $U_0 \in A$ and assume that $b \geq 2$ and $\sigma > 0$. Then $|U(t)|^2 \leq K$ for all $t \in \mathbb{R}$ where

$$K = \frac{b^2(r + \sigma)^2}{4(b - 1)} \approx 1540.27. \quad (1.4)$$

**Proof.** We define a Lyapunov function as

$$K = \{(X, Y, Z) \mid X^2 + Y^2 + Z^2 \leq R^2\} \quad (1.5)$$

and consider the evolution of the sphere as determined by the Lorenz equations:

$$\frac{1}{2} \frac{dK}{dt} = \sigma X^2 + Y^2 + bZ^2 - b(r + \sigma)Z.$$ 

$K$ is a decreasing function in time, and so trajectories will move closer to the point $(0, 0, 0)$. Solving the constrained maximization problem

$$K = \max \{K(X, Y, Z) \mid \sigma X^2 + Y^2 + bZ^2 - b(r + \sigma)Z \geq 0\}$$

by using the Lagrange multiplier method with

$$f(X, Y, Z, \lambda) = X^2 + Y^2 + Z^2 + \lambda(\sigma X^2 + Y^2 + bZ^2 + b(r + \sigma)Z)$$

we may bound the radius of a sphere into which all solutions flow as $t \to \infty$ where $K_{max} = R^2$. \qed
1.4 The Coupling Method

In [2] Azouani, Olson, Titi introduced equation (1.2) as a simple data-assimilation method that constructs an approximating solution $u$ from partial observations of an exact solution $U$ over time. A numerical study of this algorithm was performed by Gesho [8], see also Gesho, Olson and Titi [9]. The effects of noisy measurements were treated by Bessaih, Olson and Titi in [10], which led to a stochastic version of equation (1.2) that included a $Q$-Brownian motion on the right hand side to represent measurement errors. In those works the model problem was the two-dimensional incompressible Navier–Stokes equations and $\mathcal{R}(U)$ represented nodal measurements of an Eulerian velocity field as might be taken by an array of idealized weathervane anemometers that can measure the exact velocity at a discrete set of points in space at any point in time.

Synchronization of coupled copies of the Lorenz system was first observed by Pecora and Carroll in [20]. The Lorenz system was considered in the context of data assimilation by Hayden [11] and by Hayden, Olson and Titi [12] and to understand the effects noise in the observational measurements by Law, Shukla and Stuart in [14]. This thesis explores the case in which the available information about the true state of $U$ is further contaminated by a moving time average with an averaging window of size $\delta$. This type of moving average is physically relevant to real-world scientific instrumentation that approximates, for example, the measurement of velocity at any instant in time by an average velocity taken over a very small time interval.

Another physically relevant property of most instrumentation is that measurements are produced at a sequence of discrete moments in time rather than continuously in time. Sequences of discrete-in-time observations for the Lorenz system have been considered by Hayden [11], by Hayden, Olson and Titi in [12] and by Cecilia, Jolly, Foias and Titi in [7]. In order to focus on time averages we do not
treat discrete-in-time data in the present thesis.

We now describe the observational measurements of $U$ denoted by $\tilde{R}(U)$. Coupling through either the $X$ or $Y$ variable can lead to synchronization in the Lorenz system, see [11]. In this thesis the coupling is on $X$ because the analysis is easier and more closely represents the analysis used when treating the synchronization of dynamics given by partial differential equations. Let $\tilde{R}: \mathbb{R}^3 \to \mathbb{R}^3$ in equation (1.2) be given by one of

$$R = \mathcal{L} \circ \mathcal{O}, \quad \tilde{R} = \mathcal{L} \circ \tilde{\mathcal{O}}, \quad R_\delta = \mathcal{L} \circ \mathcal{O}_\delta \quad \text{or} \quad \tilde{R}_\delta = \mathcal{L} \circ \tilde{\mathcal{O}}_\delta,$$

where $\mathcal{O}$, $\tilde{\mathcal{O}}$, $\mathcal{O}_\delta$ and $\tilde{\mathcal{O}}_\delta$ represent observations with and without noise/time-averages and $\mathcal{L}$ represents an interpolation of the observations back into the phase space. Note that $\delta$ is the size of the averaging window present in the observational measurements of $U$. Specifically, we take $\mathcal{O}: \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{L}: \mathbb{R} \to \mathbb{R}^3$ to be given by

$$\mathcal{O}(X, Y, Z) = X \quad \text{and} \quad \mathcal{L}(X) = (X, 0, 0) \quad (1.6)$$

throughout our study. Thus, $\mathcal{L}(U)$ represents exact observations of $U$ obtained instantaneously at the time $t$ and $\tilde{\mathcal{O}}(U)$ a noisy version of $\mathcal{O}(U)$ which, following [10], satisfies

$$\tilde{\mathcal{O}}(U)dt = \mathcal{O}(U)dt + \epsilon dW_t \quad (1.7)$$

where $W_t$ is a standard one-dimensional Brownian motion with underlying probability space $\Omega$. The corresponding noise-free and noisy moving time averages are then given by

$$\mathcal{O}_\delta(U) = \frac{1}{\delta} \int_{t-\delta}^t \mathcal{O}(U(s)) \, ds \quad (1.8)$$

and

$$\tilde{\mathcal{O}}_\delta(U) = \frac{1}{\delta} \int_{t-\delta}^t \tilde{\mathcal{O}}(U(s)) \, ds. \quad (1.9)$$

For notational convenience we define $\mathcal{O}_0 = \mathcal{O}$ and $R_0 = R$ so that $\tilde{\delta} = 0$ indicates measurements for which the averaging is zero.
As the noise is unknown, the possible choices for $\mathcal{R}$ used in the feedback control $\mathcal{R}(u)$ can not include noise. This ensures the evolution of the driven solution is given only in terms of things which are observable. It may also happen that the size of the averaging present in the observations is also unknown. In this case, we let $\delta$ represent the time averaging in the feedback control. Thus, the possible choices for $\mathcal{R}$ are given given by one of $R$ or $R_\delta$ where $\delta$ is an approximation of $\hat{\delta}$.

When the observations are noise-free and $\delta = \hat{\delta}$ we obtain conditions under which $\|U - u\| \to 0$ as $t \to \infty$. When the observations are noisy $u$ becomes a stochastic process. In particular, $u$ solves a stochastic differential equation when $\hat{\delta} = 0$ and a random differential equation when $\hat{\delta} > 0$. In this case we obtain bounds of the form

$$\limsup_{t \to \infty} \mathbb{E}[\|U - u\|^2] < B_1. \quad (1.10)$$

Chebyshev’s inequality applied to the bound (1.10) implies there exists $T > 0$ such that

$$\mathbb{P}\{\|U - u\|^2 \leq B_2 \epsilon^2\} \geq 1 - B_1/B_2 \quad (1.11)$$

for every $B_2 > B_1$ and $t > T$. Thus, the error has a 90 percent chance of being bounded by $B_2 = 10B_1$ at any fixed point in time.

We also consider the general case when $\delta \neq \hat{\delta}$, that is, when the time averaging present in the observations is unknown. In this case, the equation for $u$ becomes

$$\frac{du}{dt} = \mathcal{F}(u) - \mu(\bar{R}_\delta(U) - R_\delta(u)), \quad (1.12)$$

In the absence of noise $\epsilon = 0$ and we obtain as Proposition 3.3 conditions and a bound $B_3$ such that

$$\limsup_{t \to \infty} \|U - u\|^2 < B_3(\hat{\delta} - \delta)^2. \quad (1.13)$$

This analysis suggests it may is possible to recover $\hat{\delta}$ using successively better guesses. When there is noise we obtain our final result Proposition 4.3 which
provides bounds $B_4$ and $B_5$ such that

$$\limsup_{t \to \infty} E[||U - u||^2] < B_4(\hat{\delta} - \delta)^2 + B_5 \varepsilon^2.$$  \hspace{1cm} (1.14)

In this case, $\delta$ can be tuned to match $\hat{\delta}$ only up to the point where the second term starts to dominate the bound.
Chapter 2

Preliminaries

In this chapter we prove a local existence theorem which will be used later to show our data assimilation equations are globally well posed as well as an integro-Gronwall inequality which will be used to estimate the accuracy of the synchronization.

We now turn our attention to proving existence and uniqueness of solutions for the integro-differential equations which define the driven system as a result of the time averages present in the observational measurements. Let

\[ f: [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad g: [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n \]

be continuous in the first variable and uniformly Lipschitz in the second. Consider the integro-differential equation

\[
V' = \begin{cases} 
    f(t, V) & \text{for } t < \delta \\
    f(t, V) + \frac{1}{\delta} \int_{t-\delta}^{t} g(s, V(s)) \, ds & \text{for } t > \delta
\end{cases}
\]

(2.1)

where \( V \in \mathbb{R}^n \) is a continuous function. Note that \( V' \) may not be defined at time \( t = \delta \), unless

\[
\int_{0}^{\delta} g(s, y(s)) \, ds \neq 0.
\]

As this is not in general true, then we consider as solutions to equations (2.1) continuous functions which are piecewise differentiable. This equation, similar to the Volterra equations studied by Levin in [15], is locally well posed as shown by

**Lemma 2.1.** Suppose \( V: [0, T] \to \mathbb{R}^n \) satisfies equations (2.1), then there exists \( \Delta t > 0 \) such that \( V \) can be uniquely extended to a function on \( [0, T + \Delta t] \) which also satisfies equations (2.1).
Proof. If \( T < \delta \), standard existence and uniqueness results for ordinary differential equations show there exists \( \Delta t > 0 \) such that the solution \( V \) can be extended to the interval \([0, T + \Delta t)\). If \( T \geq \delta \), the result follows from a straight-forward fixed point argument which we include here for completeness.

Define

\[
D = \max \{ \|V(t)\| : t \in [0, T] \} + 1
\]

and

\[
M_f = \max \{ \|f(t, v)\| : t \in [0, T + 1] \text{ and } \|v\| \leq D \}.
\]

Let \( \gamma_f \) be the Lipschitz constant such that

\[
\|f(t, v_1) - f(t, v_2)\| \leq \gamma_f \|v_1 - v_2\| \quad \text{for } \|v_1\|, \|v_2\| \leq D.
\]

Corresponding to the function \( g \) similarly define \( M_g \) and \( \gamma_g \). Let

\[
\Delta t = \min \left( 1, \frac{1}{M_f + M_g}, \frac{1}{2(\gamma_f + \gamma_g)} \right).
\]

(2.2)

and define

\[
\mathcal{X} = \left\{ v \in C([0, T + \Delta t], \mathbb{R}^n) : v(t) = V(t) \text{ for } t \leq T, \text{ and } \|v(t)\| \leq D \text{ for } t \in [T, T + \Delta t] \right\}.
\]

(2.3)

Consider the mapping \( \mathcal{J} : \mathcal{X} \to C([0, T + \Delta t], \mathbb{R}^n) \) given when \( t \leq T \) by \( \mathcal{J}(v)(t) = V(t) \) and when \( t > T \) by

\[
\mathcal{J}(v)(t) = V(T) + \int_T^t f(\tau, v(\tau)) \, d\tau + \int_T^t \frac{1}{\delta} \int_{\tau-\delta}^{\tau} g(s, v(s)) \, ds \, d\tau.
\]

We claim that \( \mathcal{J} : \mathcal{X} \to \mathcal{X} \) and that \( \mathcal{J} \) is a contraction. Estimate

\[
\|\mathcal{J}(v)(t)\| \leq \|V(T)\| + \int_T^t \|f(\tau, v(\tau))\| \, d\tau
\]

\[
+ \int_T^t \frac{1}{\delta} \int_{\tau-\delta}^{\tau} \|g(s, v(s))\| \, ds \, d\tau
\]

\[
\leq (D - 1) + \Delta t M_f + \Delta t M_g \leq D.
\]
Therefore, \( J : \mathcal{X} \to \mathcal{X} \). Moreover,

\[
\|J(v_1)(t) - J(v_2)(t)\| \leq \int_T^t \|f(\tau, v_1(\tau)) - f(\tau, v_2(\tau))\| d\tau \\
+ \int_T^t \frac{1}{\delta} \int_{\tau - \delta}^{\tau} \|g(s, v_1(s)) - g(s, v_2(s))\| ds d\tau \\
\leq \Delta t(\gamma_f + \gamma_g) \max\{\|v_1(s) - v_2(s)\| : s \in [T, T + \Delta t]\} \\
\leq (1/2) \max\{\|v_1(s) - v_2(s)\| : s \in [T, T + \Delta t]\}
\]

shows that \( J \) is a contraction. Now, by the contraction mapping theorem, there is a unique fixed point \( v \in \mathcal{X} \) such that \( J(v) = v \). Since this fixed point clearly satisfies equations (2.1), the lemma is proved.

We now prove the following integro-differential Gronwall inequality.

**Lemma 2.2.** Suppose

\[
\frac{dV(t)}{dt} + V(t) \leq Ah \int_{t-h}^{t} V(s) \, ds + B 
\tag{2.4}
\]

where \( A > 0, B > 0 \) and

\[
0 < h < \min \left\{ \sqrt{\frac{1 - e^{-1}}{2A}}, 1 \right\}.
\]

Then \( \lim \sup_{t \to \infty} V(t) < eB \).

**Proof.** By hypothesis there exists \( \gamma \in (0, 1) \) such that

\[
h < \sqrt{\frac{e^{-\gamma} - e^{-1}}{2A}}.
\]

Therefore

\[
e^{-(1-\gamma)} + 2Ah^2e^{\gamma h} < e^{-(1-\gamma)} + 2Ae^{-\gamma} - e^{-1} e^{\gamma} = 1
\tag{2.5}
\]

and

\[
2e^{-1} + 2Ah^2(1 - e^{-1}) - e^{-2} < e^{-1} + e^{-\gamma}(1 - e^{-1}) = \beta < 1.
\tag{2.6}
\]

Moreover,

\[
\beta + (e^{-1} - e^{-2})(\beta^{-1} - 1) \leq 1.
\tag{2.7}
\]
Inequality (2.7) follows by taking $\phi(\beta) = \beta + (e^{-1} - e^{-2})(\beta^{-1} - 1)$ and differentiating to get $\phi'(\beta) = 1 - (e^{-1} - e^{-2})\beta^{-2}$. Therefore $\phi$ is increasing when $\beta > \sqrt{e^{-1} - e^{-2}}$. Since $\gamma \in (0, 1)$ implies $\beta \geq 2e^{-1} - e^{-2} > \sqrt{e^{-1} - e^{-2}}$ then $\phi(\beta) \nearrow 1$ as $\beta \nearrow 1$.

Let $m$ be the smallest integer such that $1 + h \leq m$ and $B = e\beta B$. Since $\mathcal{V}$ is continuous there is $R$ such that $\mathcal{V}(s) \leq Re^{-\gamma s} + B$ for all $s \in [0, m]$. For induction, suppose $\mathcal{V}(s) \leq Re^{-\gamma s} + B$ for all $s \in [0, n]$. Claim $\mathcal{V}(s) < 2(Re^{-\gamma s} + B)$ for all $s \in [n, n+1]$. If not, then there is some $t \in (n, n+1]$ such that $\mathcal{V}(t) \geq 2(Re^{-\gamma t} + B)$ and $\mathcal{V}(s) \leq 2(Re^{-\gamma s} + B)$ for $s \in [n, t]$. Multiplying inequality (2.4) by $e^t$ and integrating from $t - 1$ to $t$ we obtain

$$
\mathcal{V}(t) \leq \mathcal{V}(t - 1)e^{-1} + e^t Ah \int_{t-1}^{t} e^{\rho} \int_{t-h}^{\rho} \mathcal{V}(s) ds d\rho + (1 - e^{-1})B.
$$

Since

$$
\int_{t-1}^{t} e^{\rho} \int_{t-h}^{\rho} \mathcal{V}(s) ds d\rho \leq \int_{t-1}^{t} e^{\rho} \int_{t-h}^{\rho} 2(Re^{-\gamma s} + B) ds d\rho
$$

$$
\leq 2h \int_{t-1}^{t} e^{\rho}(Re^{-\gamma(t-h)} + B) d\rho
$$

$$
\leq 2Re^{\gamma h} \int_{t-1}^{t} e^{(1-\gamma)\rho} d\rho + 2Bh \int_{t-1}^{t} e^\rho d\rho
$$

$$
\leq 2Re^{\gamma h} e^{(1-\gamma)t} + 2Bh(e^{t} - e^{t-1}),
$$

then using estimates (2.5) and (2.7) we obtain

$$
\mathcal{V}(t) \leq (Re^{-\gamma(t-1)} + B)e^{-1}
$$

$$
+ Ah(2Re^{\gamma h} e^{-\gamma t} + 2Bh(1 - e^{-1})) + (1 - e^{-1})B
$$

$$
= Re^{-\gamma t}(e^{-(1-\gamma)} + 2Ah^2 e^{\gamma h})
$$

$$
+ B(2e^{-1} + 2Ah^2(1 - e^{-1}) - e^{-2})
$$

$$
+ B((e^{-1} - e^{-2})(\beta^{-1} - 1))
$$

$$
\leq Re^{-\gamma t} + B(\beta + (e^{-1} - e^{-2})(\beta^{-1} - 1)).
$$

which contradicts $\mathcal{V}(t) \geq 2(Re^{-\gamma t} + B)$. Therefore $\mathcal{V}(s) \leq 2(Re^{-\gamma s} + B)$ for $s \in [n, n+1]$ and it immediately follows from equation (2.8) and induction that $\mathcal{V}(t) \leq Re^{-\gamma t} + B$ for $t > 0$. Consequently $\limsup_{t \to \infty} \mathcal{V}(t) \leq B < eB$. \qed
We remark that the proof of Lemma 2.2 is still valid when \( B = 0 \) except for the strict inequality in the last step. Consequently we also obtain

**Corollary 2.1.** Suppose the hypothesis of Lemma 2.2 are satisfied except that \( B = 0 \). Then \( \mathcal{V}(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

Combining Lemma 2.1 with Lemma 2.2 leads to a proof of global existence of the driven solution \( u \) for all time when the \( h \) appearing in Lemma 2.2 is small enough. The main issue is showing that the size of the \( \Delta t \) from Lemma 2.1 is uniformly bounded below at any point in time. Since the size of \( \Delta t \) given in (2.2) is constrained by \( M_f \) and \( M_g \), it is enough to show that these two constants don’t grow without bound over time.

We assume, as is the case for the Lorenz system focused on in this thesis, that

\[
L_f = \sup\{f(t,0) : t > 0\} \quad \text{and} \quad L_g = \sup\{g(t,0) : t > 0\}
\]

are finite. In particular, when there is no noise, the \( f \) and \( g \) which correspond to the Lorenz system are time independent. When there is noise we obtain probabilistic bounds on \( f \) and \( g \) which are uniform in time because the variance of the noise process is stationary. Since

\[
\|f(t,v)\| \leq \|f(t,v) - f(t,0)\| + \|f(t,0)\| \leq \gamma_f \|v\| + L_f \]

then

\[
M_f \leq \gamma_f D + L_f \quad \text{and similarly} \quad M_g \leq \gamma_g D + L_g.
\]

Consequently, the only requirement to prevent \( \Delta t \) from shrinking is that \( \|u\| \) stays below \( D \).

Now, if \( h \) is small enough then taking \( \mathcal{V} = \|U - u\|^2 \) in Lemma 2.2 (or in the case of noise taking \( \mathcal{V} = \mathbb{E}[\|U - u\|^2] \)) implies that

\[
\|U - u\|^2 \leq eB.
\]
Therefore taking $D = \sqrt{eB} + \sqrt{K}$ yields

$$\|u\| \leq \|U - u\| + \|U\| \leq D \quad \text{for} \quad t \in [0, T + \delta t].$$

We conclude that $\|u\|$ remains uniformly bounded and therefore exists globally in time. In the case where there is noise, the same result follows almost everywhere after an application of the Borel-Cantelli lemma.
Chapter 3

Coupling without Noise

When there is no noise in the coupling we show that the difference between the freely evolving system and the driven system converges to zero over time.

Section 3.1 treats the case when the coupling involves the $X$ variable measured exactly at each moment in time. We include a complete analysis of this simple case as a frame of reference. Section 3.2 treats the case when the coupling involves a simple moving time-average of the $X$ variable. Provided the averaging window is small enough we again show that exact synchronization occurs. Section 3.3 treats the case when the size of the averaging window is known only approximately. In this case we obtain an upper bound on the accuracy of the synchronization that depends on $|\hat{\delta} - \delta|$ where $\hat{\delta}$ is the exact, but unknown, size of the averaging window used to form the moving average of $X$ and $\delta$ is an the size of the averaging window used on $x$ in the feedback control.

3.1 Noiseless Instantaneous-in-Time Coupling

This chapter treats the simplest case of synchronization for the Lorenz system in which there are no time averages present in the observational measurements. Thus, $\hat{\delta} = 0$, $\delta = 0$ and $\varepsilon = 0$ in equation (1.12).

Substituting $\mathcal{R} = \mathcal{L} \circ \mathcal{O}$ into equation (1.2) where $\mathcal{L}$ and $\mathcal{O}$ are given by
equations (1.6) yields the driven system

\[
\begin{align*}
\dot{x} &= -\sigma(y - x) + \mu(X - x) \\
\dot{y} &= -\sigma x - y - xz \\
\dot{z} &= -bz + xy - b(r + \sigma)
\end{align*}
\]  
(3.1)  
(3.2)  
(3.3)

We now show for \( \mu \) sufficiently large that the approximating solution \((x, y, z)\) converges to the free-running solution \((X, Y, Z)\) as \( t \) tends to infinity.

**Proposition 3.1.** Suppose \( \delta = 0, \hat{\delta} = 0 \) and \( \varepsilon = 0 \). Let \( \mu > K/4 - \sigma \), then \( ||U - u|| \to 0 \) as \( t \to \infty \). In particular, synchronization is obtained by equation (1.1).

**Proof.** Define \( V = \Delta X^2 + \Delta Y^2 + \Delta Z^2 \) where \( \Delta X = X - x \), \( \Delta Y = Y - y \) and \( \Delta Z = Z - z \). Since

\[
\begin{align*}
\Delta \dot{X} &= -(\sigma + \mu)\Delta X + \sigma \Delta Y \\
\Delta \dot{Y} &= -\sigma \Delta X - \Delta Y - \Delta XZ - x\Delta Z \\
\Delta \dot{Z} &= -b\Delta Z + \Delta XY + x\Delta Y
\end{align*}
\]  
(3.4)  
(3.5)  
(3.6)

we have

\[
\frac{1}{2} \dot{V} + (\sigma + \mu)\Delta X^2 + \Delta Y^2 + b\Delta Z^2 = -\Delta X\Delta YZ + \Delta XY\Delta Z.
\]

Since \( \mu + \sigma > K/4 \) then there exists \( \alpha \in (0, 2) \) such that

\[
\sigma + \mu = \frac{\alpha}{2} + \frac{K}{2(2 - \alpha)}.
\]  
(3.7)

Using Young’s inequality we have

\[
-\Delta X\Delta YZ \leq \frac{Z^2\Delta X^2}{2} \left( \frac{1}{2 - \alpha} \right) + \frac{\Delta Y^2}{2(2 - \alpha)}
\]

and

\[
\Delta XY\Delta Z \leq \frac{Y^2\Delta X^2}{2} \left( \frac{1}{2b - \alpha} \right) + \frac{\Delta Z^2}{2(2b - \alpha)}.
\]

Therefore

\[
\frac{1}{2} \dot{V} + \left( \sigma + \mu - \frac{Y^2}{2(2b - \alpha)} - \frac{Z^2}{2(2 - \alpha)} \right) \Delta X^2 + \frac{\alpha}{2} \Delta Y^2 + \frac{\alpha}{2} \Delta Z^2 \leq 0.
\]
Since our working assumption is that $U$ lies on the global attractor, then Theorem 1.1 and the fact that $b \geq 1$ implies that
\[
\frac{Y^2}{2(2b - \alpha)} + \frac{Z^2}{2(2 - \alpha)} \leq \frac{K}{2(2 - \alpha)} = \sigma + \mu - \frac{\alpha}{2}
\]
Hence
\[
\frac{1}{2} \dot{V} + \frac{\alpha}{2} \Delta X^2 + \frac{\alpha}{2} \Delta Y^2 + \frac{\alpha}{2} \Delta Z^2 \leq 0.
\]
and consequently $\sigma + \mu \geq 1$ implies that $\dot{V} + \alpha V \leq 0$. Integrating over the interval $[0, t]$ yields $V(t) \leq V(0)e^{-\alpha t}$.

3.2 Noiseless Time-averaged Coupling

We now examine convergence of the trajectories using the time-averaged observation operator when $\hat{\delta}$ is known without noise. Taking $\delta = \hat{\delta}$ and substituting $\mathcal{R}_\delta = \mathcal{L} \circ \mathcal{O}_\delta$ into equation (1.2) yields the driven system given by
\[
\dot{x} = -\sigma(y - x) + \mu(\bar{X} - \bar{x})
\]
where
\[
\bar{X} = \frac{1}{\delta} \int_{t-\delta}^{t} X(s) \, ds \quad \text{and} \quad \bar{x} = \frac{1}{\delta} \int_{t-\delta}^{t} x(s) \, ds
\]
with $\dot{y}$ and $\dot{z}$ given as before by equations (3.2) and (3.3). Thus equations (3.5) and (3.4) remain the same but the evolution of the $\Delta X$ given by equation (3.4) now becomes
\[
\Delta \dot{X} = -\sigma \Delta X + \sigma \Delta Y - \mu \Delta \bar{X}
\]
where $\Delta \bar{X} = \bar{X} - \bar{x}$. We show for $\mu$ sufficiently large and $\delta$ small that the approximating solution converges to the reference solution as $t$ tends to infinity.

Before proving the main result in this chapter, we first turn our attention to an estimate that compares time-averaged and instantaneous-in-time feedback terms.
Lemma 3.1. Let $\Delta X = \bar{X} - \bar{x}$. We have

$$|\Delta X - \Delta \bar{X}|^2 \leq 4\delta(\sigma + \mu)^2 \int_{t-2\delta}^t \{|\Delta X|^2 + |\Delta Y|^2\}.$$ 

Proof. By definition

$$|\Delta X - \Delta \bar{X}| = \frac{1}{\delta} \left| \int_{t-\delta}^t \{\Delta X(t) - \Delta X(s)\} \, ds \right|$$

$$\leq \frac{1}{\delta} \int_{t-\delta}^t \int_s^t |\Delta \bar{X}(\rho)| \, d\rho \, ds \leq \int_{t-\delta}^t |\Delta \bar{X}(\rho)| \, d\rho$$

$$= \int_{t-\delta}^t | - \sigma \Delta X(\rho) + \sigma \Delta Y(\rho) - \mu \Delta \bar{X}(\rho)| \, d\rho$$

$$\leq \sigma \int_{t-\delta}^t |\Delta X| + \sigma \int_{t-\delta}^t |\Delta Y| + \mu \int_{t-\delta}^t \int_{t-\delta}^t |\Delta \bar{X}(\xi)| \, d\xi \, d\rho$$

$$\leq \sigma \int_{t-\delta}^t |\Delta X| + \sigma \int_{t-\delta}^t |\Delta Y| + \mu \int_{t-2\delta}^t |\Delta \bar{X}(\xi)| \, d\xi$$

$$\leq (\sigma + \mu) \int_{t-2\delta}^t |\Delta X| + \sigma \int_{t-\delta}^t |\Delta Y|.$$ 

Therefore,

$$|\Delta X - \Delta \bar{X}|^2 \leq 2(\sigma + \mu)^2 \left( \int_{t-2\delta}^t |\Delta X| \right)^2 + 2\sigma^2 \left( \int_{t-\delta}^t |\Delta Y| \right)^2$$

$$\leq 4\delta(\sigma + \mu)^2 \int_{t-2\delta}^t \{|\Delta X|^2 + |\Delta Y|^2\},$$

which finishes the proof. \qed

We are now ready to show the convergence of the driven solution to the free running solution.

Proposition 3.2. Suppose $\delta = \hat{\delta}$ and $\varepsilon = 0$. Let $\alpha \in (0, 2)$, $\mu \geq K/(2 - \alpha)^2 - \sigma$ and

$$\delta < \frac{1 - e^{-1}}{2} \cdot \frac{\alpha}{16\mu(\sigma + \mu)^2}.$$ 

Then $||U - u|| \to 0$ as $t \to \infty$.

Proof. The proof proceeds as the proof of Proposition 3.1. After multiplying by $\Delta X$ the only difference is the additional term on the right hand side that which
we estimate using Lemma 3.1 as
\[ \mu(\Delta X - \Delta \bar{X}) \Delta X \leq \frac{\mu}{2\alpha} |\Delta X - \Delta \bar{X}|^2 + \frac{\alpha \mu}{2} |\Delta X|^2 \]
\[ \leq \frac{2\delta \mu(\sigma + \mu)^2}{\alpha} \int_{t-2\delta}^{t} \{ |\Delta X|^2 + |\Delta Y|^2 \} + \frac{\alpha \mu}{2} |\Delta X|^2. \]
Therefore
\[ \frac{1}{2} \dot{V} + \left( \sigma + \mu - \frac{\alpha \mu}{2} - \frac{Y^2}{2(2b - \alpha)} - \frac{Z^2}{2(2 - \alpha)} \right) \Delta X^2 + \frac{\alpha}{2} \Delta Y^2 + \frac{\alpha}{2} \Delta Z^2 \]
\[ \leq \frac{2\delta \mu(\sigma + \mu)^2}{\alpha} \int_{t-2\delta}^{t} \{ |\Delta X|^2 + |\Delta Y|^2 \}. \]
By hypothesis
\[ \frac{\alpha \mu}{2} + \frac{K}{2(2 - \alpha)} \leq \mu + \sigma \left( 1 - \frac{\alpha}{2} \right). \]
Therefore
\[ \dot{V} + \alpha V \leq \frac{4\delta \mu(\sigma + \mu)^2}{\alpha} \int_{t-2\delta}^{t} V(s) \, ds. \]
Define \( \mathcal{V}(\tau) = V(\tau/\alpha) \). It follows that
\[ \frac{d\mathcal{V}(\tau)}{d\tau} + \mathcal{V}(\tau) \leq \frac{4\delta \mu(\sigma + \mu)^2}{\alpha^3} \int_{\tau-2\alpha \delta}^{\tau} \mathcal{V}(s) \, ds. \]
By hypothesis \( \delta \) is small enough that we may apply Corollary 2.1 with
\[ A = 4\mu(\sigma + \mu)^2/\alpha^3, \quad B = 0 \quad \text{and} \quad h = 2\alpha \delta \]
to conclude that \( V \to 0 \) as \( t \to \infty \). \( \square \)

Note that by taking \( \alpha \) small we obtain synchronization for values of \( \mu \) close to the limiting value \( K/4 - \sigma \) given in Proposition 3.1. However, as \( \alpha \) vanishes the length \( \delta \) of the time averaging window also vanishes. On the other hand, as \( \alpha \) approaches 2, then \( \mu \) approaches infinity and since \( \delta \) depends inversely on \( \mu^{3/2} \) then again \( \delta \) vanishes. Thus, there is some value \( \alpha \in (0, 2) \) for which our analytic bound on \( \delta \) is maximal. Simple calculus yields this optimal value to be
\[ \alpha^* = -\rho \cos(\phi) + \frac{22}{15} + \rho \sqrt{3} \sin \phi \approx 0.28413 \]
where
\[ \rho = \frac{4\sqrt{25370}}{75} \quad \text{and} \quad \phi = \frac{1}{3} \arctan \left( \frac{19\sqrt{1731443695}}{167665} \right). \]

For \( \alpha = \alpha^* \) it follows that we may take
\[ \mu \approx 513.153 \quad \text{and} \quad \delta \approx 0.0000063216. \]

### 3.3 Unknown Noiseless Time-averaged Coupling

In the previous chapter we assumed the time averaging window of the observations was known exactly. We then used this same averaging window in the feedback control of the driven system to obtain exact synchronization. In this chapter we examine the case when the time averaging in the observations of the free running solution is unknown. Specifically, we have

\[
\dot{x} = -\sigma(y - x) + \mu \left( \frac{1}{\delta} \int_{t-\delta}^{t} X(s)ds - \frac{1}{\hat{\delta}} \int_{t-\hat{\delta}}^{t} x(s)ds \right)
\]

with \( \dot{y} \) and \( \dot{z} \) given as before by equations (3.2) and (3.3). Recall that \( \hat{\delta} \) is the unknown size of the averaging window in the observations of the free running solution and \( \delta \) be the averaging used in the feedback control of the driven system. Recall that \( \hat{\delta} \) is the unknown size of the averaging window in the observations of the free running solution and \( \delta \) is the averaging used in the feedback control of the driven system. In this case

\[
\Delta X = -\sigma \Delta X + \sigma \Delta Y - \frac{\mu}{\delta} \int_{t-\delta}^{t} X + \frac{\mu}{\hat{\delta}} \int_{t-\hat{\delta}}^{t} x
\]

\[=-(\sigma+\mu)\Delta X + \sigma \Delta Y + \mu(\Delta X - \Delta \bar{X}) + \mu \mathcal{E}_1 + \mu \mathcal{E}_2 \tag{3.9} \]

where
\[
\mathcal{E}_1 = \frac{1}{\delta} \int_{t-\delta}^{t} X \quad \text{and} \quad \mathcal{E}_2 = \left( \frac{1}{\delta} - \frac{1}{\hat{\delta}} \right) \int_{t-\hat{\delta}}^{t} X
\]

represents the errors resulting from the unknown time averaging window. Note that when \( \delta = \hat{\delta} \) this is exactly the coupling method treated in Proposition 3.2.
When $\delta \neq \hat{\delta}$ exact synchronization does not occur; however, the error between $u$ and $U$ can be bounded by a constant which tends to zero as $\delta$ approaches $\hat{\delta}$.

We begin with a slightly modified version of Lemma 3.1 that reflects the fact that $\hat{\delta} \neq \delta$.

**Lemma 3.2.** We have

$$|\Delta X - \Delta \bar{X}|^2 \leq 4\delta(\sigma + \mu)^2 \int_{t-2\delta}^{t} \left\{ |\Delta X|^2 + |\Delta Y|^2 \right\} + 8\mu^2 \delta^2 K \left( \frac{\delta - \hat{\delta}}{\delta} \right)^2.$$ 

**Proof.** The proof proceeds as in Lemma 3.1. Thus

$$|\Delta X - \Delta \bar{X}| \leq \int_{t-\delta}^{t} |\Delta \bar{X}(\rho)|d\rho \leq I_1 + I_2 + I_3$$

where

$$I_1 = (\sigma + \mu) \int_{t-2\delta}^{t} |\Delta X|, \quad I_2 = \sigma \int_{t-\delta}^{t} |\Delta Y|$$

and

$$I_3 = \mu \int_{t-\delta}^{t} \frac{1}{\delta} \int_{\rho-\delta}^{\rho+\delta} |\Delta \bar{X}(\xi)|d\xi d\rho + \mu \int_{t-\delta}^{t} \left( \frac{1}{\delta} - \frac{1}{\hat{\delta}} \right) \int_{\rho-\delta}^{\rho+\delta} |\Delta \bar{X}(\xi)|d\xi d\rho$$

$$\leq \mu \left| \int_{t-\delta}^{t} \frac{1}{\delta} \int_{\rho-\delta}^{\rho+\delta} \sqrt{K} \xi d\xi d\rho \right| + \mu \left| \int_{t-\delta}^{t} \left( \frac{1}{\delta} - \frac{1}{\hat{\delta}} \right) \int_{\rho-\delta}^{\rho+\delta} \sqrt{K} \xi d\xi d\rho \right|$$

$$= 2\mu \sqrt{K} \delta \left| \frac{\delta - \hat{\delta}}{\delta} \right|$$

Squaring the results as

$$|\Delta X - \Delta \bar{X}|^2 \leq (I_1 + I_2 + I_3)^2 \leq 2I_1^2 + 4I_2^2 + 4I_3^2$$

yields the desired result. \hfill \Box

**Proposition 3.3.** Given $\alpha \in (0, 2)$, let

$$\mu \geq J K / (2 - \alpha)^2 - \sigma \quad \text{where} \quad J = \sigma (1 + \sqrt{2}) \left( \frac{\hat{\delta}}{2} + \delta \right) + 1.$$

and

$$\delta < \sqrt{\frac{1 - e^{-1}}{2}} \cdot \frac{\alpha}{16 \mu (\sigma + \mu)^2}. \quad (3.10)$$
Then
\[ \limsup_{t \to \infty} ||U - u||^2 < \frac{5e(2 - \alpha)\mu^2}{\alpha} \left( \frac{\hat{\delta} - \delta}{\delta} \right)^2. \]

**Proof.** The proof proceeds as before. Upon multiplying equation (3.9) by $\Delta X$ we estimate the additional terms as
\[ \mu_\epsilon \Delta X \leq \frac{\mu_1}{2} \epsilon_1^2 \Delta X^2 + \frac{\mu}{2\eta} \quad \text{for} \quad i = 1, 2. \]
Since
\[ \frac{Y^2}{2(2b - \alpha)} + \frac{Z^2}{2(2 - \alpha)} \leq \frac{1}{2(2 - \alpha)} (K - X^2) \]
we obtain
\[ \frac{1}{2} \dot{V} + \left( \sigma + \mu - \frac{\alpha \mu}{2} + \frac{X^2}{2(2 - \alpha)} - \frac{\mu \eta}{2} (\epsilon_1^2 + \epsilon_2^2) - \frac{K}{2(2 - \alpha)} \right) \Delta X^2 \]
\[ + \frac{\alpha}{2} \Delta Y^2 + \frac{\alpha}{2} \Delta Z^2 \leq \frac{2\delta \mu (\sigma + \mu)^2}{\alpha} \int_{t-\delta}^{t} \{ |\Delta X|^2 + |\Delta Y|^2 \} \]
\[ + 4\mu^2 \delta^2 K \left( \frac{\delta - \hat{\delta}}{\hat{\delta}} \right)^2 + \frac{\mu}{\eta}. \]
Now, by the Cauchy–Schwartz inequality
\[ \mu \eta \epsilon_1^2 \leq \frac{\mu_1}{\delta^2} |\delta - \hat{\delta}| \left| \int_{t-\delta}^{t} X^2 \right| \quad \text{and} \quad \mu \eta \epsilon_2^2 \leq \frac{\mu_1}{\delta^2} |\delta - \hat{\delta}|^2 \left| \frac{1}{\delta} \int_{t-\delta}^{t} X^2 \right|. \]
Upon taking
\[ \eta = \frac{1}{2(2 - \alpha)} \cdot \hat{\delta}^2 \cdot \frac{1}{\mu} \]
we obtain
\[ \frac{X^2}{2(2 - \alpha)} - \mu \eta \epsilon_1^2 \geq \frac{-1}{2(2 - \alpha)} \left| \frac{1}{\delta - \hat{\delta}} \int_{t-\delta}^{t} (X(t)^2 - X(s)^2) ds \right| \]
and
\[ \frac{X^2}{2(2 - \alpha)} - \mu \eta \epsilon_2^2 \geq \frac{-1}{2(2 - \alpha)} \left| \frac{1}{\delta} \int_{t-\delta}^{t} (X(t)^2 - X(s)^2) ds \right|. \]
Since
\[ |X(t)^2 - X(s)^2| = \left| 2 \int_{s}^{t} X \dot{X} \right| \leq 2\sigma \int_{s}^{t} (X^2 + |XY|) \]
\[ \leq \sigma (1 + \sqrt{2}) \int_{s}^{t} (X^2 + Y^2) \leq \sigma (1 + \sqrt{2})(t - s) K \]
then
\[
\frac{X^2}{2 - \alpha} - \mu \eta (\xi_1^2 + \xi_2^2) \geq -\sigma (1 + \sqrt{2}) \left( \frac{\delta + \delta}{2} + \frac{\delta}{2} \right) K = \frac{1 - J}{2(2 - \alpha)} K.
\]
Consequently
\[
\frac{1}{2} \dot{V} + \left( \sigma + \mu - \frac{\alpha \mu}{2} - \frac{JK}{2(2 - \alpha)} \right) \Delta X^2 + \frac{\alpha}{2} \Delta Y^2 + \frac{\alpha}{2} \Delta Z^2
\leq \frac{2 \delta (\sigma + \mu)^2}{\alpha} \int_{t-2\delta}^{t} \{ |\Delta X|^2 + |\Delta Y|^2 \} + \frac{4 \mu^2 \delta^2 K}{\alpha} \left( \frac{\delta - \hat{\delta}}{\hat{\delta}} \right)^2 + \frac{\mu}{\eta}.
\]
By hypothesis
\[
\frac{\alpha \mu}{2} + \frac{JK}{2(2 - \alpha)} \leq \sigma \left( 1 - \frac{\alpha}{2} \right).
\]
Therefore
\[
\dot{V} + \alpha V \leq \frac{4 \delta (\sigma + \mu)^2}{\alpha} \int_{t-2\delta}^{t} V(s) ds + \frac{8 \mu^2 \delta^2 K}{\alpha} \left( \frac{\delta - \hat{\delta}}{\hat{\delta}} \right)^2 + \frac{2 \mu}{\eta}.
\]
We apply Lemma 2.2 as in the proof of Proposition 3.2 except this time taking
\[
B = \frac{8 \mu^2 \delta^2 K}{\alpha} \left( \frac{\delta - \hat{\delta}}{\hat{\delta}} \right)^2 + \frac{2 \mu}{\alpha \eta}.
\]
The result follows. \(\square\)

### 3.4 Numeric Results for Noiseless Coupling

In all our numerical experiments the free-running solution \(U\) was obtained by taking
\[
U_0 = (-5.5751, -3.24345, -11.0728).
\]
As this value of \(U_0\) lies very close to the global attractor, the resulting trajectory satisfies the conditions of Theorem 1.1 to within a certain numerical error and therefore we suppose the bound (3.11) holds. A sharper bound may be obtained \textit{a posteriori} as the maximum value of the actual numerical solution. In particular, the trajectory for \(U\) considered here satisfies
\[
\|U(t)\|^2 \leq 1247.0911 \quad \text{for} \quad t \in [0, 500] \quad (3.11)
\]
Figure 3.1: Illustration of time-averaged exact measurements of $U$ for different values of $\delta$.

which is about 20 percent smaller than the theoretical bound.

We compute $u$ using equations (1.2) with $\widetilde{R} = R = R_\delta$ and $u_0 = (0, 0, 0)$ for different values of $\delta$ with $\mu = 30$. This value of $\mu$ is comparable to the value $\mu = K/40 \approx 38.5$ used for Figure 5 of [14]. Figure 3.4 illustrates the phase shift that occurs for $\delta > 0$ which delays the measurements of $U$ in time.

Figure 3.2: Results when coupling time-averaged exact measurements with an averaging window of size $\delta$ when $\delta = \hat{\delta}$.
In particular, Figure 3.2 shows the evolution of the error $\|U-u\|^2$ when coupling the free-running solution to the driven solution using noise-free time averaged measurements of $X$ with $\mu = 30$ when $\delta = \delta$ for $\delta \in \{0, 0.25, 0.375\}$. For $\delta = 0$ and $\delta = 0.25$ the error converges to zero within the limits of the floating point arithmetic used for the computations. We point out, however, that the rate of convergence appears slightly slower when $\delta = 0.25$. On the other hand the error remains large for $\delta = 0.375$.

Figure 3.3: Results when coupling time-averaged exact measurements with an averaging window of size $\delta$ near $\delta_c$ for $T < 2000$ when $\delta = \delta$.

Further computations show that the conditions for convergence are intractable when $\delta \approx 0.33$. Intuitively, we suspect there is a value $\delta_c$ such that for $\delta < \delta_c$ the rate of convergence is very slow, but synchronization is eventually achieved and the error decays to zero, and for $\delta > \delta_c$ the systems fail to synchronize. Our calculations suggest $\delta_c \in [0.31, 0.33]$, provided such a critical value exists. What we do know, is that numerically for values $\delta > 0.33$ there is no sign of synchronization, and for values $\delta < 0.31$ the two solutions synchronize to within the limits of the rounding error of the double precision arithmetic used for our calculations. Figure 3.3 shows instability of the synchronization for $\delta = 0.318$, where the error $\|U-u\|^2$ oscillates over time through 20 decimal orders of magnitude.
Figure 3.4: Results when coupling time-averaged exact measurements with an averaging window of size $\delta = 0.334$ using increasing values of $\mu$ where $T < 600$ when $\hat{\delta} = \delta$.

While the theory requires $\mu$ to be greater than 375.07, a slightly less restrictive bound on $\mu$ may be obtained by using the value for $K$ suggested by the bound (3.11). Either bound, however, is significantly more restrictive than the value chosen here which works when the size of the averaging window is sufficiently small. In this case where $\delta > 0.33$, there are no values of $\mu$ for which the error converges to zero, illustrated in Figure 3.4.

By further varying the value of $\delta$ it appears that $\delta \leq 0.31$ is sufficient to ensure $\|U - u\| \to 0$ numerically as $t \to \infty$. We further compare the theoretical size of $\delta$ to this numerically determined value. To find the largest value of $\delta$ that Proposition 3.2 guarantees will lead to synchronization we maximize the upper bound in (3.10) with respect to $\alpha$. Note that as $\alpha$ vanishes in the bound (3.10) the length $\delta$ of the time averaging window also vanishes. Moreover, $\delta$ also vanishes as $\alpha$ approaches 2, because $\delta$ depends inversely on $\mu^{3/2}$ and $\mu$ approaches infinity as $\alpha$ approaches 2. Simple calculus now yields the optimal value of $\alpha$ to be

$$\alpha^* = -\rho \cos(\phi) + \frac{22}{15} + \rho \sqrt{3} \sin \phi \approx 0.28413$$
where

\[ \rho = \frac{4\sqrt{25370}}{75} \quad \text{and} \quad \phi = \frac{1}{3} \arctan \left( \frac{19\sqrt{1731443695}}{167665} \right). \]

For \( \alpha = \alpha^* \) it follows that we may take

\[ \mu \approx 513.153 \quad \text{and} \quad \delta \approx 0.0000063216. \]

Using the value for \( K \) suggested by the bound (3.11) leads to a 30 percent larger value for \( \delta \); however, in either case the analytic bounds are more than 10 000 times stricter than what works numerically.

In summary our numeric results show measurements of \( X \) averaged with respect to a small averaging window are just as effective for synchronizing the driven solution to the free-running solution as exact measurements taken instantaneously in time. However, if the averaging window is too large, then the information is blurred too much for \( u \) to synchronize with \( U \). This is consistent with the theoretical results of Proposition 3.2 which proves the convergence of the driven solution to the free-running solution only when \( \delta \) is sufficiently small and \( \mu \) sufficiently large. We emphasize that our analytic bounds on \( \mu \) and \( \delta \) are much more restrictive than what works numerically.

Now consider the case treated by Proposition 3.3 where the exact size of the time averaging window present in the observational measurements is unknown. Figure 3.5 describes the evolution of \( \|U - u\|^2 \) when \( \hat{\delta} = 0.25 \) and \( \delta \) is smaller but near \( \hat{\delta} \). Note that \( \delta \) greater than \( \hat{\delta} \) results in a nearly identical graph. The asymptotic error may be approximated numerically as a function of \( \Delta \delta = \delta - \hat{\delta} \) by

\[ E(\Delta \delta) = \max \left\{ \|U - u\|^2 : t \in [50, 100] \right\}. \]

From Figure 3.5, there is a decrease in the error \( \|U - u\|^2 \) as \( \Delta \delta \) decreases before reaching \( \Delta \delta = 0 \). Table 3.4 illustrates the the convergence of the driven solution, where the limit supremum is taken over the error for all \( t > T \). As expected, the
Figure 3.5: Results when coupling time-averaged exact measurements with window size $\delta = 0.25$ using a feedback term with averaging window size $\delta \neq \hat{\delta}$. Note that $\Delta\delta = \delta - \hat{\delta}$.

Error levels decrease as $\delta$ gets closer to $\hat{\delta}$. It is reasonable to determine that there exists a function that will allow us to predict the largest value $|\delta - \hat{\delta}|$ that results in convergence. We now consider the functional dependency how $E(\Delta\delta)$ depends on $\Delta\delta$.

The theoretical bounds given by Proposition 3.3 suggest that $E(\Delta\delta)$ is proportional to the square of $\Delta\delta$. In order to test this hypothesis we performed a least
Table 3.1: Limsup of $\|U-u\|^2$ for different values of $\Delta \delta$.

<table>
<thead>
<tr>
<th>$\Delta \delta$</th>
<th>$\lim \sup |U-u|^2$</th>
<th>$\Delta \delta$</th>
<th>$\lim \sup |U-u|^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2^{-2}$</td>
<td>1267.31</td>
<td>$2^{-2}$</td>
<td>2663.41</td>
</tr>
<tr>
<td>$-2^{-5}$</td>
<td>194.081</td>
<td>$2^{-5}$</td>
<td>249.552</td>
</tr>
<tr>
<td>$-2^{-8}$</td>
<td>5.61624</td>
<td>$2^{-8}$</td>
<td>5.86352</td>
</tr>
<tr>
<td>$-2^{-11}$</td>
<td>0.0906275</td>
<td>$2^{-11}$</td>
<td>0.0911086</td>
</tr>
<tr>
<td>$-2^{-14}$</td>
<td>0.00140762</td>
<td>$2^{-14}$</td>
<td>0.00143265</td>
</tr>
<tr>
<td>$-2^{-17}$</td>
<td>$2.19772 \times 10^{-05}$</td>
<td>$2^{-17}$</td>
<td>$2.24025 \times 10^{-05}$</td>
</tr>
<tr>
<td>$-2^{-20}$</td>
<td>$3.43361 \times 10^{-07}$</td>
<td>$2^{-20}$</td>
<td>$3.50073 \times 10^{-07}$</td>
</tr>
<tr>
<td>$-2^{-23}$</td>
<td>$5.36495 \times 10^{-09}$</td>
<td>$2^{-23}$</td>
<td>$5.46996 \times 10^{-09}$</td>
</tr>
<tr>
<td>$-2^{-26}$</td>
<td>$8.38272 \times 10^{-11}$</td>
<td>$2^{-26}$</td>
<td>$8.54684 \times 10^{-11}$</td>
</tr>
<tr>
<td>$-2^{-29}$</td>
<td>$1.30979 \times 10^{-12}$</td>
<td>$2^{-29}$</td>
<td>$1.33543 \times 10^{-12}$</td>
</tr>
<tr>
<td>$-2^{-32}$</td>
<td>$2.04656 \times 10^{-14}$</td>
<td>$2^{-32}$</td>
<td>$2.08664 \times 10^{-14}$</td>
</tr>
<tr>
<td>$-2^{-35}$</td>
<td>$3.19773 \times 10^{-16}$</td>
<td>$2^{-35}$</td>
<td>$3.26052 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.0</td>
<td>$2.67042 \times 10^{-25}$</td>
<td>0.0</td>
<td>$2.67042 \times 10^{-25}$</td>
</tr>
</tbody>
</table>

squares fit to find $m$ and $s$ such that

$$\log E(\Delta \delta) \approx \log m + s \log(\Delta \delta).$$

Figure 3.6 shows that the least squares fit agrees with $E(\Delta \delta)$ to remarkable precision over a range of ten decimal orders of magnitude. Since the fitted value $s = 1.99964$ is so close to 2, we presume that the $\Delta \delta$ dependency given by our analytic bound is physical even though the values for $\mu$ and $\delta$ are much more restrictive than numerically needed.

It is worth remarking that the theoretical bound in Proposition 3.3 actually depends on the relative error $\Delta \delta/\hat{\delta}$ which, as we have held $\hat{\delta} = 0.25$ constant, is proportional to the absolute error $\Delta \delta$ in the above computations. Computations using different values for $\hat{\delta}$ are a direction for future work described in Section 7.9.
Figure 3.6: The numerical data from Table 3.1 when plotted with the least squares fit of $m(\Delta \delta)^* \text{ given by } m = 378647 \text{ and } s = 1.99964$.

Preliminary calculations suggest that a better fit is obtained for bounds of the form

$$E(\Delta \delta) \approx m \left( \frac{\delta - \hat{\delta}}{\delta_c - \hat{\delta}} \right)^2$$

where $\delta_c \approx 0.33$ is the critical value of $\delta$ for which synchronization fails detailed in Figure 3.3. The numerical dependence of the bound on $\hat{\delta}$ is a point of future work.
Chapter 4

Coupling with Noise

In this chapter we examine the synchronization of the free running and driven solutions with the addition of a time-averaged Brownian motion. By synchronization we mean that the system driven by noisy observations will converge to the true solution within a tolerance determined by the level of observational noise. The expected value $E$ of the convergence is examined as we are only able to bound the error of the system with some probability for a finite amount of time, preventing us from forming a generalized proof for pathwise convergence. In particular, we bound the expected value of the error and then use Chebyshev’s inequality to find probabilistic bounds on the paths.

4.1 Noisy Instantaneous-in-Time Coupling

Consider when the observations of $U$ are noisy and satisfy equation (1.7) where $W_t$ is a one-dimensional standard Brownian motion. Thus, the noise present in the observational measurements satisfies $\text{Var}(\varepsilon W_t) = t\varepsilon^2$. In this case, equation (3.1) becomes

$$dx = -\sigma(y - x)dt + \mu(X - x)dt + \varepsilon dW_t. \tag{4.1}$$

Note that the noise present in the observations appears in equation (4.1) multiplied by the factor $\mu$. Therefore, increasing $\mu$ to strengthen the coupling between $U$ and $u$ also amplifies the noise.

The techniques employed are almost identical to those in [2] but have been detailed here to provide a concrete point of reference for comparing our main results which involve time-averaged measurements. The results in this section, though slightly sharper, are similar to those which appear in [14]. As in Section 3.1,
carefully examining the case without time-averages first provides a concrete point of reference for comparing our results which involve time-averaged measurements.

**Proposition 4.1.** Suppose $\delta = 0$, $\hat{\delta} = 0$ and $\varepsilon > 0$. Let $\mu > K/4 - \sigma$, then

$$\limsup_{t \to \infty} \mathbb{E} \left[ \|U - u\|^2 \right] \leq \frac{\mu^2 \varepsilon^2}{\sigma + \mu + 1 - \sqrt{(\sigma + \mu - 1)^2 + K}}$$

**Proof.** With the addition of the noise, equation (3.4) becomes the stochastic differential equation

$$d(\Delta X) = -(\sigma + \mu)\Delta X dt + \sigma\Delta Y dt - \mu\varepsilon dW_t.$$

Using Ito’s formula and the same estimates as in Proposition 3.1 we obtain

$$dV + \alpha V dt \leq 2\mu\varepsilon\Delta X dW_t + \mu^2\varepsilon^2 dt.$$

Integrating over the interval $[0, t]$ yields

$$V(t)e^{\alpha t} - V(0) \leq 2\mu\varepsilon\delta \int_0^t e^{\alpha s}\Delta X(s) dW_s + \frac{\mu^2\varepsilon^2}{\alpha} (e^{\alpha t} - 1).$$

Now, taking expected values obtains

$$\mathbb{E}[V(t)] \leq e^{-\alpha t}\mathbb{E}[V(0)] + \frac{\mu^2\varepsilon^2}{\alpha} (1 - e^{-\alpha t}).$$

Consequently

$$\limsup_{t \to \infty} \mathbb{E}[V] \leq \frac{\mu^2\varepsilon^2}{\alpha}.$$

Solving for $\alpha$ in equation (3.7) finishes the proof.

---

**4.2 Noisy Time-averaged Coupling**

In this section we consider the case when the noisy observations of $X$ have been averaged in time over a known window of size $\hat{\delta}$. Thus we take $\delta = \hat{\delta}$ and in equation (1.12) and equation (3.1) becomes

$$\dot{x} = -\sigma(y - x) + \mu(\bar{X} - \bar{x}) + \mu \xi \quad \text{where} \quad \xi(t) = \frac{\varepsilon}{\delta}(W_t - W_{t-\delta}).$$
Thus
\[ \Delta \hat{X} = - (\sigma + \mu) \Delta X + \sigma \Delta Y + \mu (\Delta X - \Delta \hat{X}) + \mu \xi. \]

Theoretically the presence of a small time average in the measurements does not substantially affect the results of the instantaneous in time case. Our results show that measurements averaged over small time windows lead to bounds that are similar to those given by Proposition 4.1 when there are no time averages. The expected value of the difference between the driven solution and the free running solution in both cases is bounded by a constant factor of the variance of the noise. In particular, we have

**Proposition 4.2.** Let \( \alpha \in (0, 2) \) and \( \mu \geq K/(2 - \alpha)^2 - \sigma \). If
\[
\delta < \sqrt{\frac{1 - \epsilon^{-1}}{2}} \cdot \frac{\alpha}{32 \mu (\sigma + \mu)^2},
\]

Then
\[
\limsup_{t \to \infty} \mathbb{E}[\|U - u\|^2] < \epsilon \left( \frac{8 \mu^3}{\alpha^2} + \frac{2 \mu}{\alpha^2 \delta} \right) \xi^2.
\]

**Proof.** The proof proceeds as the proof of Proposition 4.2, where the only difference is an additional noise term. Similar estimates as in the proofs of Lemma 3.1 and Lemma 3.2 yield that
\[
|\Delta X - \Delta \hat{X}|^2 \leq 4 \delta (\sigma + \mu)^2 \int_{t-\delta}^t \{ |\Delta X|^2 + |\Delta Y|^2 \} + 4 \mu^2 \delta \int_{t-\delta}^t \xi^2.
\]
Following similar estimates as before, multiply by \( \Delta X \) and use Young’s inequality as
\[
\mu (\Delta X - \Delta \hat{X}) \Delta X \leq \frac{4 \delta \mu (\sigma + \mu)^2}{\alpha} \int_{t-\delta}^t \{ |\Delta X|^2 + |\Delta Y|^2 \} + \frac{4 \mu^3 \delta}{\alpha} \int_{t-\delta}^t \xi^2 + \frac{\alpha \mu}{4} |\Delta X|^2
\]
and
\[
\mu \xi \Delta X \leq \frac{\mu \xi^2}{\alpha} + \frac{\alpha \mu}{4} |\Delta X|^2.
\]
to obtain
\[
\dot{V} + \alpha V \leq \frac{8 \delta \mu (\sigma + \mu)^2}{\alpha} \int_{t-\delta}^t (\Delta X^2 + \Delta Y^2) + \frac{8 \mu^3 \delta}{\alpha} \int_{t-\delta}^t \xi^2 + \frac{2 \mu \xi^2}{\alpha}.
\]
Since $E[\xi^2] = \varepsilon^2/\delta$ for any time $t$, then setting $V(\tau) = E[V(\tau/\alpha)]$ yields
\[
\frac{dV(\tau)}{d\tau} + V(\tau) \leq \frac{8\delta\mu(\sigma + \mu)^2}{\alpha^3} \int_{t-2\delta}^{t} V(s) \, ds + \left(\frac{8\mu^3}{\alpha^2} + \frac{2\mu}{\alpha^2\delta}\right)\varepsilon^2.
\]
We apply Lemma 2.2 taking
\[
A = \frac{8\delta\mu(\sigma + \mu)^2}{\alpha^3}, \quad B = \left(\frac{8\mu^3}{\alpha^2} + \frac{2\mu}{\alpha^2\delta}\right)\varepsilon^2 \quad \text{and} \quad h = 2\alpha\delta.
\]
The result follows.

4.3 Unknown Noisy Time-averaged Coupling

In this section we examine the case when there is noise in the measurements and the length of the time averaging window is unknown. Our main result is a version of Proposition 3.3 that also allows noise in the observations. As before let $\hat{\delta}$ be the length of the unknown averaging window in the observations of the free running solution and $\delta$ be an approximation of $\hat{\delta}$ that will be used in the feedback control of the driven system. In this case the equation governing the evolution $x$ in the driven system becomes
\[
\dot{x} = -\sigma(y - x) + \mu\left(\frac{1}{\delta} \int_{t-\delta}^{t} X(s) \, ds - \frac{1}{\delta} \int_{t-\delta}^{t} x(s) \, ds\right) + \mu\xi
\]
where $\xi$ is as in Section 4.2. Since there is now noise in the observations, exact synchronization does not occur when $\delta = \hat{\delta}$. Thus, our theoretical bounds on $\|U - u\|_2$ take the form given by equation (1.14) consisting of two terms, one which vanishes as $\delta \to \hat{\delta}$ and the other which vanishes as $\varepsilon \to 0$. In particular, we obtain

**Proposition 4.3.** Suppose $\delta \neq \hat{\delta}$ and $\varepsilon > 0$. Given $\alpha \in (0, 2)$, let
\[
\mu \geq JK/(2 - \alpha)^2 - \sigma \quad \text{where} \quad J = \sigma(1 + \sqrt{2})\left(\frac{\delta}{2} + \hat{\delta}\right) + 1.
\]
and
\[
\delta < \sqrt{\frac{1 - e^{-1}}{2} \cdot \frac{\alpha}{32\mu(\sigma + \mu)^2}}.
\]
Then
\[
\limsup_{t \to \infty} \mathbb{E}[\|U-u\|^2] < \frac{5\epsilon(2-\alpha)\mu^2}{\alpha} \cdot \frac{(\delta - \hat{\delta})^2}{\delta^2} + \epsilon \left( \frac{16\mu^3}{\alpha^2} + \frac{2\mu}{\alpha^2\delta} \right) \epsilon^2.
\]

**Proof.** Modifying the proof of Lemma 3.2 we add the term for measurement error, which yields
\[
|\Delta X - \Delta \hat{X}|^2 \leq 4\delta(\sigma + \mu)^2 \int_{t-2\delta}^t \{ |\Delta X|^2 + |\Delta Y|^2 \}
+ 16\mu^2\delta^2 K \left( \frac{\delta - \hat{\delta}}{\delta} \right)^2 + 8\mu^2\delta \int_{t-\delta}^t \xi^2.
\]

Now, modifying the proof of Proposition 3.3 as was done for the proof of Proposition 4.2 to add the term which represents measurement error yields
\[
\frac{d\mathcal{V}(\tau)}{d\tau} + \mathcal{V}(\tau) \leq \frac{8\delta\mu(\sigma + \mu)^2}{\alpha^3} \int_{\tau-2\delta}^\tau \mathcal{V}(s) \, ds
+ 16\mu^2\delta^2 K \left( \frac{\delta - \hat{\delta}}{\delta} \right)^2 + \frac{2\mu}{\alpha\eta} + \frac{16\mu^3}{\alpha^2} + \frac{2\mu\epsilon^2}{\alpha^2\delta}.
\]

Applying Lemma 2.2 finishes the proof. \qed

### 4.4 Numeric Results for Noisy Coupling

The first set computations in this section treat the case where no time averaging is present in the observational measurements. Under this hypothesis Proposition 4.1 shows that the expected value of the difference between the approximating solution and the exact solution is bounded by a factor of the variance of the noise. A similar bound was obtained as Theorem 4.1 in [14] which, using the notation here, may be written as
\[
\limsup_{t \to \infty} \mathbb{E}[\|U-u\|^2] \leq \frac{\mu^2\epsilon^2}{2-K/(2\mu)} \quad \text{for} \quad \mu > K/4. \tag{4.2}
\]

Optimizing the bound in the bound (4.2) with respect to \( \mu \) and using the value of \( K \) given by Theorem 1.1 we obtain that
\[
\limsup_{t \to \infty} \mathbb{E}[\|U-u\|^2] \leq \frac{27K^2\epsilon^2}{128} \leq (5.005 \times 10^5)\epsilon^2.
\]
Alternatively, optimizing the bound in Proposition 4.1 yields
\[
\limsup_{t \to \infty} \mathbb{E}[\|U - u\|^2] \leq (4.838 \times 10^5) \varepsilon^2
\]
which is about 3 percent smaller. Moreover, since
\[
\frac{\mu^2 \varepsilon^2}{\sigma + \mu + 1 - \sqrt{(\sigma + \mu - 1)^2 + K}} < \frac{\mu^2 \varepsilon^2}{2 - K/(2\mu)}
\]
provided \(\mu > K/4\), the bound given in the bound (4.2) is strict. Therefore, there exists \(T > 0\) such that by Chebyshev’s inequality
\[
\mathbb{P}\left\{ \omega \in \Omega : \|U - u\|^2 \leq \frac{10\mu^2 \varepsilon^2}{2 - K/(2\mu)} \right\} \geq 0.9.
\]
Computations in [14] demonstrate numerical bounds on \(\|U - u\|\) which are proportional to \(\varepsilon\) and therefore consistent with the theoretical bounds described above. In the present thesis we fix the noise level at \(\varepsilon = 0.01\) and focus on the effects of the time averaging.

For a frame of reference, first consider 500 independent realizations of Brownian motion corresponding to \(\omega_i \in \Omega\) such that \(i = 1, 2, \ldots, 500\) and compute the corresponding pathwise solutions \(u_i\) using equation (1.2) with the coupling term \(\mathcal{R} = \tilde{R}\). The exact algorithm used to compute \(u\) shall be described later in Section 5. Approximate the expected value of \(\|U - u\|^2\) by
\[
\mathbb{E}[\|U - u\|^2] \approx \frac{1}{500} \sum_{i=1}^{500} \|U - u_i\|^2
\]
and let \(L(t)\) be the value such that
\[
\text{card}\{ i \in \{1, 2, \ldots, 500\} : \|U - u_i\|^2 \leq L(t) \} = 450.
\]
Thus, \(L(t)\) bounds 90 percent of the paths at each point in time. In particular, assuming 500 samples is statistically large enough, we have
\[
\mathbb{P}\{ \omega \in \Omega : \|U - u\|^2 \leq L(t) \} \approx 0.9.
\]
Figure 4.1 describes the results of this first computation. Note that the expected value of \(\|U - u\|^2\) oscillates around 0.01 and that \(L(t)\) is about 5 times greater.
Figure 4.1: Results when coupling noisy instantaneous-in-time measurements with $\mu = 30$ and $\varepsilon = 0.01$. The expected value was computed using 500 independent realizations of Brownian motion and $L(t)$ is a computed bound on 90 percent of the paths at each point in time. The gray region represents the 500 pathwise solutions, one of which has been plotted in a darker shade for illustration.

Thus, $L(t)$ is about 2 times smaller than the bound given by Chebyshev’s inequality (1.11). Moreover, at each point in time a small percentage of trajectories are significantly more accurate than expected and a few satisfy $\|U - u_i\|^2 \leq \varepsilon^2$.

The computation described in Figure 4.1 may be directly compared to Figure 5 in [14]. Similar values of $\mu$ and $\varepsilon$ were employed in both computations and result in similar looking pathwise solutions. The difference is that our expected values have been computed by averaging over the pathwise solutions $u_i$ which result from independent Brownian motions and approximate the time-dependent expected values which appear in the analysis, whereas [14] computes expected values that do not change over time and appear to have been computed by taking a long-time average over a single pathwise solution. We now consider how the presence of time averaging in the observational measurements affects the degree
to which $u$ synchronizes with $U$.

Figure 4.2: Time-averaged noisy measurements for different values of $\delta$ corresponding to a representative Brownian path with $\varepsilon = 0.01$.

Figure 4.2 depicts the time evolution of a typical set of noisy observational measurements with and without time averaging. Note that when $\hat{\delta} = 0$ there is no time averaging and resulting observations show oscillation of the trajectory. As we are dealing with a chaotic system, we can expect that these oscillations will perturb the system enough so that synchronization is only possible within a tolerance of the measure of the noise. When $\hat{\delta} > 0$ the noise has been smoothed out at the expense of a phase shift that delays the measurements in time. We now perform a number of calculations of $u$ using measurements of the kind depicted in Figure 4.2.

Figure 4.3 depicts the evolution of the expected value of $\|U - u\|^2$ over time for $\mu = 30$, $\varepsilon = 0.01$ and $\delta \in \{0, 0.25, 0.375\}$. As in the noise-free case, the difference between $u$ and $U$ remains large when $\delta = 0.375$. However, the difference between $\delta = 0.25$ and $\delta = 0$ appears greater than in the noise-free case. Intuitively, the presence of moving time averages seems to result in decreased efficiency of the feedback control that nudges the driven solution towards the free running
solution. Without noise, only enough nudging to overcome the tendency for nearby trajectories of the Lorenz equation to diverge is needed. As the nudging must also overcome the noise in the present case, which in turn may account for the expected error levels being slightly greater when $\delta = 0.25$. Further support for this intuition may also be inferred from the already mentioned slower rate of convergence in the absence of noise $\delta = 0.25$ compared to $\delta = 0$.

Figure 4.3: Results when coupling time-averaged noisy measurements with an averaging window of size $\delta$ and a noise level $\varepsilon = 0.01$. The expected values were computed using 500 independent realizations of Brownian motion.

Figure 4.3 depicts the evolution over time of the expected value of $\| U - u \|^2$ for $\mu = 30$, $\varepsilon = 0.01$ and $\delta \in \{0, 0.25, 0.375\}$ with respect to 500 independent realizations of noisy measurements of the kind depicted in Figure 4.2. As in the noiseless case, the difference between $u$ and $U$ remains large when $\delta = 0.375$. Similarly, for $\delta = 0.25$ and $\delta = 0$ the error decreases over time; however, unlike the noiseless case the asymptotic error level for $\delta = 0.25$ is noticeably larger than for $\delta = 0$. In particular,
Figure 4.4: Results when coupling time-averaged noisy measurements with \( \varepsilon = 0.01 \) and window size \( \delta = 0.25 \) using a feedback term with averaging window size \( \delta \neq \hat{\delta} \). Note that \( \mu = 30 \) and \( \Delta \delta = \delta - \hat{\delta} \).

![Graph showing the results](image)

\[
\max \{ E[||U - u||] : t \in [50, 100] \} \approx \begin{cases} 
0.0089 & \text{for } \delta = 0 \\
0.49 & \text{for } \delta = 0.25.
\end{cases}
\]

Intuitively, the presence of moving time averages seems to result in a decreased efficiency of the feedback control that nudges the driven solution towards the free running solution. Without noise, this decrease in efficiency doesn’t matter because the difference between the free running and driven solutions tends to zero.
over time anyway, only perhaps with a slightly decreased rate of convergence. In the presence of noise, however, the nudging must also overcome the noise. This leads to a higher error level in the difference between $U$ and $u$ over time.

The last set of computations treats the case in Proposition 4.3 when the length of the averaging window used for the observations is unknown. Figure 4.4 depicts the evolution of the expected value of $\|U - u\|^2$ when $\hat{\delta} = 0.25$ and $\delta$ is smaller but near $\hat{\delta}$. The error level decreases as $\delta$ gets closer to $\hat{\delta}$, but unlike the noiseless case, there is very little difference between $\Delta \delta = 0$ and $\Delta \delta = -2^{-11}$. Bounds on the expected error may be approximated numerically as

$$\widetilde{E}(\Delta \delta) = \max \left\{ E[\|U - u\|^2] : t \in [50, 100] \right\}.$$ 

A summary of our computations are presented in Table 4.1. These numeric results show, as do our theoretical bounds, that $\delta$ can be tuned to match $\hat{\delta}$ only up to the point where the noise term dominates. Specifically, when $\hat{\delta} = 0.25$ and $\epsilon = 0.01$ it is possible to tune $\delta$ so that $|\delta - \hat{\delta}| < 2^{-8}$.

As seen in Figure 4.2 typical values of $X$ range from $-10$ to 10. Moreover, the free running solution used in our experiments satisfies

$$\left( \frac{1}{100} \int_{0}^{100} X(s)^2 ds \right)^{1/2} \approx 7.92731.$$ 

We finish by noting that $\epsilon = 0.01$ is 1.3 percent the root-mean-squared value of $X$ and that $2^{-8}$ is 1.6 percent the value of $\hat{\delta}$. The exact relationship between the value of $\epsilon$ and the degree to which $\delta$ can and needs to be tuned would be an interesting direction for future study.
Table 4.1: Values of $\tilde{E}(\Delta \delta) = \max \{\mathbb{E}[\|U - u\|^2] : t \in [50, 100]\}$ as a function of $\Delta \delta = \delta - \hat{\delta}$. Here $\hat{\delta}$ represents the unknown size of the averaging present in the observational measurements and $\delta$ an approximation of $\hat{\delta}$ used in the feedback control.

<table>
<thead>
<tr>
<th>$\Delta \delta$</th>
<th>$\tilde{E}(\Delta \delta)$</th>
<th>$\Delta \delta$</th>
<th>$\tilde{E}(\Delta \delta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2^{-2}$</td>
<td>$1.60729 \times 10^6$</td>
<td>$2^{-2}$</td>
<td>$6.09399 \times 10^6$</td>
</tr>
<tr>
<td>$-2^{-5}$</td>
<td>38419.5</td>
<td>$2^{-5}$</td>
<td>62104.1</td>
</tr>
<tr>
<td>$-2^{-8}$</td>
<td>44.133</td>
<td>$2^{-8}$</td>
<td>38.5955</td>
</tr>
<tr>
<td>$-2^{-11}$</td>
<td>0.693591</td>
<td>$2^{-11}$</td>
<td>0.56483</td>
</tr>
<tr>
<td>0</td>
<td>0.491966</td>
<td>0</td>
<td>0.491966</td>
</tr>
</tbody>
</table>
Chapter 5

Numerical Methods

This section described the numerical methods used to obtain the results in Section 3.4 and Section 4.4. We compute both the free running solution and the driven solution using Euler’s explicit method

\[ U^{n+1} = U^n + hF(U^n) \]

\[ u^{n+1} = u^n + hF(u^n) + h\mu(R_{\delta}(U^n) - R_{\delta}(u^n)) + h\mu\hat{\xi}^n. \]

in double-precision floating-point arithmetic with step size \( h = 1/2048 \). Here \( \delta = m_0h \) for some \( m_0 \in \mathbb{N} \) but \( \delta \) may be any positive floating-point number. Upon writing \( U^n = (X_n, Y_n, Z_n) \) and \( u^n = (x_n, y_n, z_n) \), it follows that \( R_{\delta} = L \circ O_{\delta} \) where

\[ O_{\delta}(U^n) = \frac{1}{m_0} \sum_{k=0}^{m_0-1} X_{n-k} \]

and by interpolation setting \( m_3 = \lfloor \delta/h \rfloor \) that

\[ O_{\delta}(u^n) = \frac{h}{\delta} \left\{ \left( \frac{\delta}{h} - m_3 \right) x_{n-m_3} + \sum_{k=0}^{m_3-1} x_{n-k} \right\}. \]

The noise term \( \hat{\xi}^n = \hat{\xi}(nh) \) is given as in Section 4.3 by

\[ \hat{\xi}(t) = \frac{\varepsilon}{\delta} (W(t) - W(t - \delta)) \]

where \( W(t) \) is a standard Brownian motion, one of the simplest of the continuous-time stochastic processes. Numerically we use the construction of Brownian motion as developed by Lévy and Ciesielski. By Lévy and Ciesielski, there is a probability space \( \Omega \) and a sequence of independently distributed standard normal random variables \( (G_n)_{n \geq 0} \) such that for \( \omega \in \Omega \) and \( t \in [0, 1] \)

\[ W(\omega, t) = \lim_{n \to \infty} W_n(\omega, t) \quad \text{where} \quad W_n(\omega, t) = \sum_{k=0}^{n} G_k(\omega)S_k(t) \]

is a Brownian motion, where \( S_n(t) \) is from the family of Schauder functions. The Schauder functions \( S_{2^j+k} \) are tent-functions with support \([k2^{-j}, (k + 1)2^{-j}] \) and a
maximal value $\frac{1}{2}2^{-j/2}$ at the tip of the tent. Using an application of Minkowski’s inequality, monotone convergence theorem, and by the completeness of the space of continuous functions, there is a subsequence of $(W_2(t, \omega))_{j \geq 1}$ which converges uniformly in $t \in [0, 1]$ for a subset $\Omega_0 \subset \Omega$. Defining $W(t, \omega)$ as a limiting function, we see that it indeed inherits the continuity of $W_2(t, \omega)$. We conclude that there exists a restriction of $W(t, \omega)$ that is a Brownian motion defined on $\Omega$.

The algorithm for the construction of Brownian motion is an adaptation based on Lévy’s original interpolation argument. Let $t_0 < t < t_1$ and let $W(t)$ be a Brownian motion. Then,

$$G' = W(t) - W(t_0) \quad \text{and} \quad G'' = W(t_1) - W(t)$$

are Gaussian random variables with mean zero and variances $t - t_0$ and $t_1 - t$ respectively. Let $\Gamma$ be a further $N(0, 1)$ random variable which is independent of $W(t_0)$ and $W(t_1) - W(t_0)$. If $t$ is the midpoint of $[t_0, t_1]$, then $\Gamma$ is a random variable with the same distribution as $W(t)$ and is a one-dimensional Brownian motion.

**Theorem 5.1.** (Levy 1940) The series

$$W(t, \omega) = \sum_{n=0}^{\infty} \left( W_{2^{n+1}}(t, \omega) - W_{2^n}(t, \omega) \right) + W_1(t, \omega), \quad t \in [0, 1]$$

converges a.s. uniformly. In particular, $(W(t))_{t \in [0,1]}$ is a Brownian motion.

$W(t)$ is almost surely continuous and can be constructed iteratively using independent increments. This algorithm is based on Lévy’s construction of Brownian motion by means of a limit as $n \to \infty$ of piecewise linear functions taken on subsequently finer and finer refinements of a dyadic grid of size $1/2^n$. Numerically, continuous time processes are sampled on a finite grid as we can only generate finitely many values in finite time. Assuming we have already constructed $W(k2^{-j})$
for some fixed $j \geq 1$ and all $k \in [0, 2^j]$, then

$$W(l2^{-j-1}) = \begin{cases} 
W(k2^{-j}) & l = 2k \\
\frac{1}{2}(W(k2^{-j}) + W((k + 1)2^{-j})) + \Gamma_{2^{j+k}} & l = 2k + 1 
\end{cases}$$

so $W_2(t)$ is therefore a piecewise linear interpolation of the Brownian path $W(t)$.

We take the size of the time steps used in our numerical simulations equal to $h = 1/2^n$ so that no further processing is needed to compute $W(t)$ at the values needed for our numerical simulations. This results in a numerical approximation of Brownian motion, where any refinement of the discretization yields a refinement of the same simulated path. The points on the discrete time grid are simulated linearly, which is possible as Brownian motion has continuous paths.

The seed of the random generator provides a unique random variable $\omega$ in the Gaussian space $\Omega$ used to create a unique Brownian motion. The refinement of an already simulated path is an interpolation in $[t_0, t_1]$ taken as dyadic points $t = k2^{-j}$ the expansion of $W(t, \omega)$ is finite and we are able to calculate the value of $W(t, \omega)$, by successive approximations $t \to W(t, \omega)$. Set $W(0, \omega)$ to zero and let $W(1, \omega)$ be an $N(0, 1)$ distributed random variable. The initialization of $\omega \in \Omega$ as an $N(0, 1)$ distributed random variable is implemented using 1024-bit Marsaglia pseudo-random number generators [17]. The random numbers generator may take a fixed seed as an input, and as such, the same Brownian path can be sampled at any resolution on the dyadic grid. In this way we were able to change the time-scale of the simulation without changing the results from previous experiments.

We initialize the Lévy-Ciesielski algorithm with an initial refinement, then refine as needed to support our averaging scheme. The algorithm as described in the chapter on simulation by Björn Böttcher in Schilling and Partzsch [23], can be described as follows,

Note that taking $\hat{\delta} = h$ reduces $\hat{\xi}^n$ to exactly the term used in the Euler-Maruyama method for the simulation of the stochastic differential equation given
Algorithm: (Lévy-Ciesielski)
Let $J \geq 1$ we the order of refinement.

Initialize $w_0 := 0$

Generate $w_1 \sim N(0, 1)$

for $j = 0$ to $J - 1$ do
  for $l = 0$ to $2^j - 1$ do
    1. Generate $y \sim N(0, 1)$
    2. Set $w_{(2l+1)/2^{j+1}} = \frac{1}{2}(w_{l/2^j} + w_{(l+1)/2^j}) + 2^{-\left(\frac{j}{2} + 1\right)}y$
  end
end

Then $(w_0, w_{1/2}, w_{2/2}, \cdots, w_1) \sim (W_0, W_{1/2}, W_{2/2}, \cdots, W_1)$

by equation (4.1). Thus, taking $\hat{\delta} = h$ in our computations corresponds to noisy instantaneous-in-time coupling in our analysis. Although the sample paths of our Brownian motions do not depend on $h$, the sensitive dependence of the Lorenz system on initial conditions and the resulting deterministic chaos exhibited implies that the free running solution $U$ strongly depends on $h$. Therefore, we have fixed $h = 1/2048$ to ensure that the same reference solution $U$ is considered in all our simulations.

Theoretically, if $\delta = \hat{\delta}$, $u_0 = U_0$ and $\varepsilon = 0$, then $u(t) = U(t)$ for all $t \geq 0$ and any value of $\mu$. This property is preserved at the discrete level in our numerics. In particular, when $\delta = \hat{\delta}$ and $\varepsilon = 0$ exact synchronization of $u^n$ with $U^n$ is possible and was observed in certain simulations. Exact bit-for-bit synchronization of $u^n$ with $U^n$ seems to result from a lucky coincidence in the rounding of the floating point arithmetic, whereas, in general synchronization is only good up to the least significant bits. Note when $\delta \neq \hat{\delta}$ or $\varepsilon > 0$ that exact synchronization is neither theoretically possible nor was numerically observed.
Chapter 6

Conclusions

This thesis considers coupling two systems of equations using noisy partial observations of the phase space that have been blurred in time by means of a moving time average. The inclusion of blur using time averages in the observational measurements is both realistic from a physical point of view as well as tractable in our mathematical and numerical frameworks. The major contributions of our study are four new theorems that cover time averages and numeric simulations that show the coupling works even better than what is guaranteed by the analysis. Namely, the original work in this thesis centers on the statements and proofs of Propositions 3.2, 3.3, 4.2 and 4.3 and the corresponding numeric simulations.

In the context the model system, the Lorenz equations, we have shown that the data-assimilation method introduced in [2] is well-posed when the observational data is contaminated by a moving average with sufficiently short averaging window. In the noiseless case, analysis and numeric simulation shows that synchronization of the driven solution to the free-running solution occurs when the time averaging window is known and small enough. Moreover, when the time averaging window is unknown, the bound on the error is proportional to $(\Delta \delta)^2$ where $\Delta \delta$ is the difference between the size of the actual averaging window and the guess for the size of the window used in the feedback control.

In the presence of noise, time averaging smooths out the noise in the observational measurements at the expense of a phase shift that delays the measurements in time. Although averaging gives no improvement in the analytic and numerical bounds on $\|U - u\|$, there is also no significant deterioration in those bounds provided the averaging window is small enough. In fact, this is the punch line of this thesis: short time averages can be compensated for in the feedback term and
as a result have minimal effect on the quality of the synchronization between the free-running solution and the driven solution. Note, however, that time averaging appear to slightly weaken the coupling between the driven and the free-running solution which, in turn, yields a slight increase in the expected error levels.
Chapter 7

Future Work

7.1 Improved Bound on Fractal Dimension of Attractor

Using the improved ellipsoid described in Appendix A, we conjecture that an improved analytic bound on the fractal dimension of the global attractor \( \mathcal{A} \) of the Lorenz equations may be found using the techniques in Doering and Gibbon [6].

7.2 Numerically Determined Upper Bound

In the case where the size of the averaging window is unknown, we conjecture that an improvement to the analytical bound given in

Noting that in Figure 4.4 the difference between \( \mathbf{E}[\|U - u\|^2] \) and \( L(t) \) is less than a factor of 2, it may be possible to compute a bound numerically that will improve the bound obtained analytically using the Chebyshev inequality. By using a non-linear least squares we postulate that an upper bound on \( \|U - u\| \) can be determined in the case where the averaging window of \( U \) is unknown.

7.3 Adapting \( \mu \) by Computing Local Lyapunov Exponents

We postulate that an improvement on convergence can be achieved through the use of an adaptive filter that tunes the \( \mu \) parameter as the system evolves by testing the rate of convergence at time \( t \).

We conjecture that an adaptive filter may be constructed by computing local
Lyapunov numbers. By determining the rate of divergence of the trajectories, we can allow for larger values of \( \mu \) when the system is 'well-behaved'. In the noiseless case, we can achieve synchronization for unstable values of \( \delta \) such as \( \delta = 0.318 \) as shown in Figure 3.3. In the noisy case, the noise in the observations can be made to be smooth enough that convergence below the level of observational noise can be achieved.

7.4 Reconstructing Attractor Dynamics

Takens’ [?] proved that the time-delayed versions

\[
[y(t), y(t-\tau), y(t-2\tau), \ldots, y(t-2n\tau)]
\]

of one generic signal would suffice to embed the \( n \)-dimensional manifold. This may be compared to the topological result by Whitney [27] in which a single observation in \( 2n + 1 \) dimensions can be replaced by \( 2n + 1 \) observations of one dimension. Namely, Taken’s proved

**Theorem 7.1.** Let \( M \) be a compact manifold of dimension \( d \). For pairs \((\phi, h)\), where \( \phi : M \to M \) is a smooth (at least \( C^2 \)) diffeomorphism and \( h : M \to R \) a smooth function, it is a generic property that the \((2d + 1)\)-fold observation map \( H_k[\phi, h] : M \to R^{2d+1} \) defined by

\[ x \mapsto (h(x), h(\phi(x)), \ldots, h(\phi^{2d}(x))) \]

is an immersion (i.e. \( H_k \) is one-to-one between \( M \) and its image with both \( H_k \) and \( H_k^{-1} \) differentiable).

The theorem can be applied to time series by taking \( \phi \) to be the time \( T \) map of the underlying (continuous time) dynamical system, i.e. \( \phi_j(x_0) = x(jT) \), where \( x(t) \) is the trajectory starting at \( x_0 \). Since \( H_k \) is a diffeomorphism its inverse is
differentiable) this reconstruction preserves the dimension of any invariant set and the Lyapunov exponents of the flow.

Figure 7.1: Lorenz attractor reconstructed in $X$ with time delay $T = 1$.

We can see from Figure 7.4 that the reconstruction of the Lyapunov attractor results in similar, but not identical geometry in the phase space. However, according to Takens’ Theorem, the Lyapunov spectrum and the correlation dimension are preserved by the embedding. It is therefore reasonable to assume that a system constructed by averages where $\delta < \delta_c$ will also preserve these properties. In this case the phase space of $U$ will be exactly the sets of observational data. It will be difficult to compute $H^{-1}$ directly or exactly. Therefore we may be able to compute $H^{-1}$ by approximating its determining form as defined in [18].

### 7.5 Unknown Dynamics of Free Running Solution

It is realistic to suppose that only an approximation $\widetilde{F}$ is known of the exact dynamics $F$ of the free running solution. A simple example for the Lorenz systems
is when the exact parameters values $\sigma$, $r$ and $b$ are unknown. In this case we write

$$F(U) = \begin{bmatrix} \hat{\sigma}(Y - X) \\ -\hat{\sigma}X - Y - XZ \\ -\hat{b}Z + XY - \hat{b}(\hat{r} + \hat{\sigma}) \end{bmatrix}, \quad \tilde{F}(U) = \begin{bmatrix} \sigma(Y - X) \\ -\sigma X - Y - XZ \\ -bZ + XY - b(r + \sigma) \end{bmatrix}.$$

where $\sigma$, $r$ and $b$ are guesses for the true values of the parameters. Just as we obtained bounds on $\|U - u\|$ when $\Delta \delta = \delta - \hat{\delta} \neq 0$ that vanished when $\Delta \delta \to 0$, we expect similar bounds in terms of $\Delta \sigma$, $\Delta r$ and $\Delta b$.

### 7.6 Existence of $\delta_c$

Our calculations suggest that $\delta_c$, if it exists, must lie somewhere between 0.31 and 0.33. The fact that $\|U - u\|^2$ oscillates through 20 decimal orders of magnitude by hitting the noise floor due to rounding error when $\delta = 0.318$ implies that extended precision arithmetic is needed to further characterize $\delta_c$. One could use the GMP Gnu multiprecision library to more accurately determine $\delta_c$. Similar techniques were used by Hayden [11] to more accurately characterize the critical sampling interval for discrete in time observations.

### 7.7 Correcting Phase Shift when $\hat{\delta} \neq \delta$

In the case where $\hat{\delta} > \delta_c$, then taking $\delta = \hat{\delta}$ does not lead to synchronization. However, taking $\delta < \delta_c$ may lead to usable bounds on $\|U - u\|$. We may improve these bounds by time-advancing the measurements by

$$\frac{\hat{\delta} - \delta}{2}$$

so that the phase shift caused by averaging is the same in both systems even though $\hat{\delta} \neq \delta$. 
7.8 Statistical Test Whether $U - u$ is Gaussian

While $u$ is driven by a Gaussian process, the dynamics of the Lorenz equation is highly non-linear. Figure 4.1 indicates that the expected value of $\|U - u\|^2$ is quite near the top of the gray region, however, this is not unexpected as it is the square of a vector in $\mathbb{R}^3$. The question is, therefore, at any fixed point in time to what extent are probability distributions of $X - x$, $Y - y$ and $Z - z$ distributed as Gaussian. Numerically, this could be tested by plotting a histogram of $X - x$, fitting it to a Gaussian, checking the goodness of fit and calculating the probability that it arose from a Gaussian distribution.

7.9 Nonlinear Least Squares Fit of $M$

We consider performing a nonlinear fit to obtain constants $M$, $\gamma$ and $\delta_c$ such that

$$E \approx M(\delta - \hat{\delta})^2/(\delta_c - \hat{\delta})^\gamma$$

Here $\delta_c$ represents the largest value for which $\delta = \hat{\delta}$ leads to convergence. In particular, for values of $\hat{\delta}$ too large, there is no good bound on $E$ even when $\delta \approx \hat{\delta}$. Note that $M$ has the same units as $E$ multiplied by units of time to the power $\gamma - 2$. Therefore taking $\gamma = 2$ yields a constant $M$ which is invariant with respect to a change of scales in time.

7.10 Tuning $\delta$ Using Only the Observations

Our results suggest it is possible to determine the size $\hat{\delta}$ of the unknown averaging window by varying $\delta$ is a way that minimizes the resulting error. In the context of data assimilation, the only information actually available concerning $U$ are the
time averaged measurements of $X$. Therefore, it is more relevant to check whether

$$\mathcal{E} = \| R_\delta(U) - R_\delta(u) \|^2 = \left| \frac{1}{\delta} \int_{t-\delta}^{t} X(s) ds - \frac{1}{\delta} \int_{t-\delta}^{t} x(s) ds \right|^2$$

can be used to numerically determine $\delta$.

### 7.11 Extension of Results to Other Dissipative Systems

The techniques used in our analysis also apply to other model problems such as the two-dimensional Navier–Stokes equations and the surface quasi-geostrophic equations. In those cases, the resulting integro-differential inequality is non-linear. Delayed time averages in the noiseless case for is treated by Jolly, Martinez, Olson and Titi for the surface quasi-geostrophic equations in [13]. In the noisy case, the resulting non-linearity makes estimating the expected values of $\| U - u \|^2$ difficult. It is possible that a slightly modified coupling algorithm which filters statistical outliers from the observational measurements could be used to treat this case.

### 7.12 Same Bounds as Stochastic Case

When $\delta$ tends to zero the differential equations with averages tend to the stochastic case covered by Proposition 4.1. However, the bounds obtained in Proposition 4.2 blow up as $\delta$ vanishes. It is possible to obtain a version of Proposition 4.2 such that the bounds remain bounded when $\delta$ vanishes and are consistent with those which cover the $\delta = 0$ case of Proposition 4.1. Namely, we write the noise term as

$$\mu \xi(t) \Delta X(t) = \mu \xi(t) \left( \Delta X(t) - \Delta X(t - \delta) \right) + \mu \xi(t) \Delta X(t - \delta)$$
and then estimate as in Lemma 3.1 to obtain
\[
|\Delta X(t) - \Delta X(t - \delta)|^2 \leq \left( \int_{t-\delta}^{t} |\Delta \dot{X}(s)| ds \right)^2 \\
\leq 4\delta(\sigma + \mu)^2 \int_{t-2\delta}^{t} V + 4\mu^2 \delta \int_{t-\delta}^{t} \xi^2.
\]
Since \(\xi(t)\) is independent of \(\Delta X(t - \delta)\) then
\[
\mathbb{E}[\mu \xi(t) \Delta X(t - \delta)] = \mu \mathbb{E}[\xi(t)] \cdot \mathbb{E}[\Delta X(t - \delta)] = 0.
\]
This implies
\[
\mathbb{E}[\mu \xi(t) \Delta X(t)] \leq \mathbb{E}[\delta \mu^2 \xi^2] + \frac{1}{4\delta} \left( 4\delta(\sigma + \mu)^2 \int_{t-2\delta}^{t} V + 4\mu^2 \delta \int_{t-\delta}^{t} \mathbb{E}[\xi^2] \right)
\]
\[
= (\sigma + \mu)^2 \int_{t-2\delta}^{t} V + 2\mu^2 \varepsilon^2.
\]
To finish the proof requires a modified version of the integro-differential Gronwall inequality to handle the case when
\[
\frac{d\mathcal{V}(t)}{dt} + \mathcal{V}(t) \leq (A_0 + hA_1) \int_{t-h}^{t} \mathcal{V}(s) ds + B.
\]
We hypothesis that the result is likely to place greater restriction on the maximum size of \(\delta\)—likely of the order \(\sqrt{\mu}\) smaller.
Appendix A

We now describe a spherical bound on the Lorenz trajectories that is better than existing bounds from Doering and Gibbon [6]. Our bound is based on an additional tuning parameter \( \eta \) and is given by

\[
X^2 + Y^2 + (Z - \eta)^2 \frac{b^2(r + \sigma - \eta)^2}{\delta(2b - \delta)}
\]

(7.1)

where \( \delta = 2(b - 1) - 2 + \epsilon \eta \) and

\[
\epsilon = \frac{-\sigma + 1 + \sqrt{\sigma^2 - 2\sigma + 1 + \eta^2}}{\eta}.
\]

Now, differentiating with respect to \( \eta \) gives the optimal value of \( \eta \approx 1.8882316669 \). Substituting this value back into inequality (7.1) we arrive at

\[
X^2 + Y^2 + (Z - 1.8882316669)^2 \leq 1456.461665,
\]

which is about 5 percent smaller than previous bounds.
Appendix B

This section contains C-language source code used in this thesis. There are two files: the first is a pseudo-random number generator and the second is the main routine used to produce our simulations.

### 7.13 Pseudo-random Number Generator

The random number generator is based on the public domain implementation by Sebastian Vigna of the 1024-bit Marsaglia Xorshift algorithm [17].

```c
#include "xonorm.h"
#include <math.h>

/* Written in 2014 by Sebastian Vigna (vigna@acm.org)

To the extent possible under law, the author has dedicated all copyright
and related and neighboring rights to this software to the public domain
worldwide. This software is distributed without any warranty.

Modified 2015 to produce normally distributed random numbers for use in
coupling simulations with noisy observations. */

static uint64_t x=12455480610190296555ULL;
static uint64_t s[16] = {
    9899521803441930453ULL, 4037224574094532358ULL,
    13192621976843192702ULL, 10923948763034344365ULL,
    388333464591572214ULL, 7485323648124204777ULL,
    8981086447562432722ULL, 1561717205308781184ULL,
    2780386954292984643ULL, 3863312353808779985ULL,
    12109031751610640964ULL, 11534438648781303991ULL,
    9340076221226405573ULL, 219462499049377941ULL,
};
```
16106326203448552520ULL, 7245732521397473901ULL;

static int p, flag;

uint64_t xorand64()
{
    x ^= x >> 12; x ^= x << 25; x ^= x >> 27;
    return x * 2685821657736338717ULL;
}

uint64_t xorand1024()
{
    uint64_t s0 = s[p];
    uint64_t s1 = s[p = (p + 1) & 15];
    s1 ^= s1 << 31; s1 ^= s1 >> 11; s0 ^= s0 >> 30;
    return (s[p] = s0 ^ s1) * 1181783497276652981ULL;
}

void xoseed(uint64_t seed)
{
    x = seed;
    for (int i = 0; i < 64; i++) xorand64();
    for (int i = 0; i < 16; i++) s[i] = xorand64();
    p = flag = 0;
}

double xonorm()
{
    static double x2;
    if (flag) {
        flag = 0;
        return x2;
    }
    flag = 1;
    double u1 = (xorand1024() + 1.0) / 18446744073709551616.0;
    double u2 = (xorand1024() + 1.0) / 18446744073709551616.0;
    double r = sqrt(-2 * log(u1)), v = 2 * M_PI * u2;
    x2 = r * sin(v);
    return r * cos(v);
The above source contains modifications to include a routine that generates normally distributed random numbers that will be used to create the Brownian paths representing the noise terms in the observational measurements in this thesis. These changes begin on line 44 with the routine `xonorm` which returns normally distributed random numbers obtained by a polar coordinate transformation.

### 7.14 Main Routine

The main routine consists of the ODE solver, averaging functions and implementation of the Lévy-Ciesielski algorithm used to create our Brownian motions. This code was used to perform all our simulations. Prior to compilation the following preprocessor definitions need to be made:

- **BETA** — The value of $\beta = 8/3$ in the Lorenz system.
- **SIGMA** — The value of $\sigma = 10$ from the Lorenz system.
- **RHO** — The value of $r = 10$ from the Lorenz system.
- **M0** — The value $\delta = M0/N$ for observation blur.
- **M3** — The value $\delta = M3/N$ for the feedback.
- **N** — The timestep $h = 1/N$ where $N = 2048$ for the ODE solver.
- **EPSILON** — The amplitude $\varepsilon$ of the noise term.
- **MU** — The relaxation parameter $\mu$ in the feedback term.
- **M** — The time of integration $T = M$.
- **PMOD** — Output data every PMOD time steps.
- **L** — Size $L = 5$ of the cache for the Brownian paths.
The exact code used for the simulations in this thesis follows:

```c
/*
 * This program uses the original form of the Lorenz equations
 * as stated in
 *
 *
 * X' = \sigma X + \sigma Y
 * Y' = -XZ + rX - Y
 * Z' = XY - bZ
 *
 * The change of variables zeta = Z - r - \sigma yields
 *
 * X' = -\sigma X + \sigma Y
 * Y' = -\sigma X - Y - X zeta
 * zeta' = - b zeta + XY - b(r+\sigma)
 *
 * We know theoretically that if (X,Y,zeta) is on the attractor, then
 *
 * X^2+Y^2+zeta^2 <= K = b^2(r+\sigma)^2/4/(b-1) = 1540.27
 */

#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "xonorm.h"

#define BETA (8.0/3)
#define SIGMA 10.0
#define RHO 28.0

#undef M0
```
#define M0 512.0
#endif
#ifndef M3
#define M3 512.0
#endif
#ifndef N
#define N 2048
#endif
#ifndef EPSILON
#define EPSILON 0.0
#endif
#ifndef MU
#define MU 30
#endif
#ifndef M
#define M 50
#endif
#ifndef PMOD
#define PMOD (N/64)
#endif

double beta=BETA, sigma=SIGMA, rho=RHO, mu=MU, epsilon=EPSILON;
double x[6]={ -5.5751, -3.24345, 26.9272, 0,0,0 }, f[6];
double xmax[6] = { FLT_MIN, FLT_MIN, FLT_MIN, FLT_MIN, FLT_MIN, FLT_MIN };
double xmin[6] = { FLT_MAX, FLT_MAX, FLT_MAX, FLT_MAX, FLT_MAX, FLT_MAX };
double vmax = 0;
double myeta = SIGMA - RHO;
double Umax = 0;
double MM0=M0, MM3=M3;

// Cache of Brownian motion paths
#define L 5
static struct {
    int j,q;
}
double b[N+1];
}
p[L];
static double e[M+1];
static uint64_t s[M];

// Return normal N(mu,sigma) distributed random variable
static double normal(double mu,double sigma2){
  return sqrt(sigma2)*xonorm()+mu;
}

// Levy–Ciesielski Algorithm on the interval [l,l+1]
static void fill(int l){
  for(int delta=N;delta>1;delta/=2){
    int K=N/delta;
    for(int k=0;k<K;k++){
      double sigma=0.25*delta/N;
      p[l].b[delta/2+k*delta]=0.5*(p[l].b[k*delta]+p[l].b[(k+1)*delta])
        +normal(0,sigma);
    }
  }
}

// Initialize the endpoints of the intervals [l,l+1] on which
// we apply the Levy–Ciesielski algorithm ahead of time
static void doinit(uint64_t seed){
  for(int l=0;l<L;l++) p[l].j=-1;
  xoseed(seed);
  e[0]=0;
  for(int i=0;i<M;i++){
    s[i]=xorand1024();
    e[i+1]=e[i]+xonorm();
  }
}
// Return standard Brownian motion $W(t_i)$ where $t_i=i/N$

double getb(int i){
    static int q=0;
    int j=i/N;
    for(int l=0;l<L;l++) if(p[l].j==j){
        p[l].q=++q;
        return p[l].b[i%N];
    }
    int lm=0,qm=p[0].q;
    for(int l=1;l<L;l++) if(p[l].q<qm) {
        lm=l;
        qm=p[l].q;
    }
    p[lm].j=j;
    xoseed(s[j]);
    p[lm].b[0]=e[j];
    p[lm].b[N]=e[j+1];
    fill(lm);
    return p[lm].b[i%N];
}

// Forcing function for the Lorenz system
void force(double f[6],double x[6]){
    f[0]=sigma*(x[1]-x[0]);
    f[1]=rho*x[0]-x[1]-x[0]*x[2];
    f[2]=-beta*x[2]+x[0]*x[1];
    f[3]=sigma*(x[4]-x[3]);
    f[4]=rho*x[3]-x[4]-x[3]*x[5];
}

// Norm of $|U-u|$ where $U=(x[0],x[1],x[2])$ and $u=(x[3],x[4],x[5])$

double norm(double x[6]){
    double r=0;

for(int i=0;i<3;i++){
    double z=x[i]-x[i+3];
    r+=z*z;
}
return sqrt(r);

// Compute time-averages observations and feedback

double avg(double mM,int m,double *xd,int n){
    double r=0;
    int k;
    for(k=n;k>n+1-m;k-=1) r+=xd[k%m];
    r+=xd[k%m]*(mM+1-m);
    return r/mM;
}

int main(int argc,char *argv[]){
    if(MM0<1) MM0=1;
    if(MM3<1) MM3=1;
    int m0=ceil(MM0)+0.5;
    int m3=ceil(MM3)+0.5;
    double x0a[m0],x3a[m3];
    int m=m0>m3?m0:m3;
    uint64_t seed=36819848450518068ULL;
    if(argc>1) seed=strtoull(argv[1],0,0);
    doinit(seed);
    double dt=1.0/N;
    int K=M*N;
    printf("#tn wn X Y Z x y z |U-u| avg(X) avg(x) noise\n");
    printf("# m0=%d\n",m0);
    printf("# m3=%d\n",m3);
    printf("# dt=%g\n",dt);
    printf("# \delta_2=%g\n",MM0*dt);
    printf("# \delta=%g\n",MM3*dt);
printf("\# epsilon=%g\n",epsilon);
printf("\# seed=%llu\n",seed);

// Euler's method to solve for U and u
for(int n=0;n++){
    x0a[n%m0]=x0;
    x3a[n%m3]=x3;
    double tn=n*dt;
    for(int i=0;i<6;i++){
        if(xmin[i]>=x[i]) xmin[i]=x[i];
        if(xmax[i]<=x[i]) xmax[i]=x[i];
    }
    double U=x0*x0+x1*x1+(x2+myeta)*(x2+myeta);
    if(Umax<U) Umax=U;
}
if(!(n%PMOD)) {
    if(n+1>=m) {
        printf("%g %g %g %g %g %g %g %g %g %g %g\n",tn,epsilon*getb(n),x0,x1,x2,x3,x4,x5,norm(x),
               avg(MM0,m0,x0a,n),avg(MM3,m3,x3a,n),
               epsilon*(getb(n+1)-getb(n+1-m0))/m0/dt);
    } else {
        printf("%g %g %g %g %g %g %g %g %g 0 0 0\n",tn,epsilon*getb(n),x0,x1,x2,x3,x4,x5,norm(x));
    }
}
if(n>=K) break;
force(f,x);
if(n+1>=m) x3+=mu*(avg(MM0,m0,x0a,n)-avg(MM3,m3,x3a,n))*dt;
for(int i=0;i<6;i++) x[i]+=f[i]*dt;
if(n+1>=m) x3+=mu*epsilon*(getb(n+1)-getb(n+1-m0))/m0;
}
printf("\# Max %g %g %g %g %g %g %g\n",}
Note the averaging routine `avg` takes values of M0 and M3 that need not be integral and performs an interpolation in line 40 to handle decimal part. The Lévy-Ciesielsky algorithm called through the function `getb` includes a least recently used cache as an optimization because the noise term

\[ \xi^n = \frac{\epsilon}{\delta}(W(t_n) - W(t_n - \delta)) \]

repeatedly samples the Brownian motion at two different points in time separated by the distance \( \delta \). This routine is seeded by the number given in line 159 by default. If a parameter is supplied on the command line, this is used for the seed instead. The program was run 500 times with different seeds on the command line to 500 output files which were then averaged together to compute the expected values \( \mathbb{E}[\|U - u\|^2] \).

Lines 152–153 treat the case \( \hat{\delta} = 0 \) as M0 = 1 and \( \delta = 0 \) as M3 = 1, which reduces to the Euler–Maruyama method from simulating stochastic differential equations. Line 184 computes a numerical bound \( K \) on the free running solution such that \( X^2 + Y^2 + Z^2 \leq K \). Data was output every 64 timesteps. The condition in lines 199 and 201 turn on the coupling only after \( t > \max\{\delta, \hat{\delta}\} \) as described in (2.1). Line 201 assumes M0 is an integer when running simulations with noise. This condition was satisfied in our simulations where the values of \( \hat{\delta} \) considered \( \hat{\delta} = .25 \) obtained when M0 = 512 and \( \delta = 0 \) which corresponds to M0 = 1.
Bibliography


