A Fourth Grade Teaching Experiment on Fraction Magnitude:
Investigating Student Reasoning Through Mathematical Discourse and Design

Research

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by

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Abstract

Students in the United States continue to struggle with fraction mastery despite the fact that understanding fractions is essential for further progress in mathematics (NAEP, 2008). This research project investigated the potential of using high-press mathematical discourse (Kazemi & Stipek, 2008) to investigate fourth graders’ emerging understanding of fraction magnitude. Fraction magnitude refers to the measurement quality of the fraction, or how much of a given unit the fraction represents.

Students who understand the properties of fraction magnitude have demonstrated a more promising trajectory for fraction mastery and for success in algebra (Fuchs et al., 2014; Schneider, Grabner, & Paetsch, 2009; Siegler, Fazio, Bailey, & Zhou, 2012).

This study investigated fourth graders’ reasoning about the measurement construct of fractions that emerged during a whole-class teaching experiment (Lamberg & Middleton, 2009; Lamberg, 2007; McKenney & Reeves, 2013). Students participated in solving tasks which involved fraction magnitude, measurement, and density. Design research methodology was used to carefully analyze the learning ecology (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The domain specific understanding of fraction magnitude was investigated using flexible design revisions based on student work and discourse which emerged during the unit. This was used to produce a fraction magnitude unit reflecting an actual learning trajectory of fourth grade students (Lamberg & Middleton, 2009; McKenney & Reeves, 2013).

An instructional unit based on developing fraction magnitude and measurement understanding was created and used in this study. The developed unit incorporated
research-based findings on using the measurement construct for instruction (e.g., Fuchs et al., 2014; Schneider et al., 2009; Siegler et al., 2012) as well as research based curriculum, such as the Rational Number Project (Behr, Cramer, Harel, Lesh, & Post, 1979). Data was collected from each teaching episode. The lessons were video recorded, student work was documented and analyzed, and teacher/researcher interviews, as well as student interviews, were recorded, transcribed, and evaluated. An iterative design was implemented in which the data collected from one lesson informed changes made in the following lesson. The purpose of the iterative design was to test emerging theories and hypotheses of what was being understood and observed (Gravemeijer & Cobb, 2006; Lamberg & Middleton, 2009).

The teaching experiment provided insights into students’ reasoning during fraction magnitude tasks and how episodes of whole class mathematics discourse supported both teacher insight and student understanding. The realized learning trajectory along with mathematical mindset norms that emerged are documented. Implications for curriculum development and teaching fraction magnitude are discussed.
Dedicated to my mom, Barbara Bertolone (1938 – 2009) who always knew I could do it.

Dedicated to Britt Mattinson (1998-2007), who had the courage to raise his hand in second grade and say, “I want to change my answer.”
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CHAPTER I: INTRODUCTION AND PROBLEM STATEMENT

Introduction

In the United States, fraction instruction begins in first grade with the discussion of halves. By the time students are in fourth grade, they are expected to have mastered all four operations with fractions and mixed numbers (National Governor’s Association/Council of Chief State School Officers 3 (NGA/CCSSO), 2010). However, fraction mastery in the elementary grades remains difficult for students in the United States (National Mathematics Advisory Panel (NMAP), 2008; Siegler et al., 2012). For example, only 50% of a nationally representative sample of US eighth graders were found to be able to correctly arrange three fractions in ascending order of magnitude on the National Assessment of Educational Progress, while in a recent study of fourth graders with a mean IQ of 116, only 59% showed correct ordering of a set of fractions (Mazzocco & Delvin, 2008). The National Mathematics Advisory Panel (NMAP, 2008) stated that “difficulty with the learning of fractions is pervasive and is an obstacle to further progress in mathematics and other domains dependent on mathematics, including algebra” (p. 28). The inability to work with rational numbers appears to limit a child’s future success in school mathematics and can hinder adults in solving real world problems involving fractions such as recipe conversions, medicinal doses based on body weight, tax and loan percentages (Booth & Newton, 2012). While it is understood that fraction knowledge is necessary for most aspects of mathematics, “Little is known about why some children master fractions quickly but others struggle even as adults” (Fuchs et al., 2014, p. 46).

Research supports several different theories on why the integration of rational numbers provides difficulties for elementary aged children (DeWolf & Vosniadou, 2015; Sfard, 1991, Siegler, Thompson, & Schneider, 2011; Torbeyns, Schneider, Xin, & Siegler, 2015). It appears
that children need both time and instruction that allows for the assimilation of the differences between fractions and whole numbers as well as continual monitoring of misconceptions of fractions that often remain unaddressed (Vosniadou, 2007). Several studies refer to difficulties caused by what researchers label whole number bias (Mack, 1995; Ni & Zhou, 2005; Siebert & Gaskin, 2006; Small, 2014). Whole number bias recognizes that children who are comfortable with whole numbers often apply whole number properties to fractions. This prevents students from seeing the fraction as a value in itself, rather than two whole numbers with a line in between (Torbeyns et al., 2015).

Researchers have identified five sub-constructs of fractions: part-whole/partitioning, ratio, quotient, measure and operator (Bottge, Ma, Gassaway, Butler, & Toland, 2014). Each construct operates in a particular way and the different representations overlap in meaning and function. Fraction instruction in the elementary grades in the United States tends to focus predominately on the part-whole construct of fractions (Alajmi, 2012). However, studies have shown that children who understand the measurement construct of fractions tend to outperform their peers in fraction computation and later in algebra (Bailey, Hoard, Nugent, & Geary, 2012; Torbeys et al., 2015). Fractional numbers describe a measurement representing the magnitude of the fraction. As Cracknell (1915) explained, “Fractional numbers enable us to measure a magnitude in terms of a unit.” Cracknell (1915) defines a fraction in this way: To form the fraction $\frac{p}{q}$, divide the unit into $q$ equal parts, and take $p$ of them to constitute the fraction. If $p > q$, more than one unit must be so divided (p. 21).

Developing a deep conceptual understanding of fraction magnitude (the measure or value of a fraction) may provide a stronger basis for reasoning about fractions (Fuchs et al., 2014; Schneider & Siegler, 2010; Schneider, Grabner, & Paetsch, 2009). For example, fraction
magnitude understanding has been measured by number line tasks— an empty number line where students are expected to place fractions according to their magnitude (see Error! reference source not found.)

Place the fraction that represents the shaded part of the shape B on the number line.

![Shape B](image)

Figure 1 Fraction Task adapted from Pantziara & Phillippou’s (2011) assessment for levels of students conceptual understanding of fractions. This task is meant to bring to light a student’s understanding or misunderstanding of the magnitude of a fractional number.

While doing a number line activity such as in Figure 1, a fourth grade student was asked to plot the fraction 2/3. The number line went from zero to one, and the student complained that it needed to go to three. Why? So he could plot 2/3 on the 2, or two wholes out of three wholes. This illustrates the presence of whole number bias (Ni & Zhou, 2005; Siegler et al., 2011).

Armed with a magnitude understanding of fractions (that the fraction $p/q$ means dividing the unit into $q$ equal parts, and then taking $p$ of those parts as the measure), this student would divide the distance between zero and one into three equal parts, and plot 2/3 at two of those equal parts. Understanding fraction magnitude appears to alleviate some of the whole number attributes children incorrectly apply to fractions (Fuchs et al., 2014; Torbeyns et al., 2015).

Whole class mathematical discourse is a method that can be used by teachers in order to engage students in argumentation and critique of mathematical reasoning (NCTM, 2000;
This approach might provide a platform for children to openly discuss and change immature schematics of fractions into an increasingly mature mathematical understanding (Mueller, Yankelewitz, & Maher, 2014; Sfard, 2008; Towers, Martin, & Heater, 2013). However, the mathematical discourse must include clear social and sociomathematical norms in order to inspire deep intellectual thinking and speaking about mathematics (Kosko, 2014). Having a classroom routine for the mathematical discourse appears to be a first and important step in this process. Whole class mathematical discourse is an opportunity for children to organize and consolidate their thinking by communicating with their peers (National Council of Teachers of Mathematics (NCTM), 2000). The Standards for Mathematical Practices (NGA/CCSSO, 2010) state the need for instruction that fosters inquiry and validation of mathematical knowledge in the classroom. The following Mathematical Practices can be accomplished by using mathematical discourse in the classroom (p. 6):

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.

The practices above encompass acts of reasoning, arguing, and critiquing mathematics within the classroom providing an opportunity for children to arrive at new mathematical understandings (Kieran, 2001; Sfard, 2007). Mathematical discourse can inspire children to change their minds about a mathematical concept, thereby providing an opportunity to reconfigure prior misconceptions. Whole class mathematical discourse can be instrumental in convincing children to change their minds about previous notions of number, and is therefore a valuable instructional tool in cases where there is great need to slowly, accurately, and intentionally lead children to a new understanding (Kazemi & Stipek, 2008; Koichu, Berman, &
Another benefit of mathematical discourse is students’ engagement in “communicating to learn mathematics” with opportunities to listen and participate in activities emphasizing mathematical reasoning rather than just memorizing and practicing procedures (Piccolò, Harbaugh, Carter, Capraro, & Capraro, 2008, p. 404).

Issues with the efficacy of whole class mathematical discourse occur because of social concerns, teacher capabilities, and whether or not sociomathematical norms are developed and upheld (Bishop, 2012; Bray, 2011; Cobb, Wood, & Yackel, 1993; Drageset, 2015). In particular, whole class mathematical discourse requires a “high-press” approach (Kazemi & Stipek, 2008). A high-press mathematical discourse necessitates (1) thoughtful teacher interventions, (2) the posing of strategic questions (by teacher and students), (3) development of a reasoning community, and (4) establishing sociomathematical norms (Mueller et al., 2014). Additionally, following a predictable routine for mathematical discourse tends to expand opportunities for increased reasoning and mathematical understanding (Kosko, Rougee, & Herbst, 2014; Mercer & Sams, 2006; Mottier-Lopez & Allal, 2007; Nachlieli & Tabach, 2012, Wood & Kalinec, 2012).
CHAPTER II: REVIEW OF LITERATURE

The Difficulty with Fractions

The CCSSM (NGA/CCSSO, 2010) recommended students in third grade begin to develop an understanding of fractions with the following content goals (NGA/CCSSO, 2010, Number & Operations-fractions section):

- Understand a fraction \( \frac{1}{b} \) as the quantity formed by 1 part when a whole is partitioned into \( b \) equal parts; understand a fraction \( \frac{a}{b} \) as the quantity formed by \( a \) parts of size \( \frac{1}{b} \).
- Understand a fraction as a number on the number line; represent fractions on a number line diagram.
- Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

By fourth grade, students are expected to expand this understanding by visually modeling and justifying equivalent fractions, fraction comparison with statements of inequality, multiplication of a fraction and a whole number, and placing fractions with different denominators on the number line (NGA/CCSSO, 2010).

This integration of the fraction number system into the whole number system begins in early elementary school and necessitates a conceptual change in children’s knowledge; the “new information requires the radical reorganization of what is already known” (Stafylidou & Vosniadou, 2004). In order to bring about conceptual change in children’s understanding of number, both instruction and learning expectations must be rigorous and conducted with the intent purpose of leading children to a new understanding of what a number can represent (Smith, Solomon, & Carey, 2005). The reorganization of what is known about whole numbers to include a new kind of number with different properties might be a major cause of the difficulty involved with mastering fractions (Fuchs et al., 2014; Sfard, 1991; Siegler et al., 2011; Smith et al., 2005). In addition, the struggle with fractions may be caused by the fact that there are many
meanings of fractions (Kieren, 1976), the unique symbolism of fractions (Post, Wachsmuth, Lesh, & Behr, 1985), or the overgeneralization of whole number knowledge onto fraction understanding (DeWolf & Vosniadou, 2014; Ni & Zhou, 2005; Siebert & Gaskin, 2006), all of which might lead to perilous misconceptions about fractions. Research revolves around several theories for poor performance of elementary aged students in fraction understanding. These theories include the theory of Integration of a New Number System (Sfard, 1991), the theory on Whole Number Bias (Mack, 1995; Ni & Yong-Di, 2005; Post et al., 1995; Siebert & Gaskin, 2006), the Integrated Theory of Whole Number and Fractions Development (Siegler et al., 2011), and finally, the Framework Theory approach to conceptual change (Stafylidou & Vosniadou, 2004; Vosniadou, 2007; Vosniadou & Skopeliti, 2014; Vosniadou, 2014). Each of these theories will be explained further for their contribution to revealing the origins of difficulties with fraction understanding.

**Theories of the Underlying Origins of Fraction Difficulties**

**Integration of a New Number System**

To learn fractions a child must have the ability to not only understand the whole number system, but also widen their knowledge to include numbers that exist between whole numbers – an infinite amount of them. The uptake of a new type of number, such as fractions, into schemata of another number system involving whole numbers may contribute to the difficulties children exhibit in understanding fractions (Sfard, 1991; Siegler et al., 2011; Siegler et al., 2011; Smith, et al., 2005). Making this shift requires “strong within child consistency in reasoning about number” (Smith, et al., 2005, p. 134). In essence, consistent reasoning about whole numbers lays the foundation for consistent reasoning about fractions.
In 1991, Sfard proposed three stages of number integration based on the steps of “concept formation” (p. 18). According to Sfard (1991), all three must be experienced in order for students to effectively understand and operate within new types of number systems. The stages move from interiorization to condensation to reification (see Figure 2).

**Figure 2 Phases of Number Integration. Adapted from Sfard (1991).**

**Interiorization.** The first step towards integration is interiorization, in which the learner becomes familiar with the processes within the system, “like counting which leads to natural numbers or subtracting which leads to negative numbers” (Sfard, 1991, p. 18). A number system is interiorized when the mental representations of the system can be used to compare, consider, and analyze numerical situations. A hallmark of the interiorization stage of fraction numbers is the ability to think flexibly about fractions and to fluidly understand the symbolic representation and all the constructs it may represent (Post et al., 1985). Consider the rational numbers 2/3 and 5/6. One might see 2/3 as two parts out of three equal parts and similarly 5/6 as five parts of six equal parts of the same size whole, thus comparing the exact area of the amount covered. Post et al. referred to this as the embodiment of the fraction symbol (Figure 3):
In the interiorization stage, the child would shift from having to mentally or physically construct an embodiment of each individual fraction in order to, for example, be able to say which one is smaller. This thought process may involve understanding that $\frac{5}{6}$ is somewhat larger without having to visualize the physical representation of each fraction, rather, relying on the ratio relationship between 2 and 3 and 5 and 6. Abstract reasoning that does not necessitate evoking the embodiment of the fraction is a sign that interiorization of the fraction number system has occurred (Charalambous & Pitta-Pantazi, 2007, Post et. al, 1985). Considering that many elementary school children have difficulty explaining why a fraction is written with two numbers (Smith et al., 2005), it is not surprising that much of the fraction work with elementary aged children occurs in this stage of development.

**Condensation.** The next stage in Sfard’s hierarchy is the condensation stage. In this stage the learner is able to condense knowledge about the new number system and begin thinking about the process as a whole without feeling the urge to go into details. Ease in dealing with different representations of a number solidifies in this stage. A student may be in the condensation stage of fraction knowledge if they are able to accurately place $\frac{3}{5}$ on a number line, see an unfinished division problem (three divided by five) as well as cognitively understand that the representation also means three parts out of five and a ratio of 3 to 5. In this stage, the ability to generate equivalent fractions becomes a fluid procedure, “squeezed” into one process and much of it is performed mentally. The relationship between the numerator and the
denominator of fractions is understood as division, and students begin to see that fractions are infinitely divisible and that there are an infinite number of them (Fuchs et al., 2014; Smith et al., 2005; Stafylidou & Vosniadou, 2004).

Reification. Reification is the final stage of number integration and this occurs when the whole concept of a number system and what it represents can be acted on as a single object. In this stage, the various representations of fractions are seen together as an abstract concept, one that can be studied and generalized. Fractions can easily slip between constructs of meaning: part of a whole, measurement, rate and ratio, as an operator and finally, as a division problem (Charalambous & Pitta-Pantazi, 2007; Pantziara & Philippou, 2012) without causing mental angst. Students can ponder the infinite divisibility of fractions and realize there are an infinite number of fractions, create analogies between the different domains of number and use thought experiments to investigate properties and generalizations of fractions (Smith et al., 2005). The existence of the new number system is no longer in question, and how the system can lead to further understanding of the nature of all numbers is the question tackled in the reification stage (Pantziara & Philippou, 2012).

Whole Number Bias

The “Whole Number Bias” is a theory that links the misconceptions children may have about fractions with the natural ability of humans to understand whole numbers from a young age (e.g., Mack, 1995; Ni & Yong-Di, 2005; Post et al., 1995; Siebert & Gaskin, 2006). This theory states that the ease and comfort children have with the whole number system creates a resistance to broaden the idea of number to include fractions. Whole number bias is “the robust tendency to use the single-unit counting scheme to interpret instructional data on fractions” (Ni & Zhou, 2005, p. 28). Post et al. (1995) confirmed this in their teaching experiment with fourth
graders, in which even after extended conceptually based fraction instruction, children persisted in the misconceptions based on whole number ordering, i.e. that 1/9 is larger than 1/3 because 9 is larger than 3. Children tend to rely on their previous understanding of whole numbers by using the magnitude or the counting sequence to determine the magnitude of a fraction. This preference and reliance on whole number properties causes difficulty in the conceptualization of fractions for children and adults (Ni & Zhou, 2005; Smith et al., 2005). In addition, a child exhibiting whole number bias might see the embodiment of 3/8 as showing eight objects with three shaded (figure 4):

![Figure 4](image)

Figure 4. Example of a set of objects illustrating a fractional number.

He or she might not see 3/8 (three eighths) at all. Instead, this might represent three things shaded out of eight total things or a whole three over a whole eight. A common misconception in this case would be to name this embodiment as 3/5, showing three shaded and five unshaded. In this situation, 3/8 is not conceived as having its own single number identity or value (Siebert & Gaskin, 2006). Under conditions of whole number bias, the symbolic representation of a fraction is a concern because of the “difficulty mastering the symbols or the lack of conceptual referent” (Ni & Zhou, 2005, p. 39). The fraction symbol is represented with three characters (Figure 5).
To which “referent” should children ground their understanding of the fraction? This symbol has two whole numbers and a line which expresses division or a ratio between the two; undoubtedly, a fraction is “considerably more complex than a whole number” (Torbeyns et al., 2015, p. 2). Mack’s (1995) study of third graders’ ability to use the symbolic representation of fractions stemming from their informal knowledge resulted in two findings. First, while students can draw on informal knowledge to understand the meaning of a mathematical symbol, the reverse is not always true. After engaging the students’ informal knowledge of a pizza, the students were able to reason that 3/5 of the pizza meant there were five pieces of pizza and you had three. However, when shown the symbolic representation of 3/5, a prevailing explanation was there are five pizzas with three parts in each one. Second, Mack’s (1995) results suggested that students’ prior knowledge of whole numbers influenced their working with the fraction symbols. One student commented:

See, I know this other sort of math [fractions], but I know this [whole numbers] better, so I try to make it so it looks like just a regular math problem so I can figure out the answer. It's the same thing. It's just easier to figure out this way. (p. 437)

When Mack (1995) informally discussed the idea of having two friends who each ate 1/8 of a pizza, the students reasoned that two out of the eight pieces had been eaten. However, when
asked to manipulate the symbolic representation of 1/8 + 1/8, they determined the answer was 2/16. This explanation was offered:

I got the first one (meaning the real world problem) wrong. It has to be two sixteenths because you get one whole pizza with eight pieces and another whole pizza with eight pieces, so there’s two pizzas with sixteen pizzas in all. (p. 432)

The whole number bias accounts for children’s difficulty with fractions as the result of using an already established concept of whole number and relying on this understanding to process the new symbols and meaning of fractions. This appeared to be true even in instances where children could informally and correctly reason about fractions; yet when faced with symbolic representations and operations including fractions, their familiarity of whole numbers superseded their accurate reasoning.

**Integrated Theory of Whole Number and Fractions Development**

Sfard (1991, 2001) discussed the difficulty of becoming familiar with a number system and therefore supported an integrated number theory that adjusted the idea of whole number bias and took a series of steps to integrate a new system into consciousness and operate within that system and with the objects of that system. Siegler, Thompson, and Schneider (2011) added that whole number bias is “part of the story, but not the whole story” regarding the difficulties children have with fractions. Siegler et al. (2011) proposed an “Integrated Theory of Whole Number and Fractions Development” to provide a more holistic view of the process of interiorizing fractions (p. 275). This theory acknowledged that many properties of whole numbers are not true of numbers in general and the reliance on this knowledge with resistance to changing the initial concept of number is the sole focus of the whole number bias theories. According to Siegler et al. (2011), the whole number bias theory supports the ideology that learning about whole numbers is fundamentally different and unrelated to learning about other
types of numbers. The whole number bias accomplishes this by focusing on the “differences and discontinuities” between whole and fraction numbers (Siegler et al., 2011, p. 289). In contrast, the Integrated Theory of Whole Number and Fractions Development (Siegler et al., 2011) includes the development of whole number knowledge as an integral part of developing fraction knowledge. It is based on the idea that the process of numerical development is characterized by a continuous broadening of numbers that have magnitude and understanding how the new number system operates. Therefore, the integration of whole numbers and fractions follow similar developmental paths. Drawing inaccurate analogies between fractions and whole numbers causes more difficulty in learning about fractions than from drawing parallels between the two sets of numbers (Siegler et al., 2011). This is similar to the belief that each new type of number must go through stages of integration (Sfard, 1991). The integrated theory (Siegler et al., 2011) accepts the presence of whole number bias, however, and it widens the lens of the theory by focusing on “commonalities and continuities” (Siegler et al., p. 298) between whole and fraction numbers in order to integrate the two number systems.

In 2001, Wu suggested that a major fault in fraction instruction is that the relationship between whole numbers and fractions is not clearly defined for the learner. For example, one commonality that both fractions and whole numbers share is magnitude, and understanding magnitude is central to both types of number (Siegler et al., 2012; Torbeyns et al., 2015). Another difficulty is caused by the ratio or division of two whole numbers (numerator and denominator) representing one value; which is much more complex than a whole number (Torbeyns, et al., 2015). The integrated theory views interference from whole number knowledge as only one of several sources of difficulty in learning fractions. Although whole
number bias is an important part of fraction difficulties, it does not account for the complete explanation (Siegler et al., 2011).

**Framework Theory Approach to Conceptual Change**

_Central concept_. As evidenced in the above theories, students develop a “central concept” regarding whole numbers, centered on “rational activity” in their learning experiences so far (Posner et al., 1982, p. 211). The central concept of whole number is what must be changed for a student to effectively understand and operate with fractional numbers (Charalambous & Pitta-Pantazi, 2007; Smith et al., 2005). According to Posner et al. (1982), this conceptual change occurs when “these central commitments require modification” (p. 212). For example, studies have analyzed the time discrepancy in fraction magnitude comparison in both children and adults (i.e. Schneider & Siegler, 2010; Schneider et al., 2009) and concluded that whole number reasoning prevented accurate and timely responses to prompts such as: Which is bigger $1/3$ or $1/5$? Using whole number reasoning, one might presume the answer to be $1/5$, because five is larger than three. Clearly, remaining centrally committed to whole number properties while working with fractions is problematic and requires both a shift in concept and commitment to what was previously understood.

Posner et al. (1982) described this type of conceptual shift as _accommodation_, in which the student’s current concept inadequately addresses or explains the phenomenon, thus requiring a radical reorganization of what is currently known (Smith et al., 2005). Posner et al. (1982) suggested that accommodating new concepts into current schemata requires changes in “fundamental assumptions about the world, about knowledge, and about knowing…such changes can be strenuous and potentially threatening, particularly when the individual is firmly committed to prior assumptions” (p. 223). Moreover, based on Posner et al.’s (1982) study on
college students’ capacity for accommodation in a physics course, it was found that several forces on the current understanding must be present in order for reorganization to occur (see figure 6). These findings also apply to changing conceptual understanding in mathematics and will be explained thusly.

*Figure 6* Forces acting to create accommodation of a new concept; inciting conceptual change. Adapted from Posner et al. (1982).

This model will be discussed using the example presented earlier where a student wanted to place the value of 2/3 on a number line in the following fashion (figure 7):

*Figure 7. Student graphs 2/3 on a number line.*
Cognitive conflict occurs when a student is asked to reason if \( \frac{2}{3} \) is of equal value to two wholes. A simple analogy might be presented to the student: “So, you are saying that if I gave someone \( \frac{2}{3} \) of a candy bar, it is the same as giving the person 2 whole candy bars?” This may begin to create dissatisfaction with the whole number interpretation of \( \frac{2}{3} \).

If the pursuit of mathematical understanding is taken seriously by the student, this conflict begins to bring into question the basic assumptions of equality in mathematics, namely (figure 8):

\[
\begin{array}{c}
\frac{2}{3} \\
\neq \\
2
\end{array}
\]

*Figure 8. Cognitive conflict is produced with analogy and mathematical principles.*

Based on the conflict, analogy, and rationality of mathematics, the student may begin to change their prior understanding if the new information has more promise—such as maintaining the basic structure of mathematical equality or perhaps getting problems correct on a future work. Posner et al. (1982) found when faced with cognitive conflict, students also found alternatives to assimilation possibly due to the fact that fundamental revision of central concepts is difficult, and “therefore, the most unlikely approach” (p. 221). Alternatives to conceptual change are highlighted in figure 9.
Again examining the above example of placing $2/3$ on the number line at the whole number two, a student may choose different pathways to accommodate their thinking. The student may reject the cognitive conflict by concluding it is a special situation and therefore holds no merit. The student might also allow that $2/3$ could be equal to two in some cases such as when the wholes are not the same size and therefore is not of concern. Rather than change current beliefs, the student may choose to file this as an oddity of mathematics that has nothing to do with the “real world” (Posner et al., 1982, p. 221). Additionally, this conflict might slightly adjust the whole number central concept without fully breaking it apart to include fractional numbers.

Conceptual change as described by Posner et al. (1982) has been used in several studies addressing fraction understanding (e.g., DeWolf & Vosniadou, 2015; Gabriel et al., 2012; Smith
et al., 2005; Stafylidou & Vosniadou, 2004; Vosniadou, 2014). The next section will address how conceptual change and the framework theory are united to provide further explanation.

**Framework theory of conceptual change.** The construction of rational number understanding is the focus of using the framework theory approach to conceptual change (Stafylidou & Vosniadou, 2004; Vosniadou, 2007; Vosniadou & Skopeliti, 2014; Vosniadou, 2014). A key point in the conceptual change approach is that the discrepancies between the mathematical conception of fractions and the individual student’s conception of fractions are not “seen as individual deficits”, rather they are taken as typical stages of transition (Prediger, 2008, p. 4).

The framework theory is based on the observation that “early knowledge is organized in the form of naïve theories” that children generate as they interact with social, cultural, and educational constructs (Vosniadou, 2007, p.49). The framework of naïve theories allows children to create explanations and predictions and deal with unfamiliar problems. Conceptual change is the process in which a child adjusts their current naïve theory to include new information and builds a more thorough framework of knowledge (Gabriel et al., 2012; Ni & Zhou, 2005; Smith et al., 2005; Stafylidou & Vosniadou, 2004). According to scholars who support the theory of conceptual change, creating an accurate framework about rational numbers requires “instruction-induced conceptual change,” or the kind of profound conceptual change requiring “systematic instruction in order to be achieved” (Vosniadou, 2014, p. 50).

The framework theory further explains students’ difficulties with fractions on the grounds that fractions violate many of the principles that govern reasoning with natural numbers and require a reconsideration of previously learned properties of numbers. (Stafylidou & Vosniadou, 2004; Vosniadou & Skopeliti, 2014). According to this theory, during the preschool years, children
develop an initial concept of number. By third grade, most children have built a “rich and productive number concept” including assumptions and beliefs about “how numbers are supposed to behave” (Vosniadou, 2014, p. 650). For example, children are often told that multiplication makes numbers larger and is repeated addition, whereas division makes numbers smaller and is partitive in nature, and the divisor is smaller than the dividend. Another property of whole numbers is that for every whole number, there exists exactly one number greater and one number lesser than that number. Fractions violate many of the initial concepts of number and create a disequilibrium within the child’s whole number framework. The differing properties of natural numbers and fractions are summarized in Table 1.
Students struggle to assimilate incompatible information about fractions into their understanding of rational number and this process requires a transition over time rather than a spontaneous understanding (Mazzocco, Meyers, Lewis, Hanich, & Murphy, 2013; Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004, Vosniadou & Skopeliti, 2014).

A study by Stafylidou and Vosniadou (2004) investigated a range of older students (fifth through ninth grade) and their developing understanding of fractions to probe for evidence that

### Table 1: Natural Number and Fraction Comparison

<table>
<thead>
<tr>
<th>Numerical value</th>
<th>Natural number</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symbolic representation</td>
<td>One number</td>
<td>Two numbers and a line</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>( \frac{5}{8} )</td>
</tr>
<tr>
<td>Ordering</td>
<td>Sequential</td>
<td>No unique successor or preceding number</td>
</tr>
<tr>
<td></td>
<td>No number between two different numbers</td>
<td>Infinity</td>
</tr>
<tr>
<td></td>
<td>1, 2, 3…..</td>
<td>( \frac{1}{8} \ldots \infty \ldots \frac{5}{8} )</td>
</tr>
<tr>
<td>Relationship to the unit</td>
<td>Unit is smallest number</td>
<td>No unique smallest number</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>( \infty \ldots \leftrightarrow \frac{1}{8} )</td>
</tr>
<tr>
<td>Operations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Addition and Subtraction</td>
<td>Supported by sequence</td>
<td>Not supported by sequence of natural numbers</td>
</tr>
<tr>
<td></td>
<td>( 5 + 3 = ) &quot;five, six, seven, eight&quot;</td>
<td>( \frac{2}{3} + \frac{7}{8} = ) counting up not possible</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Multiplication</td>
<td>Magnifies numbers</td>
<td>Magnifies or demagnifies numbers</td>
</tr>
<tr>
<td></td>
<td>Repeated addition</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4( \frac{1}{2} \times 2 = 8\frac{1}{2} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \frac{1}{3} \times \frac{3}{4} = 2/9 )</td>
</tr>
<tr>
<td>Division</td>
<td>Demagnifies numbers</td>
<td>Magnifies or demagnifies numbers</td>
</tr>
<tr>
<td></td>
<td>Repeated subtraction</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \frac{7}{6} \div \frac{3}{4} = 1 \frac{5}{16} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 5 \div \frac{1}{8} = 40 )</td>
</tr>
</tbody>
</table>

*Adapted from Stafylidou & Vosniadou, 2004 (p. 505).*
they can overcome the “barriers imposed by their knowledge of natural numbers” and showed that over time, their naïve theories of fractions will undergo conceptual change (p. 504). A particular focus of this research was the interpretation of the numerical value of fractions. Students were asked to order the numbers 5/6, 1, 1/7, and 4/3 from smallest to largest and to explain the reasoning they used to decide upon the order. Responses to why students chose a particular order fell into four categories (Stafylidou & Vosniadou, 2014, p. 509, Figure 10):

<table>
<thead>
<tr>
<th>Order the numbers 5/6, 1, 1/7, and 4/3 from smallest to biggest. Explain why you ordered them this way?</th>
<th>The value of a fraction increases when the numbers that comprise it increase.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Example answer: 1, 4/3, 5/6, 1/7</td>
</tr>
<tr>
<td>The value of a fraction increases when the numbers that comprise it decrease</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Example answer: 1/7, 5/6, 4/3, 1</td>
</tr>
<tr>
<td>The value of the fraction increases as the size of the numerator approximates the size of the denominator</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Example answer: 1/7, 4/3, 5/6, 1</td>
</tr>
<tr>
<td>The value of the fraction increases as the size of the numerator becomes bigger than the size of the denominator</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Example answer: 1/7, 5/6, 1, 4/3</td>
</tr>
</tbody>
</table>

Figure 10. Explanations and example answers regarding fraction ordering (Stafylidou & Vosniadou, 2014).

Results suggested that the younger students were more likely to respond that the larger fractions were those with the larger numerators and denominators compared to the older students. The older students were more likely to say that the larger fractions were those with the smaller numerators and/or denominators. In addition, students were asked to give the largest and smallest fraction they could imagine. Over time (from fifth to ninth grade), students’ explanation of the density of a fraction increased in accuracy and approached the scientific view of fractions.
– that there is no limit to the magnification or demagnification of fractional numbers (infinity for all fractions).

Understanding fraction density can be a predictor of ability to reason accurately about fractions (e.g., Small, 2014; Smith et al., 2005; Stafylidou & Vosniadou, 2004). Vosniadou (2014) calls these beliefs that approach the true definition of a fraction “fragmented and synthetic conceptions” (p. 651). Figure 11 illustrates the synthetic conceptions children may develop regarding the density of fractions. Each level of conception has a piece of the truth of density; however, it does not include the full understanding.

**Levels of Synthetic Conceptions**

- **Density:** There are infinitely many numbers between any two numbers
- **Discreteness:** There are no numbers between .05 and .06
- **Advanced Discreteness:** There is a finite number of numbers between 0.005 and 0.006
- **Restricted Infinity:** Infinite decimals between decimals, but between fractions (or the opposite)
- **Restricted Infinity:** There are no fractions between decimals or decimals between fractions

Figure 11. Synthetic conceptions regarding the density of rational numbers. Adapted from Vosniadou (2014, p. 652)
Both the reasons students gave for their arrangement of fraction value and the synthetic conceptions of density of number highlight students’ attempts to integrate incompatible information about rational number into their initial concept of number (Ni & Zhou, 2005; Sfard, 1991). In order for this to occur, there needs to be a shift in children’s understanding of number properties, an ability to reason about numbers and physical quantities (Smith et al., 2005) and, in fact, the learning is much more difficult in cases of conceptual change (p. 133). Lastly, the theory allows for the fact that new conceptions of fractions are eventually constructed, however, “this new representation is not robust and does not replace the initial representation of whole number, but coexists with it” (DeWolf & Vosniadou, 2015, p. 41). The potential power of the Framework Theory of Conceptual Change are as follows (Vosniadou & Skopeliti, 2014, p. 143):

- It provides an account of the transition process from an initial to a more sophisticated understanding of counter-intuitive concepts.
- Promotes the understanding that the process will slow and gradual.
- Predicts the process of fraction understanding will give rise to fragmentation and the generation of synthetic conceptions.
- Synthetic conceptions are incorrect from a scientific point of view, however, they enable the student to move on in the knowledge acquisition process.

Children can say puzzling things about fractions that are often nonsensical, this is not surprising because their knowledge is in transition due to the radical reorganization of what is known to incorporate a new understanding (Mack, 1995; Siegler et al, 2011; Smith et al, 2005; Stafylidou & Vosniadou, 2004). This is difficult work. Each of these learning theories regarding fraction understanding shares a similar belief: gaining understanding of all types of real numbers takes time and continual processing from infancy to adolescence (Ni & Zhou, 2006; Sfard, 2001; Siegler et al, 2011; Stafylidou & Vosniadou, 2004).
**Fraction Measurement Construct**

Fractions are complex mathematical structures and aside from the confusion with whole numbers, the many uses and conceptions of fractions present another difficulty for children (Charalambous & Pitta-Pantazi, 2005; Pantziara & Philippou, 2012). As Kieren (1976) argued, “to understand the ideas of rational numbers, one must have adequate experience with their many interpretations” (p. 102). Based on Kieren’s work (1976) which named several (7) interpretations of rational numbers, researchers have developed five sub-constructs of fractions: part-whole, ratio, quotient, measure and operator (Bottge et al., 2014; Pantziara & Philippou, 2012). Each construct operates in a particular way and different representations overlap in meaning and function (Kieren, 1976). Of the five constructs, many studies address the importance of understanding the measurement construct (e.g., Fuchs et al., 2014; Lamon, 2001; Schneider, 2009; Siegler et al., 2011). Therefore, this paper will focus on the measurement interpretation of fractions. Correlation between number line estimation and mathematics achievement is a common finding in research (e.g., Booth & Newton, 2012; Fuchs, et al., 2014; Schneider, 2009; Siegler et al., 2012). Fraction measurement understanding has been assessed by number line tasks— an empty number line where students are expected to place the fractions (see Figure 12).
According to Kieren (1976), completing tasks such as these (Figure 12) involves “flexible partitioning of the unit,” (p. 131), allows for algebraic notions of equivalence to emerge, naturally focuses on the order in magnitude of the fractions, and compares fractions by length. Lamon (2001) defines the measurement construct understanding as “3/4 means a distance of 3 (1/4-units) from 0 on the number line or 3 (1/4 units) of a given area” (p. 163).

Studies support the hypothesis that accuracy in number line tasks predicts future mathematics achievement (Bailey et al., 2012; Booth & Newton, 2012; Siegler et al., 2011; Torbeyns et al.,

**Task**

a. Mark and label the points $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, and $\frac{4}{4}$ on the number line. Be as exact as possible.

![Number line task 1](image1)

b. Mark and label the point $\frac{2}{3}$ on the number line. Be as exact as possible.

![Number line task 2](image2)

c. Mark and label the points $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ on the number line. Be as exact as possible.

![Number line task 3](image3)

*Figure 12. Example of a number line task (Illustrative Mathematics, Number and Operations – Fractions)*
Siegler et al. (2011) stated, “…just as understanding of magnitudes is central to understanding whole numbers, it is equally central to understanding fractions” (p. 13). Booth and Newton (2012) suggested that it is knowledge of fraction magnitudes, or measures of fractions that relates to skills in early algebra because fraction magnitude represents a more fundamental understanding of the fraction itself. One study (Siegler et al., 2011) followed students’ achievement from fourth grade to eighth grade indicated a strong relationship between fraction arithmetic ability and measures of magnitude knowledge.

Fuchs et al. (2014) investigated the effect of fraction instruction intervention by creating three treatments for at-risk children (those that display attention, motivation and self-regulation difficulties) and those with limited cognitive resources. Cognitive resources are defined as “proper working memory, attentive behavior, and adequate processing speed” (Fuchs et al., 2014, p. 501). The control group used the district adopted text, which relied “mainly on part-whole understanding” (p. 503). The two intervention conditions worked with the measurement model of fractions and task-oriented behavior was promoted with a reward system. The discrepancy between the intervention conditions existed in the last five minutes of each session. One group received fluency practice in which they were expected to quickly generate answers to problems. The second intervention condition included five minutes of conceptual practice where students used models to build fractional amounts and explained their thinking to their classmates. Results showed that both measurement interventions outperformed the control group on fraction assessment items. In addition, the measurement interventions also showed an increase in fraction calculations. Overall, both interventions resulted in “superior fraction learning” for at-risk fourth graders (Fuchs et al., 2014).
The measurement construct may also be necessary for developing conceptual understanding of the additive operations on fractions (Kieren, 1976; Lamon, 2000; (Pantziara & Philippou, 2012). The measurement interpretation is less intuitive than part-whole understanding and only students who are particularly good at mathematics develop magnitude knowledge without explicit instruction of strategies to accurately place the fractions on the number line (Fuchs et al. 2014; Torbeyns et al., 2015). In addition, the measurement construct can include more than number lines; it can be expanded into tasks that include work with measuring cups and rulers. It basically involves tasks where students have to compare and order fractions based on their magnitude. Measurement understanding is an important part of supporting students’ conceptual understanding of fractions.

**Conclusions on Difficulties with Fraction Understanding**

Studies investigating fraction mastery in elementary school show that students have difficulties with rational numbers for a variety of reasons. Students must be able to integrate and understand the properties of fractions parallel to the properties of whole numbers. De Wolf and Vosniadou’s (2015) Framework Theory and Conceptual Change illustrated the importance of allowing children time and opportunities to reconstruct naïve theories of number to include fractions. Instruction, therefore, must lead to shifts in understanding, or what Stafylidou and Vosniadou (2004) referred to as a radical reorganization of what is known. Several studies reported that lack of fraction knowledge limits ability to do algebra in later grades and impacts mathematical achievement overall (e.g., Siegler et al., 2012).

The current “business as usual” approach to teaching fractions (Fuchs et al., 2014) does not appear to be alleviating this issue. To support student learning of fractions, teachers must be aware of the misconceptions children have about fractions, and the difficulty children face in
reorganizing their understanding of number. To generate conceptual fraction understanding, children need time and experiences to make the transition, thereby adjusting accurately their original naïve theories about number (DeWolf & Vosniadou, 2015; Posner et al., 1982; Sfard, 2001).

Allowing children time during instruction to voice their fraction conceptions and misconceptions in a whole class discussion might allow students to struggle with conceptual change as a whole class (Fuchs et al., 2014; Kosko, 2012; Mueller et al., 2014). Therefore, research on whole class discussion will be reviewed to address (1) the definition of whole class discussion or whole class mathematical discourse, and (2) how it can be used to facilitate student sense-making of difficult mathematical concepts.
Whole Class Mathematical Discourse

What is mathematical discourse?

Mathematical discourse in the elementary classroom is the exchange of mathematical ideas and information, in a learning environment, using either formal or informal mathematical language (McCrone, 2005). Communication about mathematics that brings student thinking to the center of the discussion and allows children to consider the ideas of their peers creates a mathematical discourse. Mathematical discourse, therefore, requires children to describe, explain, defend, and justify their ideas about mathematics (Kosko, 2012). Whole class discussion can involve students justifying their own mathematical reasoning and critiquing that of their peers (McCrone, 2005; Mueller et al., 2014; NCTM, 2000). The true definition and function of mathematical discourse in the classroom is an ongoing discussion of the mathematical education community (e.g. Cobb, Yackel, & Wood, 1992; Caspi & Sfard, 2012; CCSSM, 2010; Drageset, 2015; Kosko, 2012; Mueller et al., 2014; NCTM, 1989, 2000, 2007; Sfard, 2001, 2008, 2012; Wood, Williams, & McNeal, 2006; Wood & Kalinec, 2012).

Several scholars believe that mathematics itself can be seen as a discourse, or a specific type of communication, and that by practicing and emulating the accepted discourse of the mathematical community; mathematics is learned (Greeno, 1991; Pimm, 1987; Sfard, 2008, 2012). From this perspective, both talking and thinking are considered examples of communication with others and with self (Kieran, 2001). While mathematical discourse occurs in several elementary settings (i.e. with self, student to student, small groups, student and teacher, whole class), this review will focus on whole class mathematical discourse.

Mathematical discourse has gained notice because it may be a means to provide students a bridge between their independent problem solving and potential problem solving level with
opportunities to discuss and challenge mathematical thinking in the classroom environment.
When children are on the precipice of a deep mathematical understanding, discussing the idea with peers and under adult guidance is an essential step in progressing mathematical development (e.g., Barron, 2003; Brown, 2014; Erickson, 1996; Greeno, 1991; Kazemi & Stipek, 2008; Koichu et al., 2007; Mueller et al., 2014; Piccolo et al., 2008; Reid & Mgombelo, 2014). Because of the potential for a deeper mathematical understanding available to students through mathematical discussion, leadership organizations such as the National Council of Teaching Mathematics (NCTM) and the Common Core State Standards for Mathematics (CCSSM) include this type of communication as an essential underpinning to quality teaching. NCTM (2000) clearly explains the reason for increased discourse in the classroom:

To organize and consolidate their mathematical thinking through communication; to communicate their mathematical thinking coherently and clearly to peers, teachers, and others, to analyze and evaluate the mathematical thinking and strategies of others, and to use the language of mathematics to express mathematical ideas precisely. Students need opportunities to test their ideas on the basis of shared knowledge in the mathematical community of the classroom to see whether they can be understood and if they are sufficiently convincing. When such ideas are worked out in public, students can profit from being part of the discussion. (p. 60)

The CCSSM were developed in 2010 and are currently adopted by forty-five states as a common guide for K – 12 mathematics learning (NGA, 2010; ASCD, 2015). CCSSM has developed a list of Mathematical Practices that reflect procedures for inquiry and validation of mathematical knowledge in the classroom. Mathematical Practices are ways of engaging with mathematics that should be expected and practiced by all students. The following Mathematical Practices indicate the necessity for mathematical discourse in the classroom (p. 6):

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others
Mathematical discourse in the elementary classroom has a positive impact on mathematical achievement (Kazemi & Stipek, 2008; Koichu et al., 2007; Kosko & Miyazaki, 2012; Mercer & Sams, 2006). A mathematical discourse of this nature is not a classroom dialogue that is funneled by the teacher to a predetermined procedure or answer; rather it is a “genuine mathematical dialogue between the students” initiated and guided by the teacher (Cobb et al., 1993, p. 93). During the discourse, the student’s original idea or belief may be transformed by the social argumentation and debate (Kieran, 2001). For example, a student’s naïve framework of fraction magnitude might be adjusted when they receive important feedback from peers (Shotter & Billig, 1998). Sfard (2007) added that without other people’s input and critique, children may not be inspired to change their minds about mathematical ideas. Sfard (2007) argued that students need to engage in mathematical discourse to be aware of new possibilities and arrive at a new understanding. Mottier-Lopez and Allal (2007) noted that mathematical discourse can be dynamic and cumulative learning, and that it allowed students to change their minds based on collectively discussing a problem. Sfard (2007) claimed, “we often need a change in how we talk before we can experience a change in what we see” (p. 575). Without challenges to our thinking, we can falsely believe that what we are thinking is correct (Posner et al., 1982). As Schultz (2011) observed:

What is true of our collective human pursuits is also true of our individual lives. All of us outgrow some of our beliefs. All of us hatch theories in one moment only to find that we must abandon them in the next. Our tricky sense, our limited intellects, our fickle memories, the veil of emotions, the tug of allegiances, the complexity of the world around us; all of this conspires to ensure that we get things wrong again and again (p. 7).
Mathematical Discourse verses Traditional Classroom Discussion

Mathematical discourse can be confused with the typical and prevalent initiation-response-feedback (IRF) or initiation-response-follow-up (IRF) pattern (Sinclair & Coulthard, 1975) found in many classrooms. In this pattern, the teacher makes an initiation move, a learner responds, the teacher provides feedback or evaluates, and then moves on to a new initiation. Often, the IRF pattern can evolve into an extended sequence, where the evaluation of the statement appears to be missing. Brodie (2011) calls this an extended sequence of initiation-response pairs. In this communication, the teacher uses repeated initiation until the class gives the response s/he is seeking. At this point, the teacher positively evaluates the response. It is this prolonged sequence of questioning between teacher and students that can be mistaken for true mathematical discourse because the teacher continues to hold the mathematical authority (DeJarnette & González, 2015).

In 2000, Brendefur and Frykholm identified four levels of communication which aid in illustrating the difference between reform-oriented mathematical discourse and tradition classroom discussion (see Table 2).
Table 2.

Four Levels of Communication

<table>
<thead>
<tr>
<th>Level of Communication</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uni-directional Communication</td>
<td>Most often teacher dominated discussion that tends towards lectures, closed questions (IRF), and allows few opportunities for students to communicate ideas, strategies or thinking.</td>
</tr>
<tr>
<td>Contributive Communication</td>
<td>Conversations among students and with teacher limited to assistance or sharing, with litter or no deep thought. Teachers may provide opportunities for students to discuss tasks, present solution strategies, or assist each other.</td>
</tr>
<tr>
<td>Reflective Communication</td>
<td>Students share ideas, strategies and solutions with peers and teachers. Teachers and students use mathematical discourse as “springboards for deeper investigations”.</td>
</tr>
<tr>
<td>Instructive Communication</td>
<td>Involves modification of students’ mathematical understanding. Teachers use student discourse to deepen their understanding of the students’ mathematical thinking, strengths, and weaknesses. In turn, teachers adjust instruction accordingly.</td>
</tr>
</tbody>
</table>

Note: Four Levels of Communication. Adapted from Brendefur & Frykholm (2000, pp. 126–128.)

The main goal of discourse in a traditional mathematics classroom is to transmit “facts about the discipline” and demonstrate for students “those techniques that are elegant and efficient” as determined by text books or the teacher (Manouchehri & St. John, 2006). This may also be considered a representational approach to teaching mathematics where students are given perfect mathematical representations in order to practice and, hopefully, emulate these representations (Cobb, Yackel, & Wood, 1992). Scenario 1 seeks to illustrate this type of discussion. Note that while there is communication between teacher and student, the teacher controls the content of the response.
Scenario 1:

*Considering the problem of $\frac{1}{2} + \frac{4}{5}$:*

*Teacher:* Does anyone notice something about the denominators in this problem?
*Student 1:* Well, there is a 2 and a 5.
*Teacher:* What do we do when the denominators are different?
*Student 2:* We need to find the number thing, the common…
*Teacher:* Does anyone remember what it is called?
*Student 3:* Umm, common… multiples?
*Teacher:* Not quite. Who recalls what we do to the denominators in order to make them the same? What are we trying to find?
*Student 1:* I know, the common denominator?
*Teacher:* Yes, exactly. Now, what can we do to create common denominators? Does anyone have an idea for the first step?

This style of engaging children has a function in the elementary classroom as it can be an efficient way to move a lesson along or correct problems together as a class, for example, homework papers (Waring, 2009). Researchers have found that using the IRF approach to discussion can be problematic for learners as teachers may regard a single student’s answer as a representation of the knowledge of the whole class and will often continue the lesson without exploring individual student’s thinking and learning (Amiripour, Amir-Mofidi, & Shahvarani 2012; Drageset, 2015, Gal, Lin, & Ying, 2009). Teachers often do not acknowledge the silent participants of the class and this can lead to lesson plans that do not accommodate the “full spectrum” of learners (Gal et al., 2009, p. 424). As Cobb (1988) warned, “teachers can never know with absolute certainty what goes on inside each students’ head” (p. 92). Therefore, caution must be taken when inferring students’ understanding and teachers must be aware of the possible errors in the inferences.

Envisions of mathematical reform through mathematical discourse have an entirely different instructional goal (Bray, 2011; NCTM 1989, 2000, 2007; CCSSM, 2011). The purpose of the discourse should be for reflexive and instructive purposes (see Table 2) and should focus
on important mathematical ideas, and the structure should encourage the “fostering of mathematical argumentation as opposed to merely the delivery of facts and procedures” (Drageset, 2015, p. 258).

Another desired outcome of mathematical discourse is having students engage in “communicating to learn mathematics” with opportunities to listen and participate in activities emphasizing mathematical reasoning rather than just memorizing and practicing procedures (Piccolo et al., 2008, p. 404). A contrasting scenario is provided to illustrate this shift in discussion goals. In scenario 2, the teacher is supporting students in “sharing mathematical authority” (DeJarnette & González, 2015).

**Scenario 2:**

*Considering the problem of* $\frac{1}{2} + \frac{4}{5}$:

*Teacher:* What do you wonder about this problem?  
*Student 1:* Well, there is a 2 and a 5 on the bottom, so...  
*Teacher:* Yes, what else?  
*Student 2:* We have to add the fractions. I wonder how we can do it with the denominator numbers different.  
*Teacher:* Hm. I wonder too. Talk to your partner about your ideas for this problem. (Students discuss with each other.)  
*Teacher:* Who would like to share an idea with the class?  
*Student 3:* We were thinking we could just add the tops and the bottoms together.  
*Teacher:* Any other thoughts?  
*Student 4:* I’m not sure that would work. It doesn’t seem to make sense, it would be $\frac{5}{10}$?  
*Student 2:* That would be $\frac{1}{2}$, and we’re adding something to $\frac{1}{2}$. It seems like it should be more.  
*Student 1:* Yeah, that makes sense. It should be more than $\frac{1}{2}$ if we add more to it.  
*Student 5:* Hey, didn’t we do something like this before, don’t we have to change something to make them match?  
*Teacher:* Does anyone have a suggestion?  
*Student 1:* I think we need to do something like find a number that both 2 and 5 will divide into... like 10?
In the example above, students immediately identify that something is different about this fraction addition problem, namely that the denominators differ. They struggle with the notion that something feels not quite right when a class member recommends to simply add the tops and bottoms (numerators and denominators) to find the answer. Student 2 uses mathematical reasoning to spur the class to a different conclusion. Student 1 analyzes and supports this reasoning. The students realize something must be done, and grapple with the concept of “changing something”. In both scenario 1 and scenario 2 above, students grapple with the idea that they need to change something about the bottom numbers.

Manouchehri and St. John (2006) succinctly highlighted the difference in the structure, content, purpose and product of traditional mathematics discussion and mathematical discourse. Structure refers to the flow of the discourse and patterns it may follow. Content denotes what the dialogue will consist, which is often considered the objective of the lesson. The purpose of the dialogue speaks to the driving intention for the discussion itself; and the product is what might be gained by both student and teacher from participation in the discourse. A summary of this work helps to illuminate the vision of reform-oriented documents such as NCTM Standards (2000) and the CCSSM (2010) when they speak of communication in the classroom. Traditional discussion in the classroom relies on teacher authority for the structure, content, purpose, and product (see figure 13).
In comparison, mathematical discourse requires different participation and actions from both the student and the teacher. The teacher is expected to share the mathematical authority with the students while providing mathematical knowledge to help synthesize the discussion topics (DeJarnette & González, 2015; Towers et al., 2015. The structure, content, purpose and product take on different meanings in mathematical discourse (see figure 14).
The teacher envisioned in figure 14 is not abandoning the class to ruminate or invent mathematical ideas which they do not know (Kosko et al., 2014). It is important to note that the teacher is a part of the discourse and also has the responsibility of summarizing, synthesizing and adding to their mathematical knowledge. It is clear that typical mathematical discussion varies widely from mathematical discourse in structure, content, purpose and product.


*Figure 14. Summary of Mathematical Discourse Characteristics. Adapted from Manouchehri & St. John (2006, p. 546)*
It can allow students to check their beliefs, those that are true and those that are in error, against a backdrop of the mathematical knowledge held by their peers and teachers, and adjust their mathematical understanding accordingly. As Brown (2014) reflected, “within the group we progress with the strength of everyone, seeing more than we could on our own” (p. 192).

**Evidence of Mathematical Reasoning in Mathematical Discourse**

NCTM’s Process Standards (NCTM, 2000) focuses attention on the importance of developing mathematical reasoning in the elementary classroom with the following statement:

People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts (p. 56).

When students share ideas and observations about mathematics, teachers and researchers find evidence of mathematical reasoning during mathematical discourse (e.g. Caspi & Sfard, 2012; Martin & Towers, 2011; McCrone, 2005; Pimm, 1987; Wood et al., 1991; Wood & Kalinec, 2012). For example, McCrone (2005) closely followed a teacher’s year-long effort to create an environment for discourse that included justification, critique and argument. She found that over the course of time, students became more sophisticated with their reasoning and willingness to make public conjectures and effectively critique their peers’ ideas. Initially, the students offered mathematical evidence (fact stating) or procedural steps for solving the problems. Students were able to grow in “communicative competence” throughout the year and by February were offering conjectures, arguments and generalizations (McCrone, 2005, p. 132).

Mercer (2008) conducted an intervention study in which students received the treatment of working in a classroom where the teacher gave students opportunities and encouragement to question, state points of view, and comment on ideas and issues that arise while responding to
literature. Mercer (2008) hypothesized that using language as a tool for collective reasoning could shape individual reasoning through the “guided, structured experience of reasoned argument, children might become better at arguing and reasoning alone” (p. 96). Results indicated that children in the target classes made significantly greater gains on independent literary analysis, math, and science. These results strongly suggested that the children who learned “how to better use talk for thinking collectively, also improved their individual reasoning capacities” (p. 98).

In both the McCrone (2005) study and the Mercer (2008) study, students were carefully initiated and inculcated to a culture of discourse that led to greater reasoning capacity. Several studies echo the finding that while children may be able to talk about mathematical objects, such as a square or a line graph, participating in an effective mathematical discourse requires practice for both the student and the teacher (Barron, 2003; Drageset, 2015; Heyd-Metzuyanim & Sfard, 2012; Kazemi & Stipek, 2008; McCrone, 2005; Puckett, Hansen, & Shea, 2011; Wood & Kalinec, 2012; Cobb & Yackel, 1996). This research highlights a key factor for encouraging mathematical reasoning during a mathematical discourse in the classroom. Namely, it is a procedure that must be consciously and carefully done (Bray, 2011; Cobb & Yackel, 1996; Kazemi & Stipek, 2008; Kosko, 2012; Mercer & Sams, 2006; Mottier-Lopez & Allal, 2007; Mueller et al., 2014; Piccolo et al., 2008; Wood et al., 2006). More pointedly, several scholars have emphasized that student talk is not enough to facilitate mathematical reasoning; its effectiveness depends on both the content and the structure of the discourse (Bray, 2011; Martin & Towers, 2011; Nachlieli & Tabach, 2012; NCTM, 2000, 2007; Piccolo et al., 2008; Wood et al., 2006; Wood & Kalinec, 2012). Mueller et al. (2014) suggested that in order to create a
community in a classroom that develops and expects mathematical reasoning, the following conditions need to be in place:

1. Thoughtful teacher interventions
2. Posing of strategic questions
3. Development of a community that supports reasoning and the co-construction of ideas
4. The establishment of sociomathematical norms. (p. 2)

Studies on mathematical discourse in the classroom reflected these same conditions (e.g., Bray, 2011; Cobb & Yackel, 1996; Cobb et al., 1992; Kazemi & Stipek, 2008; Mercer & Sams, 2006; Nachlieli & Tabach, 2012; Towers et al., 2015). Therefore, each condition will be discussed due to the significant contributions it made towards strengthening the potential for mathematical reasoning during mathematical discourse.

**Thoughtful teacher intervention.** In a seminal work by Cobb, Wood, and Yackel (1993), a year-long teaching experiment was conducted in a second grade classroom. This research seemed to be related to Cobb’s (1988) earlier assertion of the two forms of classroom communication: transmission and constructivist. This particular study focused on the potential benefits of the constructivist approach applied within the classroom. Researchers and the teacher of the class met each week to discuss the emergence of mathematical discourse, classroom culture and reasoning among the students. The researchers concurred that the most radical dissention from traditional teaching methods was that the teacher in the classroom did not require the students to meet a set objective for her mathematical discourse. Instead, she guided genuine mathematical dialogues between the students. The students said how they actually interpreted and solved tasks and this influenced the course of the discussion. The teacher did not attempt to funnel the students to standard algorithm or computational procedures; instead she framed the children’s interpretations and understandings as actual topics of the discussions. She was able to
reformulate their explanations and justifications into terms that reflected the mathematically accepted discourses of society, and yet, were “still accepted by the children as descriptions of what they had done” (p. 93).

The year following this research, 17 more teachers joined the study and their students’ conceptual understanding of arithmetic was “significantly superior” (approximately one standard deviation) to students not in the study (p. 94). The project students were more “likely to believe that one succeeds in mathematics by attempting to make sense of things” and less likely to believe that mathematical success comes from “conforming to their teacher’s or other children’s solution methods, from attempting to be superior to classmates, or from being lucky, neat, or quiet” (p. 100).

Studies in the continuing years echoed the importance of thoughtful teacher interventions, and several repeated the approach and using a case study in a classroom where the teacher had a talent for executing discourse (e.g. Brown, 2014; Kazemi & Stipek, 2008; Hufferd-Ackles, Fuson, & Sherin, 2004; Sfard & Kieran, 2001; Sfard, 2007; Towers et al., 2015). In some cases, scholars used comparisons between teachers who could manage effective discourse and those who could not to further define how teacher interventions sustained a rich mathematical discourse (e.g. Bray, 2011; Manouchehri & St. John, 2006).

Treatment of error. Bray (2011) found that three out of the four classroom teachers she followed were not able to lead robust mathematical discussions. Treatment of student error by the teachers who could not run a mathematical discourse was significantly different than the one who could. Unsuccessful teacher treatment of errors included (p. 27):

- Ignoring student mistakes in order to avoid student embarrassment
- Maintaining tight control over the classroom discussion
- Did not anticipate student errors or possible student created solutions
- Did not identify key mathematical ideas when trying to address flawed responses
These teachers also shared that they believed that including flawed solutions in class discussions would confuse low-achieving math students. In contrast, the observed behaviors and beliefs of the teacher who was able to lead effective classroom discourse considered student error as valuable. This teacher employed the following thoughtful interventions (p. 30):

- Convinced students that errors were a natural part of learning and that there was no shame in making errors as long as one learned from them.
- Made examination of student errors an essential part of class discussion because it helped students build robust mathematical knowledge from their own and their classmate's errors.

**Following a discourse routine.** Another thoughtful intervention used by teachers is the creation of a routine for the discourse and overtly teaching expectations for the way students justify, analyze and critique their peers (Kosko et al. 2014; Mercer & Sams, 2006; Mottier-Lopez & Allal, 2007; Nachlieli & Tabach, 2012; Wood & Kalinec, 2012). A mathematical discourse requires a teacher to allow, expect and ensure that all students may safely participate, and at the same time expect that students take action by listening to others and changing their own ideas (Manouchehri & St. John, 2006).

Without clear guidance, discourse can degenerate into struggles of control, failure to understand one another, repeated attempts to explain, and rejections of the explanations (Barron, 2003). Barron (2003) observed that getting and keeping the floor in front of the class can be problematic. If a student begins their turn to share with hesitation, shrugging of the shoulders or an inaudible voice, this is what Erickson (1996) called a “damaged turn” (p. 36). Damaged turns can be a signal to those who wish to talk a chance to take away the turn, or as Erickson called them, “conversational turn sharks” (p. 38). Conversely, the turn shark could be rescuing the speaker who is unsure, providing scaffolding by voicing and revoicing an idea (p. 38). Erickson
(1996) realized that, without teacher intervention, students become masters at detecting when a turn is easily taken away and will do so, like sharks “on the lookout for blood in the water” (p. 38).

Another interesting phenomenon emerged from Heyd-Metzuyanim and Sfard’s (2012) work with identity struggles in the mathematics classroom. During classroom observations, they noted that one student in particular, Ziv, asserted his “general advantage as a person” by using sounds and gestures such as signing sarcastically when someone admitted a difficulty or rolling his eyes at his classmates’ failure to understand him (p. 141). Ziv could be considered a negative turn shark, and the teacher compounded matters by continually asking students to pay attention to Ziv, as she believed he was the only one who understood the problem. Heyd-Metzuyanim and Sfard concluded that teacher’s refrain from taking leadership created a “harmful void” in which students “need to fill but may not know how to do so on their own” (p. 144). Or as Kosko et al. (2014) observed, teachers may view “facilitating mathematical argumentation as equivalent to yielding control of such discussions entirely to the students” (p. 470). Lack of intervention by teachers may inadvertently harm the process of learning as well as students’ self-identity as an effective mathematician and valued member of the discourse (Ziv included), counterproductive to the intent and benefits of mathematical discourse.

A well-established routine can be an effective survival technique for both students and teachers immersed in mathematical discourse. It gives a teacher a reliable pattern to follow, rather than relying on teacher-identified students to get the class through the discourse. A routine can support both those students who are competent and those who are learning because of the predictive quality of the discourse structure (Mottier-Lopez & Allal, 2007). Within the routine, examples emerge of how to reason mathematically for those who are unclear or unsure
of how to participate. While their first attempts may be a thoughtful imitation of others, the hope is to inculcate all members of the class towards sophisticated participation (Erickson, 1996; Nachlieli & Tabach, 2012; Sfard, 2008; Wood & Kalinec, 2012).

A basic structure of the routine for sharing mathematical ideas can be based on Toulmin, Reike and Janik’s (1979) framework for practical reasoning (Cazden, 1995; Stephan et al., 2001). A student makes a claim, and states the position being argued. Data, the supporting facts, are given. Finally, a warrant, the assumption or generalization that connects the data to the claim is given. Within this framework, students are obliged to explain and justify reasoning, attempt to understand others’ explanations, ask clarifying questions for greater understanding, and indicate when they find a solution invalid and explain the reasons (Stephan et al., 2001). Mathematical tasks that have varied solution strategies appear to be important for initiating a deep discussion and seemed to encourage explanation, justification and generalization (Brown, 2014; Cobb et al., 1995; NCTM, 2000).

Orchestrating a productive mathematical discourse can be difficult for teachers and requires a specific approach and underlying belief system (Bishop, 2012; Bray, 2011; Cobb, 1986; Cobb et al., 1993; Drageset, 2015; Hufferd et al., 2004; Kazemi & Stipek, 2008; Manouchehri & St. John, 2006; Mercer & Sams, 2006; Stephan et al., 2001). In 1987, Pimm referred to mathematical discourse as a topic continuously advocated by researchers but rarely implemented by teachers. In 1988, Cobb asserted that leading mathematical discourse requires far more from the teacher than traditional classroom discussion. According to Cobb, a teacher needs a “deep relational understanding” of mathematics, must continually look for indications that students might have constructed unanticipated alternative meanings during the discourse shared, and undergo “a conceptual revolution of his or her own” (p. 100). Twenty-three years
later, Martin and Towers (2011) generated the following teacher characteristics for leading effective mathematical discourse:

- Willing to share control
- Allow events to unfold, yet has a responsibility to act at critical moments
- Sophisticated improvisational competence- flexibility in the moment rather than deliberate funneling of ideas (IRF approach to discussion)
- Listen to and connect with the improvisational actions of students
- Possess a sophisticated capacity to step back until the collective action requires the teacher to share mathematical knowledge or synthesize ideas to further the discourse

Teacher interventions supported by certain ideology seem to be essential for leading a mathematical discourse in ways that create reasoning opportunities for students.

**Posing of strategic questions.** By posing strategic questions the teacher can discover if students are ready for the next discursive layer, strengthening the opportunity for cognitive shifts and mathematical reasoning (Cobb et al., 1992; Kosko et al., 2014). Strategic questions may aid a student in cognitively rehearsing their mathematical reasoning while at the same time being nurtured by the teacher’s mathematical expertise. Hufferd-Ackles, Fuson and Sherin (2004) identified three levels of significant questioning shifts that occurred in classroom observations. The teacher in the classroom was able to move from a strategy sharing dynamic to an inquiry/argument culture (Wood et al., 2006). The increasingly strategic questioning strategies seemed to allow this movement. Dialogue from the classroom is shared here to illustrate the significant changes in the classroom discourse and questioning approach used by the teacher (pp. 93 – 95).
Level I Questioning: Teacher pursues student thinking.

Ms. Martinez: Now, who can tell me how many boxes of cereal I have in this container? (She points to the three-by-three array she has drawn on the board.)  
How many boxes of cereal do I have in this container, Carl?  
Carl: Nine.  
Ms. Martinez: Nine. How did you figure that out, Carl?  
Carl: Because I counted them. I counted them by 3s.  
Ms. Martinez: You counted them. You counted by 3s. Can you come up and show us? (The teacher is assisting the student to give a fuller explanation.)  
(Carl goes to the board to illustrate by pointing to the drawing.)  
Carl: I counted by 3s. There is 3 right here (row I of boxes). Right there (row 2). And there's 3 right here (row 3).  
Ms. Martinez: So, it is like you are saying $3 + 3 + 3$. What is another way we can count? Does anyone have another way we can count? Jimmy?  
Jimmy: Um, go like this. Go like this, 3, 6, 9.

Level 2 Questioning: Students begin to question.  

(In the original, portions of the discussion were in Spanish. It has been removed for ease of understanding, but is significant that the discourse included the children’s’ first language.)

(Liz has written this labeled equation)

<table>
<thead>
<tr>
<th>A</th>
<th>d</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \times 2 = 5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Ms. Martinez: Okay, Santos?  
Santos: I wonder why she put the 5 in there.  
Ms. Martinez: Can you ask your question to Liz? (Teacher assists student-to-student talk.)  
Santos: (To Liz.) Why did you put the 5 in there?  
Liz: Because it says, "How many are there all together?"  
Saul: How come there is a "d" under the 3?  
Ms. Martinez: Can you repeat the question to Liz? (Teacher assists student-to-student talk.)  
Saul: (To Liz.) How come there is a "d" under the 3?
Liz: Because it is for the dolls.
Helena: Is it a plus?
(Liz nods in agreement.)
Helena: Why did you put the three and the two together?
Liz: Because here they go together. (Note, "J" in Liz's work stands for "all together"
Angel: (To Liz.) Why did you put the 2?
Liz: For double.

Level 3 Questioning: Students initiate the questioning.

(Ms. Martinez is in the rear of the classroom, Jamie is stationed at the blackboard. He has been called on by Ms. Martinez to share his comments about whether or not it is the same to add columns of numbers left to right or right to left with the class.)

Jamie: No, because if you're taking away any numbers you gotta take away from the other ones. Are you gonna start from the right?
Santos: What do you mean?
Jamie: Right when you're taking away, yeah, subtraction, sometimes you gotta take away from the other numbers.
Maria: Sometimes you can start from the right or the left. Jamie: How? Are you going to take one from the left, I mean from the right?
Maria: Sometimes it helps to write, like, when it's subtraction, from the right or sometimes from the left.
Roberto: Either way, none of the numbers are going to change. Just do the same thing you're gonna do from left to right, subtract the same thing you're gonna do from right to left.
Jamie: Yeah, but that's not gonna be the same answer.
Roberto: If you start from right to left, you're gonna subtract something and you can subtract the same thing if you go from left to right.
Angel: And when you go from left to right, it's gonna be the same answer.
Ms. Martinez: Are you still not convinced, Jamie?

In these examples, Ms. Martinez is increasingly able to share control of the mathematical authority with her students. As she moves from Level 1 to Level 3 in her questioning, there is evidence that she is training her students to take on the mathematical authority and responsibility of the mathematical discourse (Cobb et al., 1996; McCrone, 2005; Mercer, 2008). Probing sorts of questions intended to bring to light student thinking and ideas tend to be more beneficial to students (Franke, Webb, Chan, Ing, Freund, & Battey, 2009; Kazemi & Stipek, 2008).
Students benefit from teacher questioning that presses for individual mathematical reasoning rather than a specific answer predetermined by the teacher (Hufferd-Ackles et al., 2004; Lynch & Bolyard, 2012; Mueller et al., 2014; Pimm, 1987). Mueller et al. (2014) analyzed data from afterschool sessions which involved using mathematical discourse as a learning strategy. Findings illuminated three types of moves that teachers made to establish “listening, sharing, and promoting student justification characterized by various forms of reasoning” (p. 13). The types of moves were: (a) those that made students' ideas public, (b) those that brought forth and extended students' ideas, and (c) those that encouraged explanations and justifications (p. 13). Asking strategic questions appears to be an effective and essential component for helping students learn to share and extend ideas as well as provide justification for their mathematical reasoning.

**Development of a reasoning community.** In order to create an environment that supports mathematical reasoning through discourse, teachers must ensure a socially safe situation for all students in order to successfully lead a mathematical discourse (Bishop, 2011; Resnick, O’Connor, & Michaels, 2007). In essence, if a child feels they will be ridiculed for making errors in front of the class or for being too smart in mathematics, if social status trumps mathematical justification, or if the teacher cannot demand that all ideas are valuable and worth considering, the discourse tends to do more harm than good (Bishop, 2012). When threatened, people may defend a position that doesn’t make sense to avoid the perceived pain of a drop in status (Rock, 2008). Therefore, considerations of social interactions in the classroom as well as mathematical reasoning are of importance.

By taking a closer look into the classroom, Wood, Williams, and McNeal (2006) were able to define four types of classroom cultures that developed a reasoning community. They did
this by analyzing 42 lessons in five classrooms of seven and eight year olds. The study investigated the flow of the lessons and the student behaviors observed. Evident in this delineation are the four types of communication noted by Brendefur and Frykholm (2000), yet it was developed from multiple elementary classroom observations and may allow deeper insight regarding actual classroom interactions. The four types of cultures are summarized in Table 3. The types of communication from Brendefur and Frykholm (2000) have been added to illustrate the similarities. Only in the Strategy Reporting Culture and the Inquiry/Argument Culture does argument and inquiry occur (percentages of time occurring per 100 minute classroom observation provided). As Wood and colleagues (2006) found, “reform-oriented classes fell into two major types, strategy reporting and inquiry/argument classrooms” (p. 225, italics in original). Therefore, this study may indicate that in order to create mathematical discourse as envisioned by NCTM and the CCSSM, the type of communication and culture in an elementary classroom is an essential factor.
Table 3.

**Classroom Cultures**

<table>
<thead>
<tr>
<th>Classroom Culture</th>
<th>Interaction Patterns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional Textbook Culture</td>
<td>Teacher test questions (closed-ended)</td>
</tr>
<tr>
<td><strong>Uni-directional Communication</strong></td>
<td>Student response</td>
</tr>
<tr>
<td></td>
<td>Teacher Evaluation</td>
</tr>
<tr>
<td></td>
<td>Teacher leads students to correct answer through series of questions</td>
</tr>
<tr>
<td></td>
<td>Students give known answers or predetermined information</td>
</tr>
<tr>
<td>Conventional Problem Solving Culture</td>
<td>Teacher dominated participation</td>
</tr>
<tr>
<td></td>
<td>Open-ended questions</td>
</tr>
<tr>
<td></td>
<td>Teacher hinted at solution method in ways that removed mathematical challenge</td>
</tr>
<tr>
<td></td>
<td>Students gave expected information</td>
</tr>
<tr>
<td></td>
<td>Exploring methods existed</td>
</tr>
<tr>
<td><strong>Contributive Communication</strong></td>
<td>Students tell others their strategy for solving problem (allowed for more opportunities to participate in discourse)</td>
</tr>
<tr>
<td>Strategy Reporting Culture</td>
<td>Opportunities to explore methods increased</td>
</tr>
<tr>
<td></td>
<td>Students explain solutions (one solution or many per problem)</td>
</tr>
<tr>
<td></td>
<td>Teacher elaborates and extends a child's solution to convey important ideas</td>
</tr>
<tr>
<td></td>
<td>Argument (9% of time)-teacher and students resolving differences about strategies or answers</td>
</tr>
<tr>
<td><strong>Reflective Communication</strong></td>
<td>Inquiry (5% of the time)- teacher and students questioning to clarify strategies or ideas</td>
</tr>
<tr>
<td>Inquiry/Argument Culture</td>
<td>Exploring methods frequently observed</td>
</tr>
<tr>
<td></td>
<td>Strategy reporting</td>
</tr>
<tr>
<td></td>
<td>Argument (16% of time)</td>
</tr>
<tr>
<td></td>
<td>Inquiry (9% of time)</td>
</tr>
<tr>
<td></td>
<td>Teacher elaboration much LESS</td>
</tr>
<tr>
<td></td>
<td>Children take over questioning and argument, teacher not as involved- HUGE SHIFT in discussion</td>
</tr>
<tr>
<td><strong>Instructive Communication</strong></td>
<td>Students build consensus</td>
</tr>
<tr>
<td></td>
<td>Teacher checks for consensus</td>
</tr>
</tbody>
</table>

*Note: Classroom Cultures adapted from Wood, Williams and McNeal (2006).*

Creating an inquiry/argument culture requires thoughtful teacher intervention and questioning as discussed above. In addition, the teacher must intentionally and ideologically support students in speaking up about their mathematical thinking. Cobb and Yackel (1996) found that when teachers encouraged shared responses that reflected mathematical reasoning,
students constructed “increasingly sophisticated” solutions and became aware of “more conceptually advanced forms of mathematical activity” (p. 465).

Barron (2003) stated that some members in a community don’t naturally display an intent to collaborate and “true communication takes co-regulation: a willingness and openness to be influenced by the other” (p. 337). Barron studied the interactions and effectiveness of small group problem solving in the mathematics classroom. Groups that could not effectively solve the problem suffered from communication that was oriented toward dominating the situation and reflected a need for students to protect their self-identity of strong problem solvers. This type of “turn shark” (Erickson, 1996) behavior resulted in failure of the group to find a correct solution. In fact, based on students’ individual scores on tasks similar to those done in the group, the results found that if students were in a problematic group, they might be better off working alone. However, students in successful groups significantly outperformed their peers working in unsuccessful groups.

DeJarnette and González (2015) found similar positioning among students to be problematic. Students in this study either positioned themselves as the expert, who were frequently deferred to and granted the authority to determine the accuracy of the work, the novice, who deferred to the expert and positioned themselves as less able, or the facilitator, who made sure that tasks were completed and group members participated (Esmonde, 2009). DeJarnette and González (2015) found that when several of the group members positioned themselves as the expert and refused to consider the contributions of others as correct, little movement was made towards the co-construction of ideas to solve a novel task. Small groups who were successful in completing the mathematical task in this study were able to share the role of mathematical authority and engage in the “give and take” of ideas by allowing group members
the opportunity to shift from novice to expert on a moment-to-moment basis (DeJarnette & González, 2015, p. 413). Developing a reasoning community in the classroom requires the teacher to regulate the potential damage caused by expert positioning without the intention of openly accepting others’ ideas, or the dangers of self-assigning the role of novice. Both of these positions may limit students’ ability to grow in their reasoning capacity (Heyd-Metzuyanim & Sfard, 2012).

**Establishment of sociomathematical norms.** A reasoning community will support students in learning and practicing how to listen to each other, consider differing explanations, argue their point, and justify their thinking. Sociomathematical norms refers to ways in which the reasoning community will approach the mathematics they are investigating. Cobb and Yackel (1996) defined the combination of social and mathematical activity as “sociomathematical norms” (p. 458). This formative work distinguished the unique qualities of mathematical reasoning combined with social interaction within a discourse community and is frequently referred to in the discussion of mathematical discourse (e.g. Kazemi & Stipek, 2008; Levenson, Tirosh, & Tsamir, 2009; Michaels, O’Connor, & Resnik, 2007; Mottier-Lopez & Allal, 2007).

A sociomathematical norm extends a social norm (i.e. students who share solutions should present something different than what was already shared) to a sociomathematical norm- (i.e. students should understand what actually constitutes a mathematically different solution). To illustrate further, a social norm in an inquiry-based classroom might be to treat errors as opportunities to investigate student misunderstandings (Bray, 2011). A sociomathematical norm in this case would be mistakes are opportunities to “re-conceptualize a problem, explore contradictions to a solution approach, and try out alternative strategies” (Kazemi & Stipek, 2008,
Hence, an error in a solution is an entry point for continued discourse that involves argumentation, justification and verification (Kazemi & Stipek, 2008).

Comprehending what counts as “mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant” in a classroom are examples of sociomathematical norms (Cobb & Yackel, 1996, p. 461). In essence, a student is expected to pay attention not only to what is being shared in the discourse, but also to the significance of the mathematical reasoning involved. Towers et al. (2013) referred to sociomathematical norms as a “way of being with the mathematics” and that specific knowledge of mathematics is a “kind of by-product of this other, more significant process” (p. 429).

Kazemi and Stipek (2008) studied fourth and fifth grade whole class mathematical discourse and found that it appeared that there were similar levels of discussion within the classroom and a positive social environment (reasoning community). Their results, however, revealed that one set of teachers was more likely than others to have mathematical reasoning emerge from classroom discourses. The difference appeared to be whether or not the teacher created and sustained sociomathematical norms. One way in which the teachers accomplished this was to create a “high press” interactions in which the teacher required students to examine mathematical similarities and differences among multiple strategies, while in “low press” discourse, strategies were shared in rapid succession without attention to the mathematical aspects of the work (p. 126). Under “high press” conditions, students knew they were responsible for more than stating agreement or disagreement; they strove to verify their statements often using the triangulation of words, visual representations and numerical strategies. Kazemi and Stipek (2008) discovered that the following sociomathematical norms were essential for achieving a deeper level of mathematical reasoning:
• An explanation consists of a mathematical argument, not simply a procedural description or summary
• Mathematical thinking involves understanding relations among multiple strategies
• Errors provide opportunities to re-conceptualize a problem, explore contradictions in solutions, or pursue alternative strategies
• Collaborative work involves individual accountability and reaching consensus through mathematical argumentation.

Sociomathematical norms hold the children and teacher responsible for focusing on important mathematical ideas and mathematical argumentation as opposed to “merely the delivery of facts and procedures” (Drageset, 2015, p. 258). These norms help regulate the discourse by requiring mathematical participation, action and a flexibility in understanding from the students. Learning opportunities are influenced as the teacher takes student ideas seriously and supports the mathematics being shared (Cobb & Yackel, 1996; Wood et al., 1991). Keeping and maintaining sociomathematical norms increases the press for learning- students are expected to make use of their own mathematical capabilities as they work with the community towards a deeper understanding (Barron, 2003; Kazemi & Stipek, 2008; Mercer, 2008; Yackel & Cobb, 1996). When combined with thoughtful teacher intervention, the posing of strategic questions, and the development of a reasoning community, students are drawn into a learning space that demonstrates, expects and encourages mathematical reasoning during mathematical discourse.

**Summary on Whole Class Mathematical Discourse**

Mathematical discourse has the potential to allow children the opportunity to learn and reason about mathematics. What is clear from the research is that mathematical discourse does not happen easily or without specific intention.

The lion’s share of this work falls to the teacher, who must establish a reasoning community and enforce the sociomathematical norms to ensure that all students can learn. Mathematical discourse may help clear up mathematical misconceptions, especially those to
which students are blind. Schultz (2011) called this “error blindness” and identified its unique difficulties, “it does feel like something to be wrong, it feels like being right” (p. 8). In order to reform our beliefs, we must first recognize that they are wrong.

Used in conjunction with fraction instruction, mathematical discourse might have the potential to change naïve theories by inducing conceptual change. Mathematical discourse might provide a platform for multiple ideas about fractions to be tested and verified. It might allow children to fully investigate the conceptual domain of mathematics—especially the domain of fractional numbers. During mathematical discourse, students wander into the mathematically obvious, mathematically elegant, mathematically incorrect, mathematically sophisticated and the mathematically efficient. Bishop (2012) stated that the practices and ways in which we communicate in the classroom can teach us more than mathematics, it can be a place that “we learn who we are” (p. 36).
Rational for Research Questions

Understanding and operating with fractions is difficult for most children. Middle school mathematics, such as algebra, hinges upon fraction understanding. If “business as usual” (Fuchs et al., 2014) instruction prevails and students continue to be passed through the grades without making a profound cognitive shift that adds rational numbers into their schemata of number, a host of issues for students, teachers, and the state of mathematics education in the United States ensues (NMAP, 2008). The literature implies that in elementary school, students need time and focused instruction on all five constructs of fractions (Bottge et al., 2014). The measurement construct has been found to be especially effective in helping students understand that fractions have their own magnitude; which aids in alleviating the bias of whole number magnitude (Schneider & Siegler, 2010; Schneider, Grabner, & Paetsch, 2009). In spite of this, text books in the United States rarely use the measurement construct (Alajmi, 2012). Students who are inherently good at mathematics tend to develop magnitude knowledge without explicit instruction of strategies to accurately place the fractions on the number line (Fuchs et al. 2014; Torbeys, et al., 2015). Interestingly, Fuchs et al. (2014) found that an intervention that combined overt instruction on the measurement construct of fractions and allowing for a five-minute student discussion on reasoning for their answers, students in the experimental group who were behind in fraction knowledge were able to catch up to their peers. The results of this study indicate that further research might include allowing students to participate in extended mathematical discourse regarding fraction magnitude to examine the effects it might have. Sfard (2001) argued that mathematics is communication. Will putting communication into the heart of fraction instruction produce opportunities for the profound shift in reasoning that is necessary for changing initial frameworks about number to include fractions?
Additionally, research on mathematical discourse finds that in order for the discourse to be effective, the teacher must possess a mindset that allows for students to share the authority on creating mathematical meaning (DeJarnette & González, 2015). Without creating an environment that supports reasoning and upholding sociomathematical norms, discourse tends to degenerate into social discussion rather than academic discourse (Barron, 2003).

The review of literature revealed that students would benefit from measurement construct instruction (Fuchs et al., 2014; Kieren, 1976; Lamon, 2000). Additionally, reasoning about the density and magnitude properties of fractions might provide the cognitive conflict students need to avoid whole number treatment of fractional numbers (Siegler et al., 2011; Siegler et al., 2012; Smith et al., 2005; Torbeyns et al., 2015). The literature also indicated that fraction understanding is complex and required concentrated instruction on shifting understanding about number (Pantziara & Philippou, 2012; Sfard, 1991, 2001; Siebert & Gaskin, 2006; Stafylidou & Vosniadou, 2004; Vosniadou, 2004). However, literature was not present that described insights regarding the effectiveness of high-press mathematical discourse, coupled with tackling the difficult concepts embedded in the measurement construct of fractions. Therefore, this study will focus on the following questions:

1. How do fourth-grade students reason about fraction magnitude when solving tasks in the classroom teaching experiment?
   - What is the whole-class learning trajectory that emerges?
   - What design decisions are made to modify the fraction magnitude tasks and why?

2. How does mathematical discourse effect shifts in student reasoning on fraction magnitude tasks?
   - What is the role of the mathematical discourse?
   - What design decisions are made to modify the mathematical discourse routine and why?
CHAPTER III: METHODOLOGY

For this study, a classroom teaching experiment was conducted in a fourth-grade classroom. A fraction measurement unit was developed and tested in iterative cycles to study how the learning took place under implementation (Lamberg & Middleton, 2009). Both qualitative and quantitative data were collected, however the majority of the analysis relied on the qualitative data. The study provided insight into how fourth graders (1) developed a conceptual understanding of the measurement construct of fractions, and (2) evolved in their thinking about fraction magnitude during the process of whole class mathematical discourse and learning activities throughout the unit. Conceptual shifts that occurred as a result of mathematical argumentation was analyzed to better understand the meaning that was negotiated as the students interacted with tasks and each other (Rasmussen & Stephan, 2008).

Rationale for Design Research

Design research strives towards a development of an intervention addressing a problem in practice and seeks to inform the work of others through theory building by conducting systematic investigations (Anderson & Shattuck, 2012; McKenney & Reeves, 2013; Steffe & Thompson, 2000; Walker, 2006). It is a complex and multi-faceted endeavor (McKenney & Reeves, 2013). This methodology has been increasingly used in educational research to “deal with messy situations, multiple dependent variables, and develop theories about domain specific learning processes within a social context using flexible design revisions” (Lamberg & Middleton, 2009, p. 233). Design research is often used in educational settings because it addresses the classroom environment, or the learning ecology (Gravemeijer & Cobb, 2006). At the heart of the research is the development of the design, which is often an innovative approach to learning, a unit of study, or perhaps a whole curriculum. Design research reflects the engineering process-where the
goal is to design products and develop systems that solve important problems (Cobb, et al., 2003; Middleton, Gorard, Taylor, & Bannan-Ritland, 2008). Engineers test their initial prototypes in order to ensure that the designed product will solve and serve in the capacity for which it was created. Design research is deeply connected to educational change, as it allows for daily adjustment and micro-experiments, while concurrently seeking broader implications to develop theories of learning that relate to what actually happens in a classroom.

A primary purpose for using design research in the mathematics classroom is that it allows researchers to experience “first-hand students’ mathematical learning and reasoning” (Steffe & Thompson, 2000, p. 267). Steffe and Thompson (2000) asserted that design research was accepted by mathematics educators for the following reasons:

- Methodology seemed intuitively correct.
- Research is seeking to understand children’s mathematical experiences.
- Builds accounts of how students learn specific mathematical concepts.
- Models were needed that accounted for the progress students make as a result of mathematical communication. (p. 268)

Furthermore, it allows the research team to make adjustments to the mathematical ecosystem of the classroom during the teaching experiment (Cobb, et al. 2003). The mathematical ecosystem includes the classroom ecology (Cobb & Gravejemier, 2006), mathematical tasks and activities, and students’ reasoning that might emerge as a result. Design research has two main goals; or what Shoenfield (2009) called the design-theory dualism. On one hand, it is used to develop creative and innovative approaches to learning (designs) to solve human teaching, learning, and performance problems, which reflects the work of a teacher. On the other hand, the researcher constructs a body of theories that can impact future understanding of the best practices in education (Herrington, Reeves, & McKenney, 2011).
Responsively grounded. Design research presupposes that there is an adequately grounded basis for designing the instructional sequence/innovative learning ecology (Gravemeijer & Cobb, 2006). It often takes a pragmatic approach, similar to the mixed methods stance in research methodology. This is characterized by a view that includes (Greene & Hall, 2010, p. 131):

- Knowledge as both constructed and as a function of organism-environment transactions
- A problem solving, action focused inquiry process

In essence, pragmatic inquirers may “select any method based on its appropriateness for the situation at hand” (Green & Hall, 2010, p. 133). Based on the tenet that the methods used will connect documents and processes, the design-based researcher frequently follows new revelations where they lead, tweaking both the intervention and the measurement as the research progresses (Hoadley, 2002). As Brown (1992) described:

Components are rarely isolatable, the whole really is more than the sum of its parts. The learning effects are not even simple interactions, but highly interdependent outcomes of a complex social and cognitive intervention. And this presents a methodological headache for traditional psychology, allergic as it is to multiply confounded experiments (p. 165).

This is not to say that design research is ungrounded and without intensive, valid purpose. The researcher commits to taking the time and dedication required to create and implement the intervention. Everything is documented so that readers of the research can judge for themselves if the study can be replicated, and possibly use the intervention in their own contexts (Anderson & Shattuck, 2012). This may generate initial theories pertaining to larger phenomenological issues or frameworks to allow further analysis in similar situations (Gravemeijer & Cobb, 2006). The end result of design research is the development of responsively grounded theory and the improvement of instructional design by testing innovative learning interventions (Cobb et al., 2003).
The Research Sample

Students from a fourth grade classroom in a rural middle school in a western state participated in this study. The sample included a total of 26 students, ages 9 to 10, who predominately come from low to middle socioeconomic backgrounds. The fourth grade class consists of students from Caucasian (C), Latina (L) and African American (A) backgrounds. There were 15 female (F) and 11 male (M) students (see Table 4.).

<table>
<thead>
<tr>
<th>Student</th>
<th>Ethnicity</th>
<th>Gender</th>
</tr>
</thead>
<tbody>
<tr>
<td>1CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>2CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>3CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>5CF*</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>6CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>7CM</td>
<td>C</td>
<td>M</td>
</tr>
<tr>
<td>8CM</td>
<td>C</td>
<td>M</td>
</tr>
<tr>
<td>9LM</td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>10LM*</td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>12CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>13LM</td>
<td>L</td>
<td>M</td>
</tr>
<tr>
<td>14CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>15CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>16CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>17CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>18CM</td>
<td>C</td>
<td>M</td>
</tr>
<tr>
<td>19CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>20CM</td>
<td>C</td>
<td>M</td>
</tr>
<tr>
<td>21LF</td>
<td>L</td>
<td>F</td>
</tr>
<tr>
<td>22LF</td>
<td>L</td>
<td>F</td>
</tr>
<tr>
<td>23CF</td>
<td>C</td>
<td>F</td>
</tr>
<tr>
<td>24AM</td>
<td>A</td>
<td>M</td>
</tr>
<tr>
<td>25CM</td>
<td>C</td>
<td>M</td>
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<td>26CM</td>
<td>C</td>
<td>M</td>
</tr>
<tr>
<td>27CM</td>
<td>C</td>
<td>M</td>
</tr>
<tr>
<td>28CF</td>
<td>C</td>
<td>F</td>
</tr>
</tbody>
</table>
*Student 11 moved after study began, Student 4 chose not to be in the study.

The research team consisted of the lead researcher, the classroom teacher and a mathematics education professor. The classroom teacher has taught elementary school for several years. She regularly participates in professional development courses, including one taught by the lead researcher on whole class mathematical discourse. In addition, she has been trained in the Common Core State Standards for Mathematical Content and Practices (NGA/CCSSO, 2010).

**Treatment: Rulers, Races and Regions: A Unit on Fraction Magnitude**

This teaching experiment used an interventionist approach to research: to finely tune a fraction measurement unit as a possible solution to difficulties children have in understanding fractions and maximize its robustness by trying it out within the actual learning situation which includes whole class mathematical discourse.

During the teaching experiment, the researcher and the teacher instructed from the fraction magnitude unit developed by the lead researcher. The lead researcher and the teacher co-taught the class during the teaching experiment. The teacher preferred this approach to the instruction. Sociomathematical norms were specifically taught and the class practiced a prescribed routine for whole class mathematical discourse (see Appendix A). The lessons occurred at a regularly scheduled time, Monday, Wednesday and Friday from 9:30 – 10:45, and Tuesday and Thursday from 11:00 to 12:15. The teaching experiment lasted for 6 weeks, as the schedule was adjusted due to special events (i.e. field trips, guest speakers) which were planned during the instruction time.

The developed fraction magnitude unit was guided by principles of Realistic Mathematics Education, or RME (Barnes, 2011; Gravemeijer, 2004). The underpinnings of
RME are based on the concept that instructional materials should be based in contextual situations so that the student can “reinvent conventional mathematics” (Gravemeijer, 2004, p. 107). The problem context does not need to be personally connected with the student, rather a student must believe that the problem could be real (Tobias, 2009). One principle of RME used in the unit is guided reinvention. Guided reinvention presents contextual tasks that children understand; which might allow them to make sense of the mathematics for themselves.

This unit is based on the context of the fictional “Rational Ruler Company” which hired the fourth grade students to use fractions to create unique rulers, plan running and bike races, and plan park spaces. Within the unit, students worked with three models: folding paper rulers, number lines, and regions (see Figure 15 for examples of each model).

Pantziara and Philippou (2012) found in their research that children had difficulty with number line activities, and were often confused about how to proceed with the task. Therefore, the unit was designed to practice or reteach the third grade standards of placing fractions on a number line, extending this standard to placing several fractions with unlike denominators on a number line. Additionally, fourth grade CCSS for Numbers and Operations- Fractions were embedded throughout the lessons.

The concepts of infinity and density of fractions (Smith et al., 2005) were informally addressed by three questions added to the pre/posttest:

D1. What equivalent fractions do you know for ½? Please write some here.

D2. How many equivalent fractions for ½ do you think there are?

D3. How many fraction numbers are there between 0 and 1? Please give an estimate.
These questions were added for further insight into the student’s conceptual understanding of the relationship between the numerator and denominator, as well as splitting and halving schemes (Wilkins & Norton, 2011).

**Rulers:** Typical task includes folding and labeling the value of the folds with fractions

![Diagram of a ruler with fractions marked at 0, 1, and 2]

**Races:** Typical task involves placing snacks and obstacles at fractions of the race (i.e. at 2/3 of the race there will be a water stand.)

![Diagram of a race track]

*Figure 15. Examples of models used during instructional unit*

The original fraction measurement unit was a hypothetical learning trajectory (Simon, 1995) “consisting of conjectures about the collective mathematical development of the classroom community” (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 117). The research team evaluated the interaction of the students with the fraction magnitude tasks, impact on learning, and the classroom ecology of each lesson. In this way, an actual learning trajectory was created based on the authentic account of the actual experience during the teaching experiment. (See Appendix C for final unit).

**Frameworks Supporting Study Design**

Several different frameworks were used to both guide the iterative analysis of the teaching experiment and also to analyze the data from the study.
**Instructional Design Research Cycle**

The study’s theoretical basis relied on a sociocultural approach to viewing mathematical learning. A sociocultural approach defines learning as participation in cultural practices (Sfard, 1998; Sfard & Kieran, 2001; Vygotsky, 1978). Saxe and Bermudez (1996) describe this perspective as viewing children's construction of mathematical understanding as interwoven with the socially organized activities in which they are participants. Consequently, the study incorporated two levels of investigation during the teaching experiment. Cobb, Stephan, McClain, and Gravemeijer (2001) label this dual approach as the “instructional design research cycle” (p. 116, see Figure 16).

![Diagram](image.png)

*Figure 16. The instructional design research cycle as it applies to this study. Adapted from Cobb et al. (2001).*

During this teaching experiment, elements of the learning ecology (Cobb et al., 2001) were addressed and modified to improve not only the fraction measurement unit, but also the development of a reasoning community (Wood, et al., 2006). Cobb and Gravemeijer (2008) reflected that the teacher is a co-designer of the classroom context, and that creates “the immediate social situation of their students’ mathematical development” (Gravemeijer & Cobb, 2006, p. 70). Maintaining focus on the reasoning community resulted in adjusting the discourse
routine, for example, based on what types of argumentation phenomena arise and how this might be adjusted to provide increased instances of mathematical reasoning and conceptual change.

The value in approaching the study in this way is that it allowed for documentation of (1) the development of the mathematical community (Cobb et al., 1992; Wood et al., 2006), and (2) the development of mathematical reasoning, and (3) feedback on instructional design (Cobb et al., 2001, p. 116).

**Interpretive Framework**

In order to account for the “taken-as-shared” ways of reasoning in a mathematics classroom, the research team used an interpretive framework (Cobb et al., 2001, p. 126) to analyze the collective learning of the classroom community. This framework focused on both the social and psychological perspective of the class, and how the combination of individual students created a unique classroom community (see Figure 17).
Cobb and colleagues (2001) considered the social and psychological perspective on mathematical learning to be reflexive. In essence, an individual’s mathematical interpretations and reasoning are located “within an evolving classroom microculture” and reflexively, the classroom microculture is an “emergent phenomenon continually regenerated by the teacher and students in the course of their ongoing interactions” (Cobb et al., 2001, p. 122). The reflexive nature of the interpretive framework supports the socio-cultural lens on learning. Lamberg (2001), and later Moss (2014), adapted the interpretive framework to reflect the relationships between the classroom context (social perspective), classroom activity (including tasks, communication, curriculum interactions, and representational tools), and the sense making of the
class (psychological perspective). See Figure 18 as a model of the complex interactions in the mathematical learning environment originally envisioned by Lamberg (2001).

![Figure 18. Interpretive framework](image)

The revised interpretive framework (sense making, classroom activity, and classroom context) was used to examine the sociocultural phenomenon that emerged in the classroom.

**Framework for Practical Reasoning**

Toulmin, Reike, and Janik’s (1979) framework for practical reasoning was used to view the development of mathematical reasoning during discourse. For Toulmin et al., the core of an argument has three parts. The student makes a *claim* and presents evidence, or *data*, that supports the claim. The *warrant* is offered as a connection between the claim and the data to further support the argument. Finally, a student might provide backing- a mathematical
explanation that is “taken-as-shared” by the classroom (Cobb et al., 2001) and has mathematical authority (Rasmussen & Stephan, 2008). Backing is provided to further support the argument, often showing a capacity to use flexible modeling of a mathematical concept (Pantziara & Philippou, 2012). Within this framework, students are obliged to explain and justify reasoning, attempt to understand others’ explanations, ask clarifying questions for greater understanding, and indicate when they find a solution invalid and explain the reasons (Rasmussen & Stephan, 2008). See Figure 19 for a model of this framework. The model illustrates how a student might argue against the idea that 2/3 should be placed at the number two on a number line.

Figure 19. Toulmin, Reike, & Janik’s (1979) framework for practical reasoning. Adapted from Rasmussen & Stephan (2008, p. 197).
This framework was used to evaluate the quality of the argumentation that occurred within the whole class mathematical discourse, and the effect the argument had on shifting central concepts and promoting conceptual change (DeWolf & Vosniadou, 2015; Posner et al., 1982).

**Levels of Conceptualization of Fractions**

Pantziara and Philippou (2012) created a fraction concept assessment based on Sfard’s (1991) number integration phases: interiorization, condensation and reification. Their research focused on the development of the conceptualization of fractions and the different levels children presented in conjunction with Sfard’s (1991) work. Based on the results of test item analysis and performance by 321 fourth grade students Pantziara and Philippou (2012) described six “difficulty levels” (pp. 75 – 76) of fraction conceptualization, with Level 1 representing the beginning stages of interiorization and Level 6 representing reification. This assessment adapted to create a pre/posttest (see Appendix B). Table 5 describes the levels of fraction conception and related test items.
Table 5.

Levels of Fraction Conceptualization

<table>
<thead>
<tr>
<th>Difficulty Level of Fraction Conceptualization</th>
<th>Students in this phase are able to...</th>
<th>Corresponding Items on test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1 Transition to Interiorization</td>
<td>Perform a simple step-by step procedure for computing sum of fractions with like denominators.</td>
<td>A7</td>
</tr>
<tr>
<td>Level 2 Interiorization</td>
<td>Perform simple fraction procedures</td>
<td>A5, A6, A7</td>
</tr>
<tr>
<td></td>
<td>Fill in the missing numerators in two equivalent fractions</td>
<td>A5</td>
</tr>
<tr>
<td></td>
<td>Compare two fractions with common denominators</td>
<td>A6</td>
</tr>
<tr>
<td>Level 3 Transition to Condensation</td>
<td>Find the fraction of a set of discrete objects</td>
<td>A2, A3</td>
</tr>
<tr>
<td></td>
<td>Select the correct fraction representation as a part of equally divided whole</td>
<td>A1, A2</td>
</tr>
<tr>
<td></td>
<td>Locate a fraction on a number line with iterations that match the fraction denominator</td>
<td>A4</td>
</tr>
<tr>
<td></td>
<td>Calculate the sum of two fractions with unlike denominators</td>
<td>B7</td>
</tr>
<tr>
<td></td>
<td>Divide a given representation of a fraction into equal parts and write fraction of the shaded area</td>
<td>B1</td>
</tr>
<tr>
<td>Level 4 Condensation</td>
<td>Alternate between different representations of a fraction</td>
<td>B2, B4, B5</td>
</tr>
<tr>
<td></td>
<td>Reconstruction of the whole from a given quantity (i.e. if 2/3 equals four objects, what is the value of 1/3? 3/3?)</td>
<td>B3</td>
</tr>
<tr>
<td>Level 5 Transition to Reification</td>
<td>Compare two fractions in more than one way</td>
<td>B6</td>
</tr>
<tr>
<td></td>
<td>Find fraction represented as a continuous quantity (3/4) and place it on a number line divided into different parts (eighths) than the ones represented by fraction’s denominator (fourths).</td>
<td>B4</td>
</tr>
<tr>
<td></td>
<td>Fill in a number line from 0 to ½, with a missing fraction that is divided into thirds.</td>
<td>C4</td>
</tr>
</tbody>
</table>
Find an improper fraction from a given set of objects which constituted the whole. B3

List several equivalent fractions to a given fraction. B8*

<table>
<thead>
<tr>
<th>Level 6</th>
<th>Find a fraction between two consecutive fractions</th>
<th>C6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reification</td>
<td>Draw process of fraction addition</td>
<td>C7</td>
</tr>
<tr>
<td></td>
<td>Place several fractions with unlike denominators on an un-iterated number line</td>
<td>C8*</td>
</tr>
<tr>
<td></td>
<td>Place several fractions on a number line including negative fractions, improper fractions, and mixed numbers</td>
<td>A8*</td>
</tr>
<tr>
<td></td>
<td>Use the variable x to represent a problem of fraction equivalence</td>
<td>C5</td>
</tr>
</tbody>
</table>

Note: Levels of fraction conceptualization (Adapted from Pantziara & Philippou (2012). See Appendix B for fraction pre/posttest. *Items A8, B8, and C8 are added items

Both the pretest and the posttest were scored in three ways, (1) overall score, (2) level of fraction conception based on how many items the student correctly answered in each level, and (3) a density understanding score based on the three questions regarding density and infinite fraction properties.
Methodological Framework

A characteristic of the teaching experiment in this study is that the design (curriculum unit on fraction magnitude) is tested in iterative cycles to study how the learning takes place under implementation (Lamberg & Middleton, 2009). See Figure 20 for an example of the iterative approach to design.

Iterative analysis permitted the research team to examine why the design (developed unit, tasks, and classroom context) was failing or succeeding, to create an intervention to the design, and test the design again. The difference between the original design and the actual enacted design is allowed to surface. It informed the team of how the design adapted to new circumstances (Cobb et al., 1993; Cobb et al., 2001). This is akin to the engineering process where the iteration of drafting and prototyping generates the building of viable solutions that
operate under different conditions (Bannan-Ritland & Baek, 2008). Figure 21 illustrates the iterative procedure used in the teaching experiment.

Figure 21. Model of a teaching experiment using design research. Adapted from Middleton, Gorard, Taylor, Bannan-Ritland (2008, p. 22) and Moss (2014).

Data Collection and Analysis

The next section will provide a summary of methods and procedures for each phase of data collection and analysis. The data gathered from this study will be substantial (Cobb et al., 2003). All sources related to the interpretive framework (classroom context, classroom activities and tasks, sense making) will be collected (Cobb et al., 2003). The voluminous collection of evidence will allow for daily adjustments in the fraction measurement unit and provide an opportunity for the retrospective analysis of the data (Gravemeijer & Cobb, 2006).

There are three phases in the teaching experiment which include (1) preparing for the experiment, (2) the teaching experiment, and (3) retrospective analysis (Gravemeijer & Cobb, 2006). Each phase will be described as it relates to data collection and analysis.
Phase One: Preparing for the Experiment

Observation. Initial observations of the classroom context occurred prior to the teaching experiment. During these observations, the researcher was able to take note of the current classroom ecology and interactions between the teacher and the students. Informal conversations occurred between the teacher and the researcher regarding the designed unit and how the research was to be conducted. Based on the teacher’s reflections regarding pressure to follow district adopted curriculum, the hypothetical learning trajectory (Simon, 1985) of the original unit was adjusted to embed tasks that were similar to the curriculum. The presence of the researcher in the room also established a working rapport between the teacher, researcher, and the class.

Pretest. On the first day of the teaching experiment, students were given a pretest based on the levels of fraction conceptualization created by Pantziara and Philippou (2012). This provided information about the central concepts (Posner et al., 1982) that exist in each student’s framework for conceptual fraction understanding. Included on this pretest are items that investigate students’ understanding of the fraction measurement construct and density understanding of fractions (see Appendix B). The pretest was scored to determine each students’ level of fraction conceptualization (see Table 5).

Phase two: The Teaching Experiment and Iterative Design

Iterative design. In phase two, the teaching experiment was completed. The teaching experiment lasted six weeks, covering twenty lessons in the original unit. During this time, the research team collaborated daily to adjust tasks, learning trajectories, and address sociomathematical norms that needed to be addressed in the classroom context. This cyclical process included the enactment of the lesson, data gathering, drawing analysis and conclusions
about the learning that occurred specifically in fraction knowledge. Iterative design in context was an exceptionally good way to uncover unanticipated consequences and then confirm or challenge them (Hoadley, 2002).

**Interviews and discussions.**

*Daily debriefing discussions.* Regular debriefing sessions with the research team were conducted by the lead researcher in order to “develop alternative explanations of events that occurred during the lesson” (Cobb et al., 2003, p. 11). Each debriefing used the Daily Teacher Lesson Reflection Protocol (See Appendix D). Each post lesson debriefing was video recorded and transcribed. Additionally, the researcher took hand written notes on the Daily Teacher Lesson Reflection Protocol to ensure accuracy in lesson development and an authentic historical accounting of what was changed, and why; and to support retrospective analysis.

*Focus group.* A small group of students (4) was selected based on the results of the pretest. The classroom teacher chose four students that varied in conceptual levels: 2WF (2.5 pretest score, Level 2), 17WF (10.5 pretest score, Level 5), 25WM (9 on pretest score, Level 3), and 20WM (17 on pretest score, Level 6).

The focus group was interviewed after the pretest and posttest using the Student Reflection Protocol (see Appendix E). As part of the protocol, the focus group was asked to verbalize their problem solving process for three problems from the pre/posttest. This interview was conducted following the same procedure at the beginning and conclusion of the study.

The focus group sat together for the first two weeks of lessons. The teacher followed her schedule for changing seating arrangements with the class and the students were moved to different locations. From this point, different groups were videotaped as they worked to give a
broader range of data. A variety of group work is represented as part of lesson video transcriptions.

*Student Reflection Protocol.* At the end of the teaching experiment, the whole class was asked to reflect on the experience by completing the Student Reflection Protocol (Appendix E).

*Final teacher interview.* At the conclusion of the teaching experiment, a final interview with the teacher was conducted (see Final Teacher Debriefing Protocol, Appendix F).

**Data Collection.** A wide range of data was collected in phase two which included video, student work, interviews, written notes, and logs. Table 6 explains the type of data collected and the procedure for collecting the data.

Table 6.

<table>
<thead>
<tr>
<th><em>Data Collected for Study</em></th>
<th><em>Type of Data</em></th>
<th><em>Collection Procedure</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest &amp; Posttest</td>
<td>Tests collected and scored.</td>
<td></td>
</tr>
<tr>
<td>Video</td>
<td>Two cameras collected video during each lesson. Camera 1 was stationed at the back of the room. Camera 2 filmed small group work focusing on discussion, whole class mathematical discourse, teacher/student interactions, and teacher/researcher daily interviews.</td>
<td></td>
</tr>
<tr>
<td>Student Work</td>
<td>Student work was collected at the end of each lesson.</td>
<td></td>
</tr>
<tr>
<td>Teacher Daily Reflection Protocol</td>
<td>Video recorded. A written record was kept of teacher interview after each lesson.</td>
<td></td>
</tr>
<tr>
<td>Teacher Reflection Table</td>
<td>The researcher kept a log during the study of changes made to lessons and why. This included email exchanges between the teacher/researcher as the lesson transformed for the next day.</td>
<td></td>
</tr>
<tr>
<td>Final Teacher Interview</td>
<td>Video recorded and logged with hand written notes.</td>
<td></td>
</tr>
<tr>
<td>Student Reflection Table</td>
<td>A written log was created based on class reflections on the Student Reflection Protocol.</td>
<td></td>
</tr>
</tbody>
</table>
The teacher was interviewed at the end of each lesson and plans were made for the next lesson. Decisions on shifting the tasks were based on mathematical discourse, student performance on tasks, and behaviors which required intervention (sociomathematical norms).

Figure 22 shows an example of the Teacher Daily Reflection Protocol.

<table>
<thead>
<tr>
<th>Day</th>
<th>Activity</th>
<th>Evidence of Student Understanding</th>
<th>What went well</th>
<th>Did discourse support student understanding</th>
<th>Socio-mathematical norms to address</th>
<th>Changes about this lesson? Tomorrow's lesson?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Ruler 1- Lesson 1</td>
<td>All the different answers and the conversations. Wondering about ( \frac{1}{3} ) of 2. Wondering about how many pieces vs. how ruler is labeled.</td>
<td>All the different answers and the conversations. Wondering about ( \frac{1}{3} ) of 2. Wondering about how many pieces vs. how ruler is labeled.</td>
<td>Yes! Gave examples to support their understanding. Needed more time to investigate it.</td>
<td>ELL kids shy to get up and talk. Keep encouraging and helping them formulate an argument. Would like to keep account of when they came and talked. Lucas took over a bit (watch and have a discussion with him). Conner felt that no one was listening to him (he was in Lucas's group). Teacher recommends talking about Shirkers, Workers, Showboaters.</td>
<td>No changes Lesson 1. Lesson 2 will include: How do rulers work? What does &quot;rational&quot; mean? Shirkers, workers, showboaters. Ruler #2. Ruler #3. Discuss using improper fractions. Independent work before partner work. Folders, rules (change top heavy to improper).</td>
</tr>
<tr>
<td>2</td>
<td>Ruler 2</td>
<td>Yes, but they still are seeing the paper as the whole; even if 1 is placed in the middle. Yes they are changing their answers because of the mathematics. C: We had some strong holders of the ( \frac{1}{3} ) that are resistant to understanding why it might be</td>
<td>Conversations and connecting with yesterday. It was faster recognizing there were two wholes. C: A greater group of kids saw it right away. B: One of those was the Pretzel 19 girl and she was voting for 1/6.</td>
<td>I think it is great having the partner discourse. ELL: Natalie engaging. Ruby not talking. One has a great partner. These two actually changed their answer.</td>
<td>Everyone is safe. C: I talked with Lucas to say I know you're thinking it was hard not to be called on. B: Glad we do the Shirkers workers and Showboaters. C: When things are hard or we're pushing them we get the &quot;WHAT?&quot; Addressing this is good, also if someone says an absurd answer we could address this.</td>
<td>I can't believe we only got one ruler done. Lesson one has to be two lessons. Terms: Difference between area and distance. Recommends moving to generalization of What is the whole? This helps them generalize. What is the big idea that can be applied to these types of problems? Third ruler get to concluding statement/generallization</td>
</tr>
</tbody>
</table>

**Figure 22.** This is an example of the log created from the video recording and the written notes from the daily lesson reflection.
Figure 23 illustrates how skills from each lesson were evaluated on a standards based scale (0=limited, not existent, 1= approaching ability, 2= meets standard). For example, student 9LM met the standard for dividing and labeling a ruler (2). The level of fraction conception and score on density question is also included.

<table>
<thead>
<tr>
<th>Class Number</th>
<th>Level of Fraction Conc.</th>
<th>Density</th>
<th>L9 IND Divide and label ruler</th>
<th>L9 IND Break 4/3 into parts</th>
<th>L9 IND Write add and mult. For 4/3</th>
<th>L9 IND Write division sentence (extra)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9LM</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>27CM</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>25CM</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>N</td>
</tr>
<tr>
<td>15CF</td>
<td>2</td>
<td>0</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>18CM</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>28CF</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>N</td>
</tr>
</tbody>
</table>

*Figure 23:* Skills demonstrated in each lesson were scored.

At the end of the unit, students were asked to reflect about the experience. Specifically, students were questioned about how they might place fractions on the number line, how confident they are when working with fractions, sense of safety participating in mathematical discourse, and whether or not the discourse in class was helpful in any way. All students written answers were recorded on this table for further understand the effect of the unit and discourse from a student point of view. Figure 24 shows a portion of the Student Reflection Protocol from the entire class. Student reflections listed according to posttest score.
The goal of retrospective analysis is to produce local instruction theory and potentially innovative and effective designs, and to inspire educational change (Akkerman, Bronkhorst, & Zitter, 2013). In this phase of the research, the data was analyzed using a grounded theory approach to provide explanations about why events occurred within the teaching experiment and what this can tell us about learning and understanding fractions (Gravemeijer & Cobb, 2006; Corbin & Strauss, 2014). According to Corbin and Strauss (2014) the “concepts out of which the theory is constructed are derived from data collected during the research process” (p. 30). The data collected from phase two allowed for the development of “an analytical approach that enables us to step back and view in broad relief what has transpired in a classroom over a time period” (Cobb et al., 2001, p. 118).

In order to accomplish this goal, the interpretive framework (see figure 16) was adapted. Figure 25 shows how the data collected in phase two informed grounded theories on: (1) description of the actual learning process, (2) chronological analysis to find evidence of shifts in mathematical reasoning, (3) identify pivotal episodes that caused changes in students’ thinking,
(4) noted activities that positively affected mathematical thinking, and (5) design ideas sparked by observations, created to use in future classroom settings (Akkerman, et al., 2013).

All lessons were transcribed and then analyzed through the Interpretive Framework. Since mathematical discourse was a focus of the study, conversations were preserved to better understand how they impacted or represented the shifts in understanding. This led to a more succinct look at each lessons. See Table 7 for an example from Lesson 5.
Table 7.

*Data Analyzed with Interpretive Framework*

<table>
<thead>
<tr>
<th>Data Lesson 5</th>
<th>Classroom Context</th>
</tr>
</thead>
</table>
| Adding to Shirker behaviors | C: What have you noticed about shirking behaviors?  
Student: When we don’t want to do our work.  
B: Not doing homework!  
Student Watching the clock  
Student: Lazy.  
C: Today we are going to ask you about working behaviors. What might happen if you work hard? What might happen if you (C reads list that kids made). What might happen? Turn to your neighbor and tell them what might happen if you worked hard?  
B: We are not using pens today, we are going to use pencils so you can adjust your thinking and when you get to your final ruler you can go over it in pen. We are going to glue it down.  
Student: WHAT?  
B: Did you hear that?  
C: Did it happen again?  
B: It happened! Yes, it was a good WHAAAT?  
C: Okay, we are going to do it again. And she is going to ask the question, I’m going to give the answer and I want you to allow your classmates to understand what is happening. When you throw out the WHAT? It tells us all we should be confused. I’m not confused. Are you confused? There are a lot of Teacher: We are just using pencils today on our first draft of our ruler.  
C: And when you get to the final draft you can go over it in pen.  
Class is quiet.  
C: Do you see how much nicer that was? My brain is allowed to think and do. Thank you so much. And working on changing that helps us be hard workers. Thank you Beau.  
C: The hard work in here is through the roof. Rubi is labeling lines. Harmony Is labeling lines. Pru is thinking.  
B: I see cautious work, slowing down and thinking.  
Several kids still asking WHAT?  
C: I hear several kids still asking what? Ask your neighbors, there are plenty of people who know what is going on.  
NOTE: I find that kids need a sense of urgency in order to keep with the program. At this point, we have repeated what to write at the top three times, I’ve written it on the board, and asked them to check with their neighbors. The (WHAT) is what we need to address. I want them to be aware of what is going on and work hard to keep up. |

<table>
<thead>
<tr>
<th>Lesson 5: Classroom Activity</th>
</tr>
</thead>
</table>
| Student: Yeah, because this is ½ (points to middle of the ruler) and it doesn’t keep going, this is the whole (runs hand along the whole ruler). If this said one (runs hand along ½ of the ruler) and this said 2 (shows the other half) then this would be a whole (runs hand along first ½) of…  
Student 2: Two.  
Student 1: Yeah, two.  
Student 2: So I think we all agree that its 1/8?  
Student 1: Finally I did it right!  
Student 2: Yeah, me too.  
NOTE: This conversation shows a SHIFT in thinking and reasoning about fractions. It was good to go back to something they were more familiar with (dealing with 1 whole, Becky also noted this in yesterday’s interview). They were also able to clearly explain how the two situations were different!  
Student 3: I think the denominator is eight because there are eight equal pieces in the whole thing and I did 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 8/8 because there are eight equal parts. At first I
had 4’s on there because I was thinking there were two units, but then I realized there wasn’t so I changed it to eights because there are eight equal places.

NOTE 9:45 excellent transition for decomposing fractions by Teacher.

B: Let’s do a whole number first. What can I break 8 into? Raise your hand if you can give me a combination of what I can break 8 into?

Student 4 has done it into nine bubbles. Moves onto 10 bubbles, we get another paper so he can staple it. Before this, Student 4 has been disconnected to the learning.

C: If you are finished, here is your challenge. Can you break it into 9 bubbles?

Student: No.

C: It’s been done! Someone is working on 10 bubbles

NOTE: Student 4 becomes class leader in thinking.

Lesson 5: Sense Making

C: We are building two rulers and doing a special job that the RRC wants you to do. Today we are using measurement thinking. We’re also going to stick with simple fractions. How many of you know there are many equivalent fractions for every fraction we put up there? (Kids raise hands). My friend Student 5 should have his hand up! Yes, he is the master of equivalent fractions. There are many we could write but we want to have the simple fractions to keep our rulers neat and tidy

NOTE: Student 5 filled each of his rulers with many equivalent fractions. He often became so absorbed in this, he didn’t want to change his mind when he mislabeled a ruler. This made him mad and upset that others didn’t understand what he was doing. He was the student who put 4/32 up for 1/8. My reason for acknowledging his skill with this was to hopefully help him be more flexible with changing his mind and realizing that we may not be asking for equivalent fractions but he has an important insight.

Student 6: No, it’s not two. The whole is not two. See, it’s a half (points to ½). So it’s a half of one. So the whole thing is one.

C: Count with me as we go along our ruler. 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 8/8. We know there are two names for ½, one is 4/8. What is the other name for one whole?

Class: 8/8

C: And that is okay, because some of us know there are lots of equivalent names for fractions.

NOTE: I try to say this as much as possible to help with density understanding.

NOTE: Using ruler helpful to show kids misunderstanding of the value of numbers. Student 6 wrote ½ when he meant ½ of 1/8. Student 7 wrote ¼ when she meant ½ of 1/8. It was good to show them what the magnitude was of the fractions they are using.

Qualitative analysis. The conceptual framework was used to analyze each transcribed lesson. Affecting episodes from the classroom context, classroom activities, and sense making were extracted. This compacted view was then analyzed first by looking for similar concepts that can be derived from data (Corbin & Strauss, 2014). The written log of the changes made in each lesson was examined to connect how the interactions in the conceptual framework caused the research team to adjust the tasks, discourse routine, and sociomathematical norms. Careful
attention was paid to the transformation of unit tasks based on discourse that emerged during the teaching episodes as well as student work samples. Additionally, attention was also paid to how sociomathematical norms were created and upheld was explored. The student reflection table was considered as support for the usefulness of mathematical discourse and how students reacted to the interacting pieces in the conceptual framework.

From this, categories were developed that attempted to reflect higher-level and more abstract content that informs a theory (Corbin & Strauss, 2014). See Figure 26 for a model of this analysis strategy and how it was implemented for this particular study.

![Figure 26](image)

*Figure 26. Constructing grounded theory from data; a first step. Adapted from Corbin & Strauss (2014, p. 104)*

Essentially, this process tracked the collective activity of the mathematics classroom focusing on “normative ways of reasoning of a classroom community” (Rasmussen & Stephan, 2008). This methodology allowed for the large amounts of qualitative data to tell the unique story of the teaching experiment and allow for emerging themes and localized theories to rise into consciousness.
Quantitative analysis. Qualitative data consisted of pre and post test scores, pre and post density/infinity understanding (based on the three questions added to the pre/posttest), and pre and post changes in levels of fraction conception (Pantziara & Philippou, 2012). This data was explored for significant changes in students’ performance. Additionally, specific items from the pre/posttest that were targeted in the unit were compared to see if there was improvement in the overall ability of the class to accurately complete these items.
CHAPTER IV RESEARCH FINDINGS

The teaching experiment and consequent analysis were conducted to investigate the following research questions:

1. How do fourth-grade students reason about fraction magnitude when solving tasks in the classroom teaching experiment?
   - What is the whole-class learning trajectory that emerges?
   - What design decisions are made to modify the fraction magnitude tasks and why?

2. How does mathematical discourse effect shifts in student reasoning on fraction magnitude tasks?
   - What is the role of the mathematical discourse?
   - What design decisions are made to modify the mathematical discourse routine and why?

The findings revealed that mathematical discourse affected both shifts in student reasoning as well as influencing the actual learning trajectory. Therefore, the learning trajectory that emerged will be documented along with the shifts in student reasoning will be discussed in the first part of the findings. The next section will discuss the classroom norms that emerged within the context of the teaching experiment that influenced the mathematical discourse within the classroom.

**Sense Making and the Learning Trajectory**

The fraction magnitude unit was designed by the researcher to address the CCSS for fourth graders in Numbers and Operations-Fractions. The context of designing a ruler for the Rational Ruler Company (rulers and races) was used. The unit contained many hands on interactive opportunities for students to explore the concept of fraction magnitude. The ruler
tasks were designed to provoke cognitive conflict (Posner et al., 1982) in order to promote conceptual change of fraction understanding. The tasks in the unit are designed to engage students in problem solving tasks such as opposed to a direct teaching approach. This approach was chosen so that it provided students with opportunities to engage in mathematical argumentation. The process provides students with opportunities to listen and participate in mathematical reasoning and discourse, essentially communicating to learn mathematics (Drageset, 2015; Piccolo et al., 2008).

Analysis of the classroom episodes, teacher reflection, and the mathematical discourse which occurred revealed several themes within the learning trajectory. Those themes are (1) fraction construct transitions (2) envisioning rulers as a continuous context (4) iteration and partitioning (4) unit as scaffolding for CCSS, and (5) concepts of density and infinity rising throughout the lessons. These categories will each be further explained and justified.

**Fraction Construct Transitions**

Rational Number Project researchers found that part-whole concepts were foundational to fraction knowledge, but to progress to a mature understanding students had greater success when using a variety of physical models representing other fraction constructs (Cramer, Post, & delMas, 2002). The learning trajectory that emerged during the teaching experiment showed that the original unit miscalculated the ability of fourth graders to transition from one construct of fraction to the other. The researcher needed to adjust lessons in order to provide scaffolding for these transitions. The actual learning trajectory of the teaching experiment was informed by efforts to aid students in making shifts in constructs in order to effectively reason about fraction magnitude.
**Part-whole construct reliance.** During the initial stages of the teaching experiment, students relied on their part-whole understanding. How students visualized the unit influenced how they reasoned about the iterations. The first lesson in the original magnitude unit called for students to fold and label three different rulers during the lesson, in actuality, students were only able to fold one. For example, Ruler 1 was split into four equal sections and started at 0 and ended at 2 (see Figure 27).

![Figure 27](image)

*Figure 27.* Fold a ruler into four equal pieces. It starts at 0 and ends at 2. What should we label the lines?

The class was asked to label the three iterations as fractions on the ruler. The teacher inquired, “What would be hard about this task?” One student said, “It starts at 0 and ends at 2 and it would be hard because what would you put as the fourths?” Students were asked to justify their reasoning and explain how they labeled their ruler.

This line of questioning emerged during the whole class mathematical discourse:

**Student 1:** *labels first line as ¼.* I think the line would be ¼ because it would be four quarters.

![Diagram](image)

**Student 2:** *erases ¼.* I think instead it’s ½ because even though it’s divided into four equal parts if you call that 1 and 2, then this would be one piece and this would be the other piece.

![Diagram](image)
Class: Ohhhh.

Student 3: I think you are both correct because these are its own categories but you still can put $\tfrac{1}{4}$ here because they cannot be their own categories. So I would put $\tfrac{1}{4}$ and $\tfrac{1}{2}$. (Arguing that on the first line, it can be labeled both $\tfrac{1}{4}$ and $\tfrac{1}{2}$)

Student 4: I actually agree with Student 1, I understand what she’s doing but it should not be $\tfrac{1}{4}$. It should be $\tfrac{2}{4}$. See, $\tfrac{1}{2}$, when it’s cut in $\tfrac{1}{2}$ it will be $\tfrac{1}{4}$ (draws a line between 0 and first line). And then you put these one fourths together and it will be $\tfrac{2}{4}$.

Student 1: I think it is a matter of perspective because it can be $\tfrac{1}{4}$ because of all 4 sides. It can be $\tfrac{1}{2}$ because there’s two wholes but it’s just like matters of perspective because there can be multiple ways you can see of the ruler.

The first student saw the iteration as $\tfrac{1}{4}$ of the whole unit. The whole was made up of 2 units. Therefore, the bar was partitioned into fourths and the first iteration represented a fourth of the bar. Student 2 saw the iteration as $\tfrac{1}{2}$ and recognized the whole as being made up of two units. The half represented one unit partitioned into two equal units. Student 3 accounted for the $\tfrac{1}{4}$ by splitting one unit into four equal parts and visualized the $\tfrac{1}{2}$ as a unit being made up of two
¼ units. The last argument from Student 1 shows a non-directional and discrete labeling of the ruler, in which each part can be ¼ labeled from either direction, or it can be ½, as argued, “It is a matter of perspective”. The student identified each segment of the whole as ¼, and pointed to the first and third fold indicating it could be counted from either side. Students envisioned the ruler as either being made up of 1 unit or 2 units. How students thought about the unit influenced how they labeled the ruler. Student 1, who pointed out that directionality did not matter, pictured the bar as an area model and was not connecting the model to the context of distance.

The tasks of labeling rulers in this way provided what Posner et al. (1982) described as a cognitive conflict. The part-whole scheme, or what Norton and Wilkins (2013) called the equipartitioning scheme (EPS), allowed students to break continuous wholes into equally sized pieces that exhaust the entire whole. However, students who applied the equipartitioning scheme to the ruler task in isolation ended up with conflicted mathematical proofs (i.e. labeling the first line as ¼, even though they can see that 4/4 is not equal to 2). Consequently, the originally designed unit was altered to provide opportunities for students to become dissatisfied with the central concept (EPS, part-whole construct) and to begin using a competing concept (measurement construct) to increase success. This was done by adding several initial lessons where the task was to build a variety of rulers (see Appendix C).

**Shift from part-whole to measurement construct.** This shift involved students recognizing a fraction as representing distance from zero. Students discussed whether the value (in Figure 27) represented ½ or ¼. The teacher began to reemphasize the attributes of a ruler, and that this task was to create a ruler. Rulers start at zero and increase in magnitude as iterations travel to the
right. Moreover, fractions on rulers are related to the individual units within the ruler. This shifted student thinking:

Student 1: I think it’s one half because on a ruler you can’t put \( \frac{1}{4} \) in between 0 and 1. Like in between 1 would be \( \frac{1}{2} \).

Student 2: I wonder why she had \( \frac{1}{4} \) on each side because you can’t have \( \frac{1}{4} \) on both sides so I thought it would be \( \frac{1}{4} \) on one side then \( \frac{1}{2} \) because there’s…you really can’t have \( \frac{1}{4} \) on both sides.

Teacher: What do we know about rulers? How do they work? As they move from left to right, what happens to the numbers?

Student 2: I think they get smaller. I think the fractions get smaller.

Student 3: Well I think that all the answers are correct but we’re making a ruler so I think we should change one of these. If we’re making a ruler I think this (the third line) should be 1 \( \frac{1}{2} \). (Changes answer).

The teacher continued to draw student thinking towards the idea of linear measurement by having students look at actual rulers and asking, “What do rulers do?” And, “What happens to the numbers as they go across a ruler? Do they get bigger or smaller?” The task of creating the ruler was further embedded in the class discourse.

Teacher: Well, the Rational Ruler Company wants the rulers to get bigger as they go across, just like these rulers. So today when we build our rulers, we’re going to start at one end and travel to the other and it will get larger. We know the rulers are going to be broken up in pieces. How many pieces did we have yesterday?

Class: Four

Teacher: We had four pieces but were we able to just label them \( \frac{1}{4} \)?

Student: No.

Teacher: Because we had to look at where it starts and where it ends. That was huge learning yesterday. So let’s look at our job today.

The context of the Rational Ruler Company and reliance on student understanding of rulers did not shift the sociomathematical norm of how to label the rulers. After creating another
ruler with six partitions that started at 0 and ended at 2, students were asked to label the value of the iterations. The discourse reveals that students are still unconvinced:

*Teacher:* What might be hard for kids as they try to build that ruler for the RRC?

*Student:* It would be confusing like last time that ½ of the strip thingy is one whole and the other half is another whole.

*Teacher:* Do you think we will run into that today?

*Class:* Yes.

When the teacher called for answers (discourse routine) asking what to label the first line on this ruler, the class has a variety of ideas: 2/3, 1/4, 1/6, 3/18, and 2/8. Each answer came with an argument which illustrates the confusion among the class:

1/4: I think it’s one fourth. Because last time we folded it and we had four squares but now we are folding it and have two more squares. So then we would have two less. If that makes any sense. So instead of it being ½, we’re going back two so I got ¼.

1/3: I think it’s 1/3 because there’s three lines and each of those represent 1/3 and if you put them all together its 3/3 or 1 whole.

1/6: Well, I think it could be 1/3 too, but I have 1/6 on mine because the first line would be 1/6 of the whole think instead of just using this half line and then restarting again.

3/18: I’m supporting 3/18. I think it’s a lot of numbers but this is just one of them. Because if you split each category into three and times it by six you will get 18 and since we are going up from here, we get 3/18 instead of 1 out of 18. You go 3, 6, 9, 15, and 18.

Because of this continued confusion and disagreement, it was decided to distinguish between the two types of labeling perspectives (Part-whole thinking and Measurement thinking). Several lessons were inserted into the unit to establish the difference between part-whole thinking and measurement thinking (see figure 28).
Figure 28. Part-whole and Measurement Thinking.

Making Part-whole and Measurement Thinking Explicit

To encourage the shift to measurement thinking, lessons explicitly focused on the two points of view emerging during the discourse. For example, those who saw the first iteration in Figure 27 as $\frac{1}{4}$ were using part-whole thinking. Those who saw the same iteration and labeled it $\frac{1}{2}$ were using measurement thinking. The students were challenged to answer the question: When is it $\frac{1}{4}$? When is it $\frac{1}{2}$? A ruler created in lesson three (see Figure 29) to promote continued discourse on this conflict. This was done by using the context of a two mile race that students were running. Two different scenarios were proposed to differentiate the differing perspectives.

Figure 29. Ruler used to represent a race.
Scenario 1: Jose is running a race. The race is two miles. He runs from the start to the first fold. How much of the race has he finished when he stops at the first fold?

In a small group conversation, students demonstrated that the context of the race was helpful for supporting beliefs about the answer.

Student 1: What did you get (Student 2)?
Student 2: I got ¼.
Student 3: Well, we think it is 1/8.
Student 2: Cause there are 4 spaces and one whole.
Student 1: Yeah, but that’s only one mile.
Student 1: This is the whole race.
Student 3: If it was split into miles then it would be ¼
Student 3: It says how much of the race.

The contextual situation also changed teacher gestures regarding the rulers. The teacher began placing her finger at the start (0) and asking students to count with her as she moved her finger across the ruler, “1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 8/8”. Another conversation revealed that the gestures made by the teacher made a difference for students to visualize distance. Students noticed the pattern of the whole race as being made up of 8 units. Each iteration represented a segment of the unit that increased in distance across the number line. Students treated the 2-unit as the whole race. 1/8 represented eighth of the hole race.

Student 1: I think it is 1/8.
Student 2: Yeah, me too.
Student 1: I think it would be
Student 2: 1/8
Student 1: Yeah, but the only reason I put ¼ right here is because there’s one right here (points to first section of ruler) and there is a 1 at the ½ place (points to middle). But I think 1/8 also makes sense but I don’t know.
Student 2: Yeah, because there is eight (Does not have any fractions on ruler). But then there’s one fourth because there’s four, points to both sides of the ruler.
Student 1: Yeah, but then she counted 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 8/8. She kind of gave us the answer because she already said 1/8 (points to first fold). So, do you agree?
Student 2: Yeah.
Scenario 1 developed a new step in labeling the iterations—identifying the whole. This small group discourse demonstrates how the labeled fraction is dependent on the context of the situation, which was the goal of presenting Scenario 1.

Student 1: I think it is 1/8 because the whole race is 2 miles. So it is like this isn’t one whole (points to the 1 mile mark). The race is two miles (points to the text). So if you count them (touches each eighth) 1, 2, 3, 4, 5, 6, 7, 8 there are eight pieces in the whole race.

Teacher: Tell us about that one section.

Student 1: It is one section of the whole race. The whole race is eight pieces.

Student 1 to her group: Yeah it would be 1/8.

Student 2: But, now that the word problem is there it says that it is the whole race.

Student 1: Yeah, 2 miles, not 1 mile, 2 miles.

Scenario 2 was introduced to the class using gestures and context combined: “Talia starts here. Put your finger on 0, the guy says go, she runs to the first fold and she says to the people giving her water, how many miles have I run?” By doing so, students were forced to think about each mile in the race as representing one unit. The gesture of moving from zero to the first fold drew students’ attention to the distance as related to the mile, as opposed to the whole race.

During whole class discourse, students refer to the race, the miles, and the action of running the race as part of their justifications for their answers. A student noticed that each unit was divided into 4 equal parts. The teacher had the class count with her as she drew her finger from the start to the finish, this time counting by miles,”1/4, 2/4, 3/4, 5/4, 6/4, 7/4, 8/4”. By doing so, students were able to notice the pattern of iterations beyond a whole unit recognizing 5/4 represented 1 fourth of a segment more than the whole. As students explained thinking, they took up the gestures modeled by the teacher by pointing to the ruler and referring to the distances. This became part of the sociomathematical norms of the class in proving arguments.

Student: I think its ¼ because she’s half (points to ½ a mile) well not quite ½ way through, but when she is, does ½, already so that’s ¼ and when you stop right
here (points to a mile) and when she gets right here she’ll have ¼ of a mile to go (points to the second mile of the race, the first line beyond 1). So, so far she already has ¼.

The teacher began to use visual cues by bracketing or circling the wholes to which they referred on the ruler. Students were instructed to do this in later lessons to cue the identification of the whole. Two visual models emerged for the part-whole and measurement thinking (see Figure 30).

**PART/WHOLE THINKING**

Label this ruler with part/whole thinking. What is each line out of the WHOLE unit?

![Visual model for part-whole thinking]

**MEASUREMENT THINKING**

Label this ruler with MEASUREMENT THINKING. What part is each line out of 1 km?

![Visual model for measurement thinking]

*Figure 30. Visual models for part-whole and measurement thinking*

After the lessons about part-whole and measurement thinking, the teacher reflected, “I am glad we explicitly taught the difference between the two (measurement and part-whole). I think if we do several that are just measurement they will totally get it. I think the kids that are struggling are having a hard time switching back and forth between the two.” Thus, the unit was adjusted to include several more tasks in which verbal and written directions cued students to use
measurement thinking. The context of a distance (a race, a drive, a bike ride) was added to future tasks to support student reasoning about the accuracy of their answers.

One outcome of establishing the expectation of using measurement thinking was that it allowed the class to circumvent lengthy discourse in which children continued to argue the part-whole answer. In the instances where students brought up an answer based on part-whole thinking, the error was identified and the class moved on. In this example, students are discussing a ruler with 12 equal partitions that starts at 0 and ends at 2 miles. Because there are two miles, each mile has six equal partitions; therefore the denominator of the partitions is six.

Teacher: How many pieces in each mile?
Class: Six.
Teacher: What should my denominator be? What is the bottom number of our fraction?
Student: Twelve.
Teacher: Could be 12, what’s 12? If I say it is 12, am I using part-whole or measurement thinking?
Class: Part-whole.
Teacher: If I’m using part-whole. I’m saying here’s the whole (pointing to entire ruler) and there is twelve pieces. Are we using part-whole or measurement?
Class: Measurement.
Teacher: So in measurement we look at each unit. So I’m going to ignore that (covers up second mile). How many pieces in this?
Class: Six
Teacher: So the bottom number will be what, 12 or 6?
Class: Six

Rulers as a Continuous Measurement

For the first three lessons in the unit, the rulers ended at two. The teacher expressed concern that students might generalize the understanding that the rulers always go to two. “I think they need to work on something with three wholes or four wholes”. Accordingly, the unit was adjusted to include rulers that had different numbers of units, for example, 0 to 4 (see Figure 31). Because of this change, students began to expand their thinking of the measurement context as being part of a continuous context as opposed to a discrete context. They began to view the
linearity of a ruler as a segment of a distance. This particularly was the case when students had to partition the ruler into multiple units which were segmented into equal pieces, and had to figure out what part each segment represented. Figure 31 shows a ruler divided into 16 equal sections where the fourth fold was labeled one.

![Image of ruler divided into 16 equal sections]

Figure 31. 16 equal pieces, start at 0 end at 4, two examples of student work.

Students began to notice the pattern that the denominator represented the number of parts the whole unit was divided into, and the numerator represented the number of parts. Within the continuous context, the role of the numerator and denominator was addressed in discourse about labeling the ruler shown in Figure 31:

*Student 1:* I labeled it ¼ because in each whole there are four different sections.
*Teacher:* Why did you do 5/4? (Questioning the label on the line after 4/4)
*Student 1:* Because I counted in improper fractions 5/4, 6/4, 7/4 and so on.
*Teacher:* Can you count it out for us?
*Student 1:* ¼, 2/4, ¾, 4/4, 5/4.
*Teacher:* What about 6/4?
*Student 1:* That would be 1 ½.
*Teacher:* And it is also 6/4. Do you see why Trevor call this 1 ½?
*Class:* Yes.
Teacher: When I see $\frac{7}{4}$, why is the 7 up there? Why do we put a 7 there? The numerator says 7.

Student 2: Well, if you just count them, we all agree that this is $\frac{1}{4}$, so it would be (pointing to each fourth on the ruler) $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{5}{4}$, $\frac{6}{4}$, $\frac{7}{4}$ and so on.

Teacher: Do you see what (Student 2) is saying? What is the job of the numerator? The numerator has a special job. The top number has a special job. It tells us what? Turn to your neighbor and tell them what the numerator tells us. Who would like to share?

Student 3: The job of the numerator is counting the fraction that is there. You have $\frac{1}{4}$ and the numerator keeps adding it, so it shows us how much of that fraction there is. (Goes to the board and uses the picture of the ruler on board). So the denominator is $\frac{1}{4}$, and um wait, the numerator shows you how much fourths here are, how much of the denominator.

Student 4: I think the numerator is saying how much of the denominator there is.

Teacher: The numerator tells us the distance from 0, (draws a line from 0 to 7 on ruler). What does the denominator do?

Student 3: Tells us the number of sections in the unit.

It cannot be argued that all students developed a verbally robust understanding of the denominator and the numerator. However, the tasks of building and labeling the rulers forced a cognitive conflict regarding choosing the right denominator for the ruler. The use of improper fractions instead of mixed numbers promoted iterative counting ($\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{6}{4}$…) and they began to notice that with each iteration, the distance increased in equal segments. The following student’s comment is an example of noticing the pattern as well as the flexibility between mixed numbers and improper fractions that developed.

Student to partner: So between 1 and 0 there are 3 equal parts and that’s thirds. So I labeled the first one $\frac{1}{3}$, the second one $\frac{2}{3}$ and this would be $\frac{3}{3}$ but it’s also 1 so I don’t want to change that because we already labeled it one, but it’s the same thing. Um, and in the rules it says we are not allowed to use mixed numbers so it wouldn’t be 1 $\frac{1}{3}$, and 1 $\frac{2}{3}$, so I made top heavy fractions. So $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$ and $\frac{5}{3}$. 
Equivalent fractions naturally developed because the rulers usually utilized more than one denominator. This is evidenced in lesson 8, when the task was to build a ruler (0 to 1) that showed halves, fourths, eighths, and sixteenths (see Figure 32).

Figure 32. Build a ruler with halves, fourths, eighths and sixteenths.

Students developed the expectation that 0/4, 0/2, 0/8, 0/16 as well as 1/1, 2/2, 4/4, 8/8, and 16/16 be included on the rulers. It also became the norm of the class to find and label equivalent fractions whether or not it was required (see Figure 33).

Figure 33. Student fills ruler with equivalent fractions.

Through the repeated labeling of the rulers, students came to expect equivalent fractions and included them in their counting.

Teacher: As I go down the ruler tell me what you labeled it.
Class chanting together: 1/6, 2/6 or 1/3, 3/6, 4/6 or 2/3, 5/6, 6/6 or 3/3 or 1, 7/6, 8/6 or 4/4, 9/6, 10/6 or 5/3, 11/6, and the next one is 12/6.
Teacher: That’s pretty fabulous.
**Rulers as a Part of a Whole**

Working with continuous units in a ruler required measurement thinking. However, when the whole unit only represented a fraction such as \( \frac{1}{4} \), it required students to conceptualize the fraction as representing a unit that could be iterated. When iterating \( \frac{1}{4} \) into four equal pieces to create one whole, it required students to visualize the missing parts of the unit and think about the segment in relation to the original whole. If a fourth is split into four equal parts, one segment represents a fourth of a fourth, or \( \frac{1}{16} \) (see Figure 34). To scaffold the visualization, this lesson occurred the day after building the ruler in Figure 32. Using the ruler broken into sixteenths, the four sections in \( \frac{1}{4} \) were sixteenths.

Within the mathematical discourse, students began to use mathematical evidence to support their claims. In this conversation, students argued about what to label the first line in a ruler with four equal pieces that starts at 0 and ends at \( \frac{1}{4} \) (see Figure 34).

![Ruler that is a part of a whole.](image)

**Student 1** argued that the first line was \( \frac{1}{16} \), and the class seemed to be in agreement for the most part. However, another student raised his hand and wanted to argue why the first line should be \( \frac{1}{4} \):

**Student 2:** I think It is \( \frac{1}{4} \) because you can’t just call it \( \frac{7}{4} \)* if you have a half right there (points to middle line). You can’t just have a random fraction on a number line with specific unit amounts. Like this has 4, so if there is 4 units that would make it \( \frac{1}{4} \) of each unit.

*'(The \( \frac{7}{4} \) is a reference to a number on the previous day’s ruler, Figure 31).

**Student 1:** I wonder why he did \( \frac{1}{4} \) (Student 2). Because if our whole is \( \frac{1}{4} \), then how would we have \( \frac{1}{4} \) on the first line?
Student 2 argued for the part-whole interpretation where the unit is partitioned into four equal parts, and the first segment is ¼ of the whole unit. Unique in this discourse is that Student I responded with a cognitive conflict, if ¼ is the whole unit, how can the first segment also be ¼? Another student further wondered, “I wonder if the unit is ¼ so it is like ¼ inside the ¼. So ¼ of ¼ is 1/16”.

**Partitioning and Iterating Race Rulers**

Understanding fractions as measures requires the operation of iterating (Kieren, 1988; Tzur, 2004). Iterating in a continuous context requires repeating a given length that is $n$ times as big as the given part (Wilkins & Norton, 2011). Tzur (2004) found that bringing forth students iterating procedures assisted in helping students to see fractions as measures. The race context in the unit provided opportunities for iterating by placing fractions with unlike denominators on an empty number line. The mathematical discourse that emerged from these types of tasks showed that (1) iterating thirds is difficult for fourth grade students, and (2) with the scaffolding of “race rulers”, most students were able to develop strategies to accurately place fractions on the number line.

In Lesson 11 of the fraction magnitude unit, students began iterating by planning races that had events at certain fractions of the race (see Figure 35).
Figure 35. Example race planning task completed by student. All fractions are put on the working draft, which later became the race ruler.

This task required students to order the fractions on the number line in relation to magnitude. Fractions with unlike denominators were given as descriptors for events in the race. This problem was carefully set to allow students to estimate distances through halving. Students were given colored pencils which proved useful to separate the iteration of unlike denominators. Students were successful at iterating halves, fourths, eighths and sixteenths as this type of
partitioning appeared to be visually capable of accomplishing this task. On the student reflection protocol, students were asked to share their strategy for putting fractions on a number line. Fourteen students mentioned finding half of the line and proceeding from there. Answers such as “You have to divide in the middle of the race plan so you don’t get confused” and “I look at it and I split it in half” indicated that halving was a common strategy. One student reported, “I make a ½ then you can make the other fractions by splitting in ½”.

Partitioning by thirds proved more visually challenging to estimate than halving. In the second race we planned, students were expected to include halves, thirds, fourths, and eighths. The teacher indicated that she felt the students would be challenged with iterating thirds and wanted to do this as a whole class. During the lesson, students were called up to place ½, 1/3, and 2/3 on the number line. We began with 1/2, and then students were asked to place 1/3 and 2/3 on the number line at the board. Figure 36 is a recreation sequence of how students iterated thirds during the activity and illustrates the difficulty of the task for the fourth graders in this class. Each number line represents a change that a student made to the line when they came up to the board during the discourse.
The last interpretation (Student 4) came fairly close to thirds and the student measured the distances with his hands, checking for accuracy. He commented, “I want to change both. Because if you count it out, earlier they weren’t equal pieces because one was small, one was big (Student 2), and one was small. So I want to change this to 1/3 right here and another third right here. While he worked to get it correct, students continued to comment on his accuracy and ask, “Can I come up and change it?” Even when working together as a class to include thirds, halves, fourths and eighths on one race ruler, the task became too difficult (see Figure 37).
Putting one half on the line appeared to skew students’ ability to envision the thirds. One issue that students had to resolve was visualizing thirds and not focusing on the half, which appeared to provide confusing information. The student work showed evidence of inaccuracies in placing the fractions. In this case, the colored pencils added to the confusion because students could not erase and adjust their fractions. It should be noted that no iterations were provided in the number line race tasks, rather students had to make their own. This added a level of complexity to the ruler task, especially when it included thirds in addition to the halves.

Because of this, the unit was changed to include two race rulers (one for thirds and one for halves) in order to ameliorate this situation (see Figure 38). Additionally, students were provided strings as a tool to measure and iterate the thirds across the line. Students responded well to having a tool and were able to physically check their mental process for envisioning the thirds (see Figure 38). This was especially helpful for students who were still not clear about equipartitioning as it reinforced the general concept of a fraction.
On the student reflection protocol, students with a higher score on the posttest allowed for the inclusion of thirds and other denominators in their strategy for the task. Comments such as: “What I do is divide it into the different numbers of the denominator” and “First, I would split the line into the number of pieces shown on the denominator of the fraction I need to place. Then, I label each place” reflect a more sophisticated and flexible approach to the number line task. At the conclusion of the unit, most students were able to create two race rulers (thirds and halves) and accurately place up to six given fractions on an un-iterated line.

Figure 38. A race ruler added for thirds and sixths.
Scaffolding for Common Core State Standards

The fraction magnitude unit transitioned from creating and labeling rulers to using fractions to plan a running or biking race. Both models developed into effective tools to scaffold and practice the following concepts:

- Fraction comparisons including statements of inequality
- Fraction decomposition
- Writing addition and multiplication sentences to represent the decomposition
- Multiplying fractions by whole numbers

**Fraction comparison.** Students were asked to write statements of equality or inequality after they created rulers and races. After creating the ruler in Figure 32 with halves, fourths, eighths, and sixteenths, students were able to give statements of equality or inequality based on the ruler.
Teacher: Can you give an inequality?
Student 1: $\frac{1}{8} > \frac{1}{16}$
Student 2: $\frac{1}{8} = \frac{1}{8}$
Student 3: $\frac{1}{16} < \frac{1}{1}$
Student 4: $\frac{1}{8} < \frac{1}{2}$
Student 5: $\frac{1}{2} = \frac{1}{2}$

Students were required to use their race rulers to create statements of inequality with partners (see Figure 40). Students were prompted to check the rulers for accuracy. The teacher continued to use the race metaphor by saying, “If I start the race and run towards the fractions, which one is farther? 1/3 or 3/8?” This was accompanied by the gesture of moving a finger from the start of the race to each fraction. Using the measurement construct in this way provided students an opportunity to compare fraction magnitudes after locating the fractions on the number line themselves. The organic process of iterating the units on the race ruler, using the race ruler as a guideline for where the fractions go on the number line, and then using the number line to comparing magnitude seemed to build a strong foundation for this task.

Figure 40. Statements of inequality created from race rulers.
Fraction decomposition and creating addition and multiplication number sentences.

Beginning in Lesson 5, rulers were used as a scaffolding tool for decomposing fractions and writing addition and multiplication number sentences. The fourth grade district adopted curriculum used number bonds (see Figure 41, 42) to compose and decompose whole numbers, and the teacher felt this would be a familiar model for a more difficult task. The teacher initiated the discussion by having students decompose the whole number 8 into equal pieces. Students were then asked to decompose the number 8/8 into pieces. The following discourse occurred which demonstrated the transfer of the whole number decomposition to fraction decomposition.

Moreover, the teacher reflected that the use of the ruler as a tool was “absolutely essential” for students.

*Teacher:* So let’s put 8/8 in a bubble. I want to break this into pieces. Let’s start with just 2.

*Teacher:* Look at your ruler, does your ruler help you?

*Teacher:* I want to break 8/8 into two pieces.

She draws a pie with eight pieces.

*Teacher:* Can I do eights this way? Is this a pie with eight pieces?

*Class:* yes

*Teacher:* I want to eat the whole thing, what 2 chunks could I take out?

*Student:* 4/8 and 4/8.

*Teacher:* So if I ate 4/8 and 4/8 would I eat the whole pie? Does that equal 8/8?

*Class:* Yes.

Teacher flips to the ruler.

*Teacher:* So if I did 4/8 and I did another, would that equal the whole? Give me another one.

*Student:* 2/8 plus 6/8

*Teacher:* Let’s count them out on the ruler. (Students count to 2/8 and then another 6/8 to land on 8/8).

*Student:* 5/8 plus 3/8
Student: 1/8 and 7/8

Teacher: Are you sure? Let’s count it, (shows how to count it out)

Teacher: Let’s do some number bonds. You were telling me them, lets write them.

Student: 5/8 plus 3/8

Teacher: Did you guys hear what she said? She said plus. So is 5/8 plus 3/8 = 8/8? (Writes equation on board).

Class: Yes.

Teacher: Absolutely. That’s just the other piece right there.

Beau: 2/8 plus 6/8

Teacher: Will you give me the addition sentence for that?

Student: 2/8 + 6/8 = 8/8

Teacher: Do you think we can do three pieces? I think we can move onto three pieces.

Students were successful in creating number bonds and writing addition sentences to describe the decomposition (see Figure 41). Students were asked to create number bonds for 8/8 with 3, 4,5,6,7, and 8 bubbles. When students finished early and the teacher wondered if they might do 9 bubbles, several students were able to accomplish this by breaking an eighth into sixteenths (Figure 42).
Figure 41. Students create number bonds to decompose fractions and write addition sentences.

Figure 42. Decomposing an eighth into sixteenths to achieve 9 addends.

The students were able to easily write addition sentences which allowed the teacher to introduce writing fraction multiplication sentences. The teacher entered this conversation using
previously understood concepts of multiplication by the class. They knew multiplication was a shortened way to write repeated addition and that a number next to a parenthesis meant multiplication, e.g., \(5(4) = 5 \times 4\). The discourse that emerged demonstrated students were ready for this shift. Figure 43 shows student work writing multiplication sentences.

*Student:* \(2/6 + 2/6 + 2/6 + 2/6 = 8/6\)

*Teacher:* I’m getting tired of writing all these \(2/6\)! What is it called when I keep adding the same number?

*Class:* Repeated addition.

*Teacher:* So, repeated addition, we like to shorten things up to turn it into multiplication.

*Teacher:* Who can tell us how they broke it up into eight equal pieces?

*Student:* \(1/6, 1/6, \text{etc.}\)

*Teacher:* What is your addition sentence?

*Student:* \(1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 8/6\)

*Teacher:* I’m tired! Someone help us with a multiplication sentence?

*Student:* \(8(1/6) = 8/6\)

*Teacher:* That’s great work! Let’s give it a round of applause!

*Figure 43.* Student work writing multiplication sentences.
From this number bond, students were able to generate an addition sentence, multiplication sentence, and some even saw the division sentence possible in the task. Figure 44 shows the group work assignment that generated the discussion below.

**We label the ruler.**

*Teacher:* One fourth is decomposed into...

*Class:* 1/16, 1/16, 1/16, 1/16

*Teacher:* Say your addition sentence:

*Class:* 1/16 + 1/16 + 1/16 + 1/16 = 4/16

*Teacher:* What would be the multiplication sentence?

*Class:* 4 times 1/16 equals 4/16.

*Teacher:* You can even do division. We took ¼ and divided into 4 equal pieces, what was the size of each equal piece?

*Class:* 1/16

*Teacher:* Brilliant!

**Figure 44.** Student work showing division of a fraction into four equal pieces.
After this group work, students were asked to take a test with two problems similar to Figure 44 using different fractions. Results of the test showed that all students were able to create the ruler and break the fraction equally (100%). Nineteen out of the 24 (79%) students who took the test wrote an accurate addition and multiplication problem and 14 students (33%) were capable of creating an accurate division problem.

The teacher was initially concerned about having the students move towards multiplication sentences. Her comment was, “So soon?” She felt the students would be overwhelmed by the complexity of assignments that required (1) creating a ruler with multiple wholes, (2) decomposing a given improper fraction, (3) writing an addition sentence, (4) using multiplication to represent repeated addition, and (5) possibly a division sentence. However, she was pleased that the students were able to accomplish this and commented, “This is what makes me want to do math all day.”

**Fractions as operators.** Students were asked to work with multiplying a whole number by a fraction in a different way when working with the race context. Students were asked to plan a bike race similar to the running races, however, in this situation, the distance of the race was included. The intention of the lesson was to ask students to reason about a fraction operating on a whole number. For example, students were asked, “If the bike race was 12 miles long, and the riders get a sandwich at ½ of the race, how many miles will they ride before they get to the sandwich stop?” The teacher and researcher discussed a model to support students to solve this problem and decided to create a tape diagram and a number line as illustrated below (see Figure 45). In this case, students were told that the race was 36 miles long and task presented was to determine how many miles a rider would have to ride to arrive at the Popsicle station (4/6 of the ride).
This model proved to be cumbersome and confusing for most of the class as can be seen by the two different examples of student work in Figure 45. Some students were able to easily see the tape diagram as a representation of 36 miles and the fraction 1/6 would operate by breaking 36 into 6 equal pieces (top example). For other students, the number 36 was too large to reason in this way. To support the understanding, the teacher met with small groups of students and used tiles to represent the 36 miles.

![Tape diagram](image)

*Figure 45. Tape diagram below number line to support fraction multiplication.*

Because of student confusion and poor performance of this model, the researcher and teacher chose to use the familiar number bond model. The teacher felt that all students needed to work with the tiles to provide physical and visual evidence for the concept. During the lesson, all
students manipulated tiles to divide whole numbers into equal groups, and fill out the number bond model.

Teacher: Now we are doing a whole new thing with fractions, we are operating with fractions. What are we doing?

Class: Operating

Teacher: A fraction can operate on a number, just like a doctor operates on a person. Doctors have to cut open people and fractions operate on numbers by cutting them apart.

Teacher: So, fractions can operate on part and wholes and they can also operate on numbers. They can divide lines equally and do distances. Now we are going to operate on numbers. The first number we are going to operate on is 10. The fraction I’m going to pick is ½. We have the numerator and the denominator. What is the denominator?

Class: 2

Teacher: The two tells us how many equal groups we should make. I want you to break your tiles into 2 equal groups. You have 5 seconds.

Students break up tiles.

Teacher: Who can tell me what goes in each bubble? (see Figure 46)

Student: 5

Teacher: Good. You should have two sections of 5. Two told me how many equal pieces to break ten into. Fractions are really interesting. When we use number bubbles for fractions they always have to be equal pieces.

Teacher: When we cut up circles, they have to be equal pieces. When we iterate our lines, it has to be equal pieces. Fractions always have to be equal pieces. That is the basic understanding of what fractions do for us, break things into equal pieces. And today they are going to be operating on numbers.

The model that emerged was a number bond labeled with fractions outside each bubble (see Figure 46). Using tiles as a numerical support, students were able to break whole numbers into equal groups and identify each group as a fraction of the whole number. The model provided scaffolding for finding more than the unit fraction of a number as students were able to calculate, for example, 3/8, ½, and even 9/8 of 32 (see Figure 45). For this particular fourth grade class, the teacher reflected that these last lessons pushed the limit of their current abilities
but she wanted to “push through the concept”. Both the teacher and the researcher felt that this model held potential for future applications and greater understanding of how a fraction can act as an operator in many different ways- on a ruler, on a number line, on a shape, or on a number.

![Figure 46. Number bond model for fraction as an operator construct.](image)

**Infinity Rising**

A key piece in integrating fractions as a number system is understanding the relationship between the numerator and the denominator of fractions as division, and beginning to see that fractions are infinitely divisible and that there are an infinite number of them (Fuchs et al., 2014; Smith et al., 2005; Stafylidou & Vosniadou, 2004). This understanding was assessed in the pretest when students were asked to answer the following questions:

**D1.** What equivalent fractions do you know for ½? Please write some here.

**D2.** How many equivalent fractions for ½ do you think there are?
D3. How many fraction numbers are there between 0 and 1? Please give an estimate.

Instruction in this unit did not specifically address the infinite property of fractions, however, the concept continued to rise throughout the tasks and discourse of the class. In the first lessons, this conversation occurred in the focus group:

Student 1: Wait…yeah but this isn’t anything it’s between 0 and 1 and there can be multiple fractions between 0 and 1, there isn’t an equal number there’s just fractions.
Student 2: Hm.
Student 1: See (pointing to the space between 0 and 1 on ruler).
Student 2: Guys, I figured out how much fractions there are in one whole. It’s infinite.
Student 1: Yes, it’s infinite!

The discussion illustrates that students were noticing that there can be an infinite number of fractions between zero and 0. The task of labeling, halving and splitting rulers and racers provided opportunities for students to see equivalent fractions. One student in particular, labored to provide an equivalent fraction for each response:

Student: I’m supporting 3/18. I think it is a lot of numbers but this is just one of them. Because if you split each category into three and times it by six you will get 18 and since we are going up from here, we get 3/18 instead of 1 out of 18. You go 3, 6, 9, 15, and 18.

In subsequent lessons, this student complained that others “didn’t understand what he was talking about” when he answered 8/32 for an answer the others saw as ¼. This student exhibited the relational understanding of the numerator and denominator which allowed him to produce equivalent fractions.

Students were asked to provide equivalent fractions based on a ruler that was created and several were able to see the pattern of producing multiple equivalent fractions using a doubling procedure (see Figure 47).
Figure 47. Student work showing doubling pattern for equivalent fractions.

Another situation where the notion of multiple representations arose was during the task of making number bonds for a certain fraction and several students wanted the challenge of breaking the fraction into more bubbles. The student work in Figure 48 demonstrates the continued halving of the ruler to attempt to create more addends for the number bond for 7/4. The final equation the student wrote was: \(4(1/32) + 3(1/16) + 8(1/8) + 1/4 = 7/4\). It should be noted that this particular student had the highest score on the pretest, but remained disengaged in the learning until this particular lesson.
Figure 48. Student uses halving of the ruler to create 16 addends for 7/4.

The exhaustion of doubling and halving in order to create all the fractions possible naturally rose during class discussion while students were labeling rulers with equivalent fractions.

Teacher: What other names are there for one whole? Will you record them on the board? (To student at the board).

Class: 3/3, 6/6, 1/1, 2/2

Teacher: How many names are there for one whole?

Class: Infinity, a million, could we ever stop?

Class: No! Well you can have some, but infinity...

Teacher: Yes, every single number that is out there plus one more. Thank you very much. Give this a round of applause.

In our final lesson discourse, the concept of infinity spontaneously arose, led by the student who provide the answer of 6/48 when the class expected 1/8 (Student 1 in this conversation). Student 2 in this discourse provided the work for Figure 48.
Teacher: Are you saying 6/48 is equivalent to 1/8.
Student 1: Yes.
Teacher: He’s saying 1/8 is equal to 6/48. Is 3/24 equal to 1/8?
Class: Yes.
Teacher: What’s your next fraction?
Student 2: 12/96
Student 3: 24/192
Teacher: How are you doing that? It’s like fraction magic!
Teacher: What are the kids doing to get these equivalent fractions?
Student 3: We’re doubling the denominator.
Student 4: We’re also doubling the numerator.
Student 2: We’re dividing the numerator and the denominator.
Student 5: 48/384
Teacher: How did you do it?
Student 4: I doubled the bottom and the top of 24/192
Student 3: 2/16
Student 5: ¼ doubles to that
Student 2: 96/768
Teacher: Could we ever stop doing this?
Class: NO, unless you want to stop.
Teacher: What if someone said keep going until the numbers don’t work?
Student 4: It’s infinite. So like there are infinity fraction between 1.
Teacher: Infinite means it never stops. What does infinite mean?
Class: It never stops.
Teacher: What’s the next one Student 2?
Student 2: 792/1556
Teacher: Could I do it again? Again? Again?
Student 5: 50 to 100, 100 to 200, 200 to 400, 400 to 800, 800 to 1600
Teacher: Does it ever stop?
Class: No
Teacher: Should we stop?

Student 2: I have another one! 384/3072

At the end of the experiment, Student 2 made his teacher take a picture of his work as he continued to investigate the infinite property of fractions (see Figure 49).

*Figure 49. Infinity rising.*
Quantitative Analysis

Twenty-six fourth grade students participated in the teaching experiment to understand the possible benefits of the fraction magnitude unit intervention as measured by the pretest and posttest scores (see Figure 50). The dependent variable was performance on the test and the independent variable was time, which had two levels. The first time point was at the very beginning of the unit, and the first lesson was to administer the pretest. The second time point was the last lesson of the unit, during which the students took the posttest, which was the same test as the pretest. The difference scores were symmetrically distributed, as assessed by a histogram. The Wilcoxon signed-rank test was used to determine whether the median difference between the two related groups (i.e., the two time points) was statistically significant. A Wilcoxon signed-rank test was chosen because the population sampled (n=26) did not meet the assumption for a t-test (n ≥ 30). Data are medians unless otherwise stated.

Of the 26 students in the study, the fraction magnitude unit elicited an improvement in the posttest in 23 participants compared to the pretest, whereas one participant saw no improvement and two participants had a lower score. A Wilcoxon signed-rank test determined that there was a statistically significant median increase in posttest scores (pretest median = 7.5, posttest median = 14.75, difference = 7.25), Z= -3.918, p < .001. Morse (1999) recommends using eta-squared as a measure of effect size for the Wilcoxon signed-rank test. SPSS was used to calculate Kendall's W, which is equivalent to eta-squared. The effect size (W= .678) suggested the magnitude unit had a high effect on the significance.
Figure 50. Comparison of pre and posttest scores.

To further analyze the possible effects of the fraction magnitude unit, those items on the test that were specifically practiced or addressed in the unit were analyzed separately. For example, item B1 represents a conceptual understanding of fractions, but was not part of the unit tasks. Conversely, test item A4 was directly addressed when students planned races (see Figure 51).

Figure 51. Pre/Posttest Items.
The difference scores were symmetrically distributed, as assessed by a histogram. The Wilcoxon signed-rank test was used to determine whether the median difference between the two related groups (i.e., overall item performance on pretest and overall item performance on posttest) was statistically significant. A Wilcoxon signed-rank test was chosen because the number of test items analyzed (n=23) did not meet the assumption for a t-test (n ≥ 30). Data are medians unless otherwise stated.

Of the 23 items analyzed from the test, 20 items showed improvement, one item saw no gain, and two items had a lower score (see Figure 52). A Wilcoxon signed-rank test determined that there was a statistically significant median increase in these items (item pretest median = 7, item posttest median = 13, difference = 6), Z= -4.334, p < .001. Eta-squared was again used as a measure of effect size for the Wilcoxon signed-rank test. SPSS was used to calculate Kendall’s W, which is equivalent to eta-squared. The effect size (W= .689) suggested the magnitude unit had a high effect on the significance.

![Pre/Posttest Items Addressed in Unit Comparison](image)

*Figure 52. Comparison of class performance on items specifically taught in fraction magnitude unit.*
Developing a Classroom Community and Sociomathematical Mindset Norms

The intent of the teaching experiment was to have students learn through communication and provide ample opportunity to problem solve in a mathematical community. One intent of this study was to use a discourse routine (see Appendix A) during the teaching experiment and analyze shifts made in the routine and document what and why changes were made. The teaching experiment data showed the discourse routine to be robust and few changes were needed to match the needs of the classroom community. Classroom lesson analysis showed that both the teacher and students implemented the discourse routine regularly. The sociomathematical norms as envisioned by Cobb et al. (2001) appeared robust in this class. However, certain behaviors that arose during teaching episodes needed to be addressed. The data showed many instances where the teacher intervened during lessons to adjust the mindset of the class towards the challenging mathematics. In the teacher interviews, the student behaviors were noted and plans were made to address them in the next lesson. A routine was established in the teaching experiment where behaviors towards mathematics were explicitly discussed before most lessons to adjust the sociomathematical mindset norms of the class. The specific sociomathematical mindset norm set up at the beginning of the lesson became an effective tool throughout the lesson to identify and encourage a positive sociomathematical mindset. The following themes arose from the data and were explicitly addressed throughout the teaching experiment:

- Floating Dialogue
- Shirkers, Workers, and Showboaters
- Transitional Anxiety
Floating Dialogue

A noticeable trait of this particular classroom community was the emergence of “floating dialogue.” Floating dialogue refers to what is said by students out of turn which narrates the class, often in a negative way with less than desirable results. The most common example of floating dialogue was the permission the class gave itself to say, “What?” or sometimes “Hmmm?” This comment was used in a variety of circumstances:

Teacher: Well, the Rational Ruler Company wants the rulers to get bigger as they go across, just like these rulers. We call that directionality. The direction in which you travel across is really important.

Student: What??

......

Student: I think it could be ½ of ¼.

Teacher: What is the fraction that’s half of fourth?

Student: Half of fourth

Class: What? Huh?

......

Teacher: Put an arrow to 3/16. Prove to your neighbor you have put an arrow to 3/16.

Teacher: Freeze…

Teacher demonstrates how to do it. We have to practice this twice.

Student: Whatttt?

......

Teacher: We are not using pens today, we are going to use pencils so you can adjust your thinking and when you get to your final ruler you can go over it in pen. We are going to glue it down.

Student: WHAT?

In these cases, when students called out “what”, this conveyed that the class did not understand teacher directions or other students’ mathematical ideas. The goal of an inquiry community is to provide a safe environment for students to wonder. In the student reflections at the end of the experiment, three students commented that, “I do not feel safe sharing my answers because it scares me someone will say “WHAT?” when I give an answer”.

Floating dialogue was also used by the class to tell each other what to do. When playing Pass the Pen, a game where students come up and add to mathematics problem silently, students
had difficulty curbing their floating dialogue. All of these comments were made out of turn and consequently distracted the student up at the board.

*Student:* You have to pass the pen to a boy.

*Student:* Make sure you change the color, you have to change the color.

During Pass the Pen, students were specifically told that they are allowed to go up and change something on the board, they just need to raise their hand. After a classmate plotted $2/3$ incorrectly on the number line, several students commented, “Can we change an answer?” And, “Are we allowed to change something on the board?” This caused the student at the board to immediately turn around and adjust the answer.

This type of dialogue was also used to correct the teacher as was evidenced in several lessons where the teacher made insignificant errors. For example, in one lesson the teacher asks the class to use their pens to write, several students float the dialogue, “Pencils, you said we were using pencils”. The researcher shortened a student’s name during a lesson, and a student added, “She doesn’t like people to call her that name.”

Taken one instance at a time, the dialogue seems harmless, collectively the floating dialogues created an unsafe atmosphere for error and learning. The teacher reflected that she “usually ignores this” and keeps teaching even though it does bother her.

This behavior was addressed by bringing to attention when it occurred. The class had become so used to communicating in this way it was difficult to break the habit as it persisted throughout the teaching experiment.

*Teacher 1:* We are not using pens today, we are going to use pencils so you can adjust your thinking and when you get to your final ruler you can go over it in pen. We are going to glue it down.

*Student:* WHAT?
Teacher 1: Did you hear that?
Researcher: Did it happen again?
Teacher 1: It happened! Yes, it was a good WHAAAT?
Researcher: Okay, we are going to do it again. And she is going to ask the question, I’m going to give the answer and I want you to allow your classmates to understand what is happening. When you throw out the WHAT? It tells us all we should be confused. I’m not confused. Are you confused? There are a lot of people who are not confused.
Teacher 1: We are just using pencils today on our first draft of our ruler.
Researcher: And when you get to the final draft you can go over it in pen.
Class is quiet.
Researcher: Do you see how much nicer that was? My brain is allowed to think and do. Thank you so much. And working on changing that helps us be hard workers. Thank you.

By the end of the teaching experiment, the floating dialogue was lessened, however there was a need to address it throughout the experiment.

Shirkers, Workers, and Showboaters

Shirkers, Workers and Showboaters is an approach to identify types of working that can be helpful, and that which is unhelpful. Shirking behaviors are the type that students exhibit during disequilibrium or from fear that they do not understand the math. They position themselves as novices in mathematics, which allows them to defer the learning to an expert. Showboating behaviors are what students use to position themselves as experts, which limits participation and learning (DeJarnette & González, 2015; Esmonde, 2009). Workers exhibit targeted behaviors of adopting a flexible and academic stance to learning.

The class exhibited shirker behaviors in the first lesson and the teacher wanted to address this right away. Floating dialogue was an issue (What?). Additionally, according to the discourse routine, students are given three to four minutes to work independently on a problem
before working with a partner. This was almost impossible for some students and they exhibited behaviors such as talking, complaining they did not understand, and getting up to go to the bathroom or get a drink. As one student complained, “This doesn’t make my brain hurt, I just don’t know how to do it!” As a result of this behavior, the independent work time of the routine was often ineffective, and the behaviors distracted others.

To draw attention to this, the class was asked to give examples of shirking that they use during math when things are difficult. They were also asked to list “working” behaviors (see Table 8).

Table 8.

<table>
<thead>
<tr>
<th>Class Reflections on Shirking and Working</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Shirking Behaviors</strong></td>
</tr>
<tr>
<td>Say to ourselves that we can’t do it</td>
</tr>
<tr>
<td>Ugh, do I have to do that</td>
</tr>
<tr>
<td>Make up excuses, bathroom, nurse</td>
</tr>
<tr>
<td>Pencil sharpening, getting drinks</td>
</tr>
<tr>
<td>Say we’ll do it tomorrow</td>
</tr>
<tr>
<td>Read a book instead</td>
</tr>
<tr>
<td>Throw fits</td>
</tr>
<tr>
<td>WHAT?</td>
</tr>
<tr>
<td>Put our heads down</td>
</tr>
<tr>
<td>Playing with things</td>
</tr>
<tr>
<td>Daydream and look around, wander around the room</td>
</tr>
<tr>
<td>Ignore the teacher</td>
</tr>
<tr>
<td>Sit in a lazy position</td>
</tr>
<tr>
<td>Watch the clock</td>
</tr>
<tr>
<td>Stop when it gets hard</td>
</tr>
</tbody>
</table>

Once this list was created, it was used in several ways to develop working behaviors. Students set goals based on the list before lessons occurred. In Lesson 6 students each chose a shirking behavior to notice and try something else, while in Lesson 7, students were asked to identify a working behavior to use during class. Before Lesson 8, the teacher had students
complement each other for working behaviors. This was an effective way to use the list. For example, one student complimented a classmate by saying that she is “working so hard”. This particular classmate had an issue several days earlier and refused to do any work. The student was noticing the change she made in her behavior.

The teacher also used the list to narrate for students what it looked like to work hard. During independent work, which was problematic at the beginning, the teacher would say, “I see five students rereading the problem to check that they understand it. I see one student focusing on the problem and not playing with things.” The teacher was able to specifically identify targeted working behaviors using the students’ contributions to the list. Over time, this language was adopted by the classroom. A student who was being interviewed about shirkers and workers commented, “It’s good to know what a shirker is because I used to do that.” Another student said, “No one wants to be a shirker, so they try to be a worker.”

**Addressing Transitional Anxiety**

The lessons in the fraction magnitude unit were different from the curriculum the students had been using all year. In addition, students were working with fractions and many of them had limited understanding of fractions at the onset. The teacher reflected about this class in particular, “I have really low students, some middle to low students and then some very high students.” She felt that this was a draw back because she noticed that for most of the class, the fourth grade CCSS was difficult. The data analysis showed that as a class, when a new concept or model was presented, students would revert back to their shirking behaviors. Floating dialogue increased and student partnerships deteriorated. During one difficult lesson a student reported that her partner told her, “You are going too slowly and don’t get it. You can copy
mine.” This occurred at key moments in the unit: transitioning from part-whole to measurement, placing thirds on a number line, and finding fractions of a whole number.

After lesson 17 (finding fractions of a whole number) two students refused to do their work in spite of teacher and peer efforts at mediation. After the lesson, both students admitted that they were frustrated because they were “really good at the races,” and then the work changed. Perhaps because of the difficulty of the concept, during this lesson, several students chose not to work or rely solely on their partners to give them the answers. One student in particular did not get started right away and when questioned, he said, “I don’t have a paper. My pencil is broken.” The teacher shared that this scenario often happens with this student and it is frustrating.

At these times, students’ conceptual beliefs were challenged and students’ tended to exhibit behavior to avoid conceptual change (Posner et al., 1982). A high press for mathematical discourse (Kazemi & Stipek, 2008) coupled with new curriculum brought about resistance from the classroom community that needed to be addressed.

Before the next lesson, we created another list about behaviors that raise and lower your math ability directly referring to the behavior of the class the day before. The first person to raise their hand to contribute to the lowering ability behaviors list was the student who refused to work at all the lesson before. She added to the list, “Be stubborn”. Students created the following document (Table 9) which was used to identify and support positive sociomathematical mindset norms.
Table 9.

<table>
<thead>
<tr>
<th>Student Created List</th>
<th>Behaviors that raise our ability to do math</th>
</tr>
</thead>
<tbody>
<tr>
<td>Behaviors that lower our ability to do math</td>
<td></td>
</tr>
<tr>
<td>Being stubborn</td>
<td>Thinking in our heads and it was hard but we did it anyways</td>
</tr>
<tr>
<td>Panicking</td>
<td>Ask for help</td>
</tr>
<tr>
<td>Arguing with each other</td>
<td>Figure out how to solve the problem</td>
</tr>
<tr>
<td>Rushing</td>
<td>Debate about answers in a good way</td>
</tr>
<tr>
<td>Shirking</td>
<td>Working with partners and agreeing</td>
</tr>
<tr>
<td>Being scared I won’t finish by recess</td>
<td>Working it out</td>
</tr>
<tr>
<td>WHAT??</td>
<td>Get a new partner when it isn’t working</td>
</tr>
<tr>
<td>Complain</td>
<td>Get other people’s answers and see how they did it</td>
</tr>
<tr>
<td>Showboat</td>
<td>Get a partner if you don’t have one- be assertive</td>
</tr>
<tr>
<td>Rush ahead</td>
<td>Caring about our partner</td>
</tr>
<tr>
<td>Do other things</td>
<td>Figure it out together</td>
</tr>
<tr>
<td>Watch the clock</td>
<td>Thinking positive</td>
</tr>
<tr>
<td>No pencil, can’t find their paper</td>
<td></td>
</tr>
<tr>
<td>I’m bored</td>
<td></td>
</tr>
<tr>
<td>Stop when it gets hard</td>
<td></td>
</tr>
<tr>
<td>Cry</td>
<td></td>
</tr>
<tr>
<td>Refuse to change our mind, get stuck on an answer</td>
<td></td>
</tr>
</tbody>
</table>

During the lesson the teacher cued the students when it was going to look different or become challenging. She posed the options, “Okay, it is going to change a bit and look different. I can complain, whine, lose my things, or I can just see what it is about and take one step towards the learning.” At the end of this session, the teacher reflected, “This was the best discussions. The kids calmed down and owned the math. They were totally okay and volunteering to explain their answers.”

On this day, the class spent one hour and 45 minutes on the math lesson. At the end of the session students were justifying their answers. One student raised her hand and said, “I need more convincing.” She then went on to pick other students to explain further how they arrived at
their answer. This will be added to the mathematical discourse routine from this teaching experiment. Questioning the class, “Who needs more convincing?” allows students to become further convinced when they are not exactly sure why an answer or concept is true. When examining the work from this lesson, this statement was found on a student’s paper (see Figure 53).

I have been convinced.

*Figure 53.* I have been convinced.
CHAPTER V: DISCUSSION

The main goal of this study was to investigate how fourth graders developed understanding of fraction magnitude. Researchers have found that working with fraction magnitudes, distance perspective, and the measurement construct may provide a stronger basis for conceptual fraction understanding of magnitude (Fuchs et al., 2014; Schneider & Siegler, 2010; Schneider, Grabner, & Paetsch, 2009). The fraction magnitude unit was tested in a fourth grade classroom and adjusted to meet the actual learning trajectory of students as they participated in tasks and mathematical discourse.

Part-Whole to Measurement Transition

Consistent with findings from research, a measurement approach provided a richer opportunity for students to work with fractions. Kieren (1988) believed that the part-whole construct and measurement construct should be melded into one approach. The collective outcomes of the unit found that students needed to transition from part-whole thinking to measurement thinking in an explicit way. This did not naturally occur. The context became important for students to distinguish between part-whole verses linear measures. Folding rulers and focusing on the distance from zero, provided opportunities to see fractions as continuous and reaching beyond one whole. Students also reasoned about partitioning a part of a whole (1/4 of ¼) which encourages the visualization of iterating a piece of a unit (1/n) n times in order to create the original whole (Norton & Wilkins, 2013). The unique series of tasks in the fraction magnitude unit provided students the opportunity to reinvent the conventional mathematics of fractions (Gravemeijer, 2004). It could be argued that the measurement focus of the unit provided students an opportunity to invent and construct an understanding of fractions that they did not have before.
Partitioning and Iterating a Number Line

The ability to visualize quantities on the number line is critical for understanding fraction magnitude and comparing fraction quantities. Research states that the ability to accurately compare and place fractions on a number line correlates with mathematics achievement (Booth & Newton, 2012; Fuchs, et al., 2014; Schneider et al., 2009; Siegler, Fazio, Bailey, & Zhou, 2012). Making iterations and being able to estimate locations on the numberline was an important task for understanding fraction magnitude. It was found that students easily mastered halving the number line and splitting the partitions to find halves, fourths, eights, sixteenths, and many could visualize the implications of continued splitting. Partitioning thirds was visually more challenging. Some students were able to generalize a strategy for partitioning based on any denominator, while others were only able to rely on halving. Therefore, when developing tasks, it is important to consider visualization and estimation of units on the numberline. It is also important to consider which fractions students are asked to order. A suggested approach is to first give fractions that involve halving (for example halving halves for fourths, eighths, sixteenths or halving thirds for sixths, twelfths, twenty-fourths). The next step would be to combine two halving schemes, for example fifths (tenths, twentieths) and thirds (sixths, twelfths). In this way, students can develop strategies for iterating number lines.

Scholars have suggested several reasons that children have difficulty with fraction knowledge. Students’ reliance of whole number understanding can lead to misconceptions when dealing with fractions (Mack, 1995; Ni & Zhou, 2005; Post et al., 1985; Siebert & Gaskin, 2006). Interestingly, this was evident in this study. For example, a student asked, “Isn’t 1/6 bigger than 1/3, because 6 is bigger than three?” Because we had constructed a ruler during the lesson with both 1/3 and 1/6 on it, students could visually and physically check this statement. Providing a
physical model that can easily be constructed out of common materials, a paper ruler or a number line on a piece of paper, provided important scaffolding to check the magnitude of fractions and correct the misconception of comparing denominators in order to determine magnitude. The other evidence of whole number bias occurred on the pre and posttest when students were asked to add fractions with different denominators. This class had not received instruction on this task yet, and many added the numerators and the denominators.

The symbolic representation of the fraction is complex, as children must understand not only what the numerator and denominator signify (Ni & Zhou, 2005), but also the infinitely dividing relationship between them (Smith et al., 2005). Labeling rulers that represented continuous contexts and iterating empty number line tasks in the magnitude unit offered repeated opportunities to develop an understanding of the symbolic representation of the fraction. The concept of infinity arose from the experiment in a profound and introspective way.

Changing Children’s Fraction Framework

Conceptual Change

Posner et al. (1982) suggested that conceptual change is necessary for students to reinvent their framework of number. They suggested that the following four situations must occur for learning to take place and to develop understanding of number: cognitive conflict, metaphors to aid explanation, commitment to sensible mathematical understanding, and a competing concept that has more promise. The fraction magnitude unit provided opportunities for the students in this teaching experiment to expand their understanding of fractions as part whole concepts to fractions as measures in order to engage in magnitude thinking. This process that took place within the unit is illustrated in figure 54.
The student performance on ruler and race tasks and consequent mathematical discourse changed the trajectory of the unit by demonstrating a need to shift students’ conceptualization of fractions from equipartitioning or part-whole to the more mature theme of the measurement construct. This is consistent with research findings that recognize the importance of equipartitioning in fraction understanding, but realized that the measurement construct plays a central role to accessing the other fraction constructs (Cramer et al., 2002; Kieren, 1988; Lamon, 2001; Norton & Wilkins, 2013).

An overarching theme in research about fraction understanding is that it takes time for students to integrate the complicated concept of fractions into their schema of knowing (Sfard, 2001, Stafylidou & Vosniadou, 2004; Vosniadou, 2007; Vosniadou & Skopeliti, 2014; Vosniadou, 2014). The opportunities for students in the fraction unit to repeatedly build, label,
and discuss emerging understanding of the measurement construct might be a salient approach to shifting central concepts and providing an extended experience within a similar context. Shifts in pre and posttest levels of fraction conception are provided to support this claim (see Table 10).

Table 10.

<table>
<thead>
<tr>
<th>Class Shifts in Levels of Conception</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Level of Conception</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Level 1</td>
</tr>
<tr>
<td><em>Transition to interiorization</em></td>
</tr>
<tr>
<td>Level 2</td>
</tr>
<tr>
<td><em>Interiorization</em></td>
</tr>
<tr>
<td>Level 3</td>
</tr>
<tr>
<td><em>Transition to Condensation</em></td>
</tr>
<tr>
<td>Level 4</td>
</tr>
<tr>
<td><em>Condensation</em></td>
</tr>
<tr>
<td>Level 5</td>
</tr>
<tr>
<td><em>Transition to Reification</em></td>
</tr>
<tr>
<td>Level 6</td>
</tr>
<tr>
<td><em>Reification</em></td>
</tr>
</tbody>
</table>

**Realized Learning Trajectories: Fractions as Measures and Fraction Iteration**

In order to improve fraction mastery among elementary aged students, research supports the need for a significant shift in the conceptualization of fractions. Sfard (1991) provided a process of integrating the new number system, and maintained that this process takes time. The integrated theory of whole number bias as proposed by Siegler et al. (2011) accounted for the difficulties applying new properties to fractions, which are radically different from whole numbers. The framework theory interprets the difficulties children have with fractions as typical stages of transition (Prediger, 2008). Essentially, current research on fraction understanding agrees that a change must occur in children’s schema of number. The purpose of using design research in this study was to conduct systematic investigations which, in turn, allowed the research team to build micro theories about how this change might occur in a fourth grade class
(Anderson & Shattuck, 2012; McKenney & Reeves, 2013; Steffe & Thompson, 2000; Walker, 2006).

The realized learning trajectory which emerged from the study was a case of challenging students’ naïve theory about fractions with the unit tasks and classroom activity, in order to build a more thorough framework of knowledge (Stafylidou & Vosniadou, 2004). The naïve theories that children initially held about fractions was effectively challenged by the transition from part-whole to measurement to magnitude, and finally using fractions as operators on numbers. Tasks that promoted the desired conceptual change were continued and adjusted, those that were confusing were eliminated. Critical questions emerged from student discourse and were capitalized on by the teacher to reframe and promote a new framework of fraction knowledge. For example, the teacher began to explain the tasks of fraction labeling, partitioning, iterating, and comparing by using the example of a race and questioning the distance from zero on the number line. It should be noted that the instruction was based on the context of the unit, working for the Rational Ruler Company by making rulers and planning races. Thus, critical questions utilize the context to support the learning. Figure 54 is a realized learning trajectory of understanding fractions as measures.
<table>
<thead>
<tr>
<th>Fraction Magnitude and Measurement Concepts</th>
<th>Tasks Promoting Change</th>
<th>Critical Questions: Reframing student thinking</th>
</tr>
</thead>
</table>
| **Part-whole to distance (magnitude, measurement construct)** | Rulers with interesting labels (i.e. 8 equal parts, put one on the fourth fold)  
Identifying pre-iterated units  
Concrete expectation of Measurement Thinking | How much of the whole race? (part-whole)  
How much of a mile? (Measurement) |
| **Discrete to Continuous (beyond one whole)** | Rulers with a variety of wholes  
Identifying pre-iterated units beyond one whole  
Labeling with improper fractions  
Pattern counting and noticing  
Function of numerator and denominator  
Decomposing fractions (greater than 1)  
Write addition and multiplication sentences | How far have I gone if I start at 0 and stop here?  
How much of a mile?  
Who has gone further? |
| **Part of a Part (when the whole is a fraction)** | Rulers that end in ¼,  
Visualizing the whole when given a part  
Identifying pre-iterated units within the part, based on the partitioning of the whole  
Scaffolding: Build a ruler shows the whole, follow with a ruler that is part of that whole with same partitioning  
Decomposing a part into parts, writing multiplication and possibly division sentences. | If this is 1/4, what is the whole? (Iterating out four fourths for one whole).  
Have you made a ruler broken into fourths and sixteenths?  
Would this help?  
*Note: As with partitioning, students were able to visualize the whole if given fractions in halving pattern. This was much more difficult if given fractions in the thirds pattern. |
| **Number Lines (showing fraction magnitude)** | Estimating iteration without pre-marked units  
Scaffolding with separate lines for thirds, halves  
Use string as a tool | How do we find 1/3 on the number line? What does the three mean? (divide line into three equal parts)  
Is 1/3 smaller than 1/6? Check your number line. |
<table>
<thead>
<tr>
<th>Fraction as an Operator (on a whole number)</th>
<th>Building Concept of Infinity and Density of Fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Use manipulatives (tiles) to create the whole number</td>
<td>- Physically folding and splitting number lines</td>
</tr>
<tr>
<td>- Regions (area) can be represented by building arrays (i.e. 6X7 rectangle)</td>
<td>- Decomposing fractions into smaller pieces</td>
</tr>
<tr>
<td>- Denominator = number of equal groups</td>
<td>- Decomposing challenges: Split 8/8 into 10 parts.</td>
</tr>
<tr>
<td>- Numerator = number of the equal groups to take into consideration</td>
<td>- Labeling and encouraging equivalent fractions</td>
</tr>
<tr>
<td></td>
<td>- Wondering about infinity: Can I ever stop?</td>
</tr>
</tbody>
</table>

<p>| | |</p>
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<th></th>
<th></th>
</tr>
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<tbody>
<tr>
<td>Put your finger on 0. Which fraction is further to run to? Which fraction will I get to first?</td>
<td>What is the denominator? Break up the tiles into ___ equal groups. What is the numerator? How many tiles in ____ of the equal groups?</td>
</tr>
</tbody>
</table>

| Figure 54. Realized Learning Trajectory for Understanding Fractions as Measures. |
An important issue that students wrestled with throughout the unit was iteration. Iteration was presented to the students in several forms: pre-iterated units of a whole, pre-iterated units of multiple wholes, pre-iterated units as part of a part, and estimating iterations on an unlabeled unit. The tasks in the unit presented opportunities to work in each of these formats and a realized learning trajectory for iteration was created (see figure 55). Additionally, students who were able to manage the tasks over time, appeared to utilize a widening variety of fraction conceptualizations (Pantziara & Philippou, 2012). See Table 5 for expanded list of skills that students exhibit in each stage of Levels of Fraction Conceptualization (Pantziara & Philippou, 2012)

<table>
<thead>
<tr>
<th>Type of Iterization</th>
<th>Example Task</th>
<th>Skill from Levels of Fraction Conceptualization (Level)</th>
<th>Additional Skills</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pre-iterated units of a whole</strong></td>
<td>Fold a ruler into eight equal parts. Label the fourth fold ½. What should we label the lines?</td>
<td>Select the correct fraction representation as a part of equally divided whole (Level 3)</td>
<td>Visualize the parts of a whole without the distractor of ½.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>List several equivalent fractions to a given fraction (Level 5)</td>
<td>Visualize 4/8 as equivalent to 1/2</td>
</tr>
<tr>
<td><strong>Pre-iterated units of multiple wholes</strong></td>
<td>Fold a ruler into 16 equal pieces. Label the fourth fold 1. What should we label the lines?</td>
<td>Alternate between different representations of a fraction (Level 4)</td>
<td><strong>Iterations beyond 1 continue fraction pattern with improper fractions</strong></td>
</tr>
<tr>
<td></td>
<td>This ruler starts at 0 and ends at 2. Divide it into 6 equal pieces. What should we label the lines?</td>
<td>Find an improper fraction from a given set of objects which constituted the whole (Level 5)</td>
<td><strong>Identifying the whole in terms of number of parts, and labeling iterations accordingly</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td>List several equivalent fractions to a given fraction (Level 5)</td>
<td></td>
</tr>
<tr>
<td>Pre-iterated units as part of a part</td>
<td>Reconstruction of the whole from a given quantity (Level 4)</td>
<td></td>
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<td>----------------------------------</td>
<td>--------------------------------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>This ruler starts at 0 and ends at ¼. Divide into 4 equal parts. What should we label the lines?</td>
<td>Visualizing the iteration continuing to a whole and realizing how many iterated parts are in entire whole (1/4 has 4 equal parts, one whole has 16) Labeling accordingly.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Estimating iterations on an unlabeled unit.</th>
<th>Locate a fraction on a number line with iterations that match the fraction denominator (Level 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Place 2/3, 2/4, 1/8, 6/6, ½, and 6/8 on an un-iterated number line.</td>
<td>Fill in a number line from 0 to ½, with a missing fraction that is divided into thirds (Level 5)</td>
</tr>
<tr>
<td></td>
<td>Create separate number lines for thirds to avoid visual distraction of ½.</td>
</tr>
<tr>
<td></td>
<td>Place several fractions with unlike denominators on an un-iterated number line (Level 6)</td>
</tr>
<tr>
<td></td>
<td>Use visual estimation and magnitude to order fractions vs. an algorithm to determine value.</td>
</tr>
</tbody>
</table>

Figure 55. Realized Learning Trajectory for Fraction Iterization

**Mathematical Discourse: Dual Purpose**

Another goal of this study was to examine how using effective mathematical discourse influenced the design of the unit and shifts in student thinking. The mathematical discourse provided continual insight on how students envisioned the unit tasks. Consequently, the teacher and researcher were able to shift the design of the unit to match the demonstrated need through close attention to class discourse. The teacher often commented that using the district adopted curriculum did not allow for discourse in the classroom due to the pressure to stay on the schedule. She also reflected that when the students can talk with their partners and with the whole group, they “own the math”. A desired outcome of mathematical discourse is to foster mathematical justification as opposed to the delivery of facts and figures (Drageset, 2015). By
purposely embracing the fraction magnitude unit with mathematical discourse, the students and
the teacher had a richer experience. On the student reflection protocol, students confirmed the
mathematical discourse was helpful because:

- I can talk about how I solved it and see if I got it wrong.
- I learn stuff I didn’t know and just got better listening to them. Plus I can get
corrected and learned.
- It makes me feel more confident.
- If I say how I got an answer out loud, I may realize my mistake.
- One time, I liked my answer but I didn’t know how I got it so when someone
went up I understood my answer.

The original unit was designed as a problem-solving approach to discovering and
utilizing the concept of fraction magnitude. It was envisioned that the mathematical discourse
would allow students to practice justification and argumentation as recommended by the CCSS
Mathematical Practices (NGA/CCSSO, 2010). At the beginning of the unit, the tasks proved
difficult for the students and because of this, the confidence needed to effectively argue a
mathematical position was weakened. Therefore, the use of mathematical discourse transitioned
to a dual purpose. Initially, the discourse opened a conversation that began at the beginning of
the lesson and continued throughout. The teacher used the discourse as a strategy to expose
student thinking during tasks, producing multiple opening and closings of micro-mathematical
discourse which attributed to the larger discourse on how to accurately think about the task.
Students played “Pass the Pen” as a silent and lesson supporting discourse, adjusting answers
silently during turns at the board. As students grew in confidence, more of the discourse was
focused on a single problem and students were able to sustain argumentation towards a correct
conclusion to which most of the class agreed. This distinguished two types of mathematical
discourse routines: discourse for solving a rich mathematical task and discourse for a continuous
conversation throughout the lesson. See figure 56 for explanation of how two types of discourse with different purpose emerged from this study.

![Diagram of discourse types](image)

**Figure 56.** Dual purpose of Mathematical Discourse

Social/Emotional verses Academic Stance towards Mathematics

An invisible learning trajectory was developing separate from the tasks of the unit or the use of mathematical discourse. This invisible journey was that of the human condition and the courage it takes to try something new-knowing that you might fail. To address this underlying current, the classroom created lists of “Shirker, Worker, and Showboater” behaviors and “Behaviors that Raise and Lower Your Ability to do Math”. The behaviors that appeared in the lists commonly occur in all classrooms. However, in addition to the lists created of behaviors,
students were asked why the behaviors occurred. This was a critical question for the classroom community. For example, when asked why we shirk, students explained in the following ways:

- We are afraid we aren’t smart enough and everyone find out
- I’m the only one who has to work hard, everyone else gets it right away
- We are afraid people will make fun of us for not knowing

Shirking behaviors were often blamed on a fear of changing social status due to a lack in mathematical ability. The comparison of self to others was also a common theme in the discourse about shirking. Showboating in this class also happened and students felt strongly about the negative results. When students were asked for the showboating behaviors within the class they listed the following behaviors:

- Telling people, “I’m done!” (Floating Dialogue)
- Shouting, “This is easy!” (Floating Dialogue)
- Saying, “I know the answer!” while raising hand to be called on (Floating Dialogue)
- Bragging about test results
- Saying, “I told you so” to their partner when the mathematical discourse proved their answer correct
- Refusing to work with a partner who they believe cannot keep up

Students reported that showboating behaviors “make everyone feel like they can’t do it” (emotional), “that they are better because they say they finish first” (social comparison), and “they make you feel upset because they are bragging and you want to beat them” (social/emotional). The class was also able to reason about why students showboated and this again uncovered a social and emotional response:

- Showboaters not intentionally doing this
- They are worried someone might not like them
- Someone teased them before class and they want to get back at them
- They over exaggerate because they feel like they have to impress people

There was often an emotional response during episodes of transitional anxiety. Several students in the class would cry in response to working independently on a problem, even when reminded that in three minutes they would work with a partner. A particular student exhibited shirking behaviors in the first few lessons. When this student was questioned about this, she admitted, “I’m worried”. She explained that she worried she couldn’t understand the mathematics and that it was too hard. Two other students refused to work during a difficult lesson. One put their head down and cried, the other avoided all work by being aloof, pretending to work, and ignoring efforts by teachers and peers for assistance. When asked what was behind this behavior, one student replied, “I was frustrated when it looked different and I didn’t get it”. The other reflected that she was angry we moved from race rulers to fractions operating on numbers because, “I was good at the race rulers and then you changed it!” This underlying, and very real, social status concern and emotional response from the students appeared to be prompted by the mathematics. The teacher was able to acknowledge these students when they chose to accept help instead of shutting down, were open to new information by continuing to work throughout the lesson, or when they chose to work with a quality partner to relieve the constant worry.

When flooded with emotion (fear, anger, frustration, worry), students were not able to take an academic stance towards the mathematics, instead, it triggered “behaviors that lower your ability to do math”. The teacher capitalized on this realization by students with this conversation:
Teacher: Can you be open to learning something new when you are doing these behaviors (pointing to lowering behaviors)?
Class: No...
Teacher: Did you see these behaviors yesterday?
Class: Yes.
Teacher: This is all the emotion that came up (pointing to lowering behaviors) and we understand you are afraid. It looks new, you haven’t done it before, but you are only in fourth grade, things are going to change in mathematics all the time.

Adjusting Sociomathematical Mindset Norms

As predicted by Posner et al., (1982), challenging current conceptual understanding with a new understanding evokes a resistance. The floating dialogue, working (and not working behaviors), and student positioning during the teaching experiment continually posed a road block to the higher level mathematics and thinking planned in the lessons. Students resisted the invitation to change their minds about fractions with an arsenal of poor behavior, especially when the mathematics changed or became academically taxing. This could be an underlying reason why research has found that running a mathematical discourse the way envisioned by reformers is difficult (Bishop, 2012; Bray, 2011; Cobb, 1986; Cobb, Wood, & Yackel, 1993; Drageset, 2015; Hufferd, Ackles, Fuson, & Sherin, 2004; Kazemi & Stipek, 2008; Manouchehri & St. John, 2006; Sams & Mercer, 2006; Stephan et al., 2001). In order to create a classroom culture that supported positive discourse experience, the underlying sociomathematical mindset norms of the class community had to be addressed and shifted. Adjusting these norms involved helping students internally develop a positive work ethic, develop confidence and cultivate a supportive learning environment among classmates. The teacher wove these concepts into daily instruction, continually working on both the individual student behaviors and the ways in which these affected the classroom community.
Design research provided the opportunity to notice and address the sociomathematical mindset norms of the class and create small interventions. The decisions were driven by teacher-noticing in the interview after each lesson. A realized timeline for adjusting these norms as they emerged was created (see Figure 57).

![Realized timeline for Setting Sociomathematical Mindset Norms](image)

**Figure 58.** Realized timeline for Setting Sociomathematical Mindset Norms

Below is a list of methods a teacher might try in a classroom to adjust a social/emotional reaction to mathematics to an academically oriented stance:

- Identifying shirking, working and showboating behaviors as a class. The list should be from students’ thinking.
• Develop a flexible mindset around the behaviors by reminding students of their choice in their behaviors.

• Work towards an understanding of what is underneath the behaviors. For example, shirkers often feel afraid that others will find out they do not understand and will choose resisting behaviors. Showboaters often position themselves as the expert to maintain the illusion that they are always correct. Underlying this may be an identity built on being smart, and it is very threatening for this to be challenged.

• When behavior occurs, address the classroom community as a whole to identify which type of behavior it represents (Is this shirking or showboating?).

• Address why “we” do it: we are afraid, it is hard, and we don’t want to be wrong….. The use of “we” is essential to setting up a community of positive mindset norms. When we (including the teacher) admit that sometimes we shirk, and sometimes we showboat, students feel that they fit into part of the whole and have a responsibility towards the community.

• Discuss how it impacts the community by limiting the ability to learn mathematics. Create a list of “Behaviors that Raise and Lower Our Ability to do Mathematics.

• The lists are instrumental in applying students’ own understandings (italicized words directly from student created lists) of shirkers and workers. This becomes a classroom document and can be used to:

  1. Adjust undesired behavior: “Three people are turning not looking at the board and are playing with things at their desk. We know that sometimes people shirk when they are afraid they won’t get it, but we are going to wait for them to be ready because they are important.”
2. Praise hard working behavior: “I see five people who have already put their name on their paper and are quietly reading the problem. We know that hard workers start their work right away.”

- Thank students constantly for choosing different behaviors and making small changes towards important things. This reflects back to students how behavior choices affect mindset, and ultimately ability to be open to new learning.

As one student reflected, “I am glad I know what a shirker is because I don’t want to be one, but sometimes I do the things on the list”. Perhaps this study can contribute further understanding towards the creation of a positive sociomathematical mindset in a classroom. Understanding and acknowledging the underlying emotional and social concerns triggered by mathematics might be a significant step in the development of a true classroom community. This tactic might also bolster an academic stance verses a social/emotional reaction to the constant change and increasing difficulty of mathematics curriculum.

**Closing**

At the outset of the experiment the classroom teacher was concerned about taking time away from the district adopted curriculum and feared falling behind. As the unit progressed, the research team realized that the collective tasks provided a foundation for students to explore fractions in an effective way. By creating a ruler, students were able to compare fractions without interference of whole number bias. Students were able to naturally decompose fractions and create addition and multiplication sentences. Some students were able to see the division of fractions, a concept that is addressed in later grades. Understanding fraction magnitude is correlated with future math achievement (Fuchs et al., 2014; Schneider et al., 2009; Siegler et al., 2012) and might aid in overcoming whole number bias (Fuchs et al., 2014; Torbeyns et al.,
 Asking students to order several fractions by magnitude or place them on an empty number line (see Figure 10) is a common task found in curriculum. This research found that given experience and strategies (using a number line for thirds and another for halves), students can be successful at this task.

This study was limited by the number of classrooms and students who participated. The intimate workings of a teaching experiment may provide unintentional bias in findings. Therefore, future research might include further testing the magnitude unit in a variety of settings, perhaps as an intervention for students in upper grades who struggle with fraction understanding. The opportunity for an in-service teacher to participate in reflective practice during a teaching experiment in order to adapt to and fully adopt curriculum reform might also be investigated. A continued examination of developing sociomathematical mindset norms in the classroom and the impact on the overall classroom mindset and individual student mindset towards mathematics would be an insightful study.

In 2014, Fuchs et al. reflected that little is known about why some children easily understand fractions and others don’t. Additionally, their research called for a change in the business as usual approach to teaching fractions (Fuchs et al., 2014). This teaching experiment uncovered micro theories on children’s misconceptions by testing a unique approach to visualizing fraction magnitude, while requiring students to engage in mathematical discourse. At the conclusion of the study, it was found that the fraction unit based on magnitude combined with mathematical discourse did shift some students’ understanding of fractions in a fourth grade class. The unit was carefully converted to match the actual needs of the class while learning about fraction magnitude using design research. Further insight was gained in what causes fraction difficulty for fourth graders and how building a measurement understanding with the
tasks in the unit provided an opportunity for rich mathematical discourse and conceptual change.

Two frameworks were proposed to add to the current body of data concerning fraction magnitude: the Framework for Understanding Fractions as Measures (Figure 54) and the Framework for Fraction Iterization (Figure 55). The study produces two enactments of mathematical discourse; continuous and for solving a rich task (see Figure 56). Both were used in the teaching experiment, and the discourse routine (see Appendix B) was used during the two types of discourse.

Through careful analysis, an underlying and negatively skewed mindset towards learning mathematics was discovered within the class. The teacher and researcher adjusted each lesson to address this issue, which was termed, “sociomathematical mindset norms”. The underlying emotions for negative sociomathematical mindset norms were also noted, and it was found that the difficulty of the mathematics often triggered emotional and social responses. The research team worked to create positive sociomathematical mindset norms and encourage an academic stance towards the mathematics. This was done in order to increase the effectiveness of the unit tasks, student participation in the discourse, and student individual work.

In closing, the magnitude approach to learning fractions was effective and could address the need to improve fraction instruction in the elementary grades. Additionally, the mathematical discourse woven through all the tasks provided ample opportunity for students and the teacher to examine and learn from the classroom understanding of fractions. The magnitude and discourse together produced positive results. As one student commented at the end of the experiment: “I love sharing my ideas with the class so they can get smarter and I can too.”
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Tobias, J. M. (2009). Preservice elementary teachers’ development of rational number understanding through the social perspective and the relationship among social and


Appendix A. Whole Class Mathematical Discourse Routine

Following a predictable routine for mathematical discourse tends to augment opportunities for increased reasoning and mathematical understanding (Kosko, Rougee, & Herbst, 2014; Mercer & Sams, 2006; Mottier-Lopez & Allal, 2007; Nachlieli & Tabach, 2012, Wood & Kalinec, 2012). The discourse routine followed in this study was created by the researcher along with a former co-teacher. The discourse routine reflects the following findings in the research regarding promoting effective mathematical discourse in the classroom.

**Reasoning Community**

In order to create an environment that supports mathematical reasoning through discourse, teachers must ensure a socially safe situation for all students in order to successfully lead a mathematical discourse (Bishop, 2011; Resnick, O’Connor, & Michaels, 2007). This routine accounts for keeping students socially safe and might encourage greater commitment to the reasoning community (Bishop, 2011). It prevents the issue of “turn sharks” (Erickson, 1996) and works towards eliminating negative student positioning (DeJarnette & Gonzales, 2015) that might influence the discourse.

**Development of Social and Sociomathematical Norms**

Mathematical understanding is addressed in tandem with social and sociomathematical norms during this discourse routine. Two examples of this are provided:

**Behaviors that raise and lower our math ability.** Students are asked to reflect on what behaviors fourth graders use to either get out of doing mathematics, or to increase their ability in mathematics. A poster is made and posted in the classroom. The benefits of this conversation with the students and the creation of the poster is that behaviors during the discourse can be addressed using this language. For example, when the fourth grade class first started with
mathematical discourse, a student shared an answer and the rest of the student laughed. Immediately, they were asked to reflect about this behavior. “When we laugh at other people’s answers, does that raise or lower our ability to do mathematics?” A student brought up that they thought that the student who spoke was trying to make the class laugh. Again, the class is questioned, “Why do we do this? Why do we try to make people laugh when we’re wrestling with important math ideas? Does this raise or lower our ability to do math?”

**Shirkers, Workers, and Showboaters.** Students have often identified themselves as smart or dumb (Bray, 2011) at mathematics and their mathematics classroom behavior often reflects this self-imposed identity. A discussion is held with the students regarding three attitudes towards mathematics. *Shirking* behaviors are actions that prevent you from learning the mathematics. These students position themselves as novice and defer to experts while positioning themselves as less able (DeJarnette & González, 2015). *Showboating* behaviors are actions that are meant to establish expert status (DeJarnette & González, 2015). Experts often refuse to consider the contributions of others as correct and have a desire to establish mathematical authority. Both shirking and showboating behaviors are detrimental to the establishment of social and sociomathematical norms (Heyd-Metzuyanim & Sfard, 2012). In the mathematics classroom, we are looking for *workers*. Working students are working hard at learning mathematics. This discussion also allows students to reflect and refrain their behavior during mathematical discourse.

**High expectations for participation.** The students are expected to listen to their peers. We call for this attention by asking for “Eyes and Hands”, this requires everything out of hands and eyes looking at the speaker. The student who is going to share a mathematical idea or
argument does not begin speaking until everyone is listening. Teachers monitor and instruct students on how to accomplish his and why it is so important.

**The Mathematical Discourse Routine**

1. Everyone gives eyes and hands. Do not start until this is so.

2. Problem is projected in a place where all students can see. They read the problem to themselves (no one should be writing or trying to solve the problem).

3. Together, students read the problem out loud. Stop and begin again if students are not reading along with you.
   - Ask students, “What about this problem might be difficult for students? What might be hard for us to understand?”
   - It is very important to imply in this question, what might be difficult for US as a class as we try to solve this problem. Avoid asking, “What might be difficult for you? What don’t YOU understand?”
   - Underline or highlight the parts of the problem that the students bring up. Resist the temptation to explain the whole problem. The difficult spots are where new things may be learned.

4. Independent Work: Students work in their journal or on a handout to begin their initial ideas on how to approach/solve the problem. Let them work for 3 to 4 minutes. Avoid answering panicked questions about the problem. Instead, *narrate the room*. This means walking around and informing the class about where others have started:
   
   I see two people making a drawing and labeling it with the numbers 15 and 13. Another person is doing a multiplication problem. Four people are taking the time to reread the problem.
Refrain from naming the individuals who are doing this. This promotes a greater sense of a cohesive mathematical reasoning community—here is how we are approaching this problem.

5. Students work in partners or triads to come to agreement on an answer to the problem. They discuss their reasoning with others regarding why they believe their answer is correct. Teachers monitor the room. Again, avoid giving away hints to the answer. As soon as the community of learners understands that answers and hints are being provided by the mathematical authority in the room (the teacher), it takes away the reason for them to work through the difficult parts of the problem.

6. When it feels like most groups have an answer, call for eyes and hands. Make sure students are not writing still. In this step, there will be a call for answers. The point of the call for answers is to get EVERY answer that children have to the problem being solved. Make a list in a visible place on the board. Keep asking for answers until everyone puts their answer on the list. Refrain from saying things like, “Good answer” or “I like that thinking” when the correct answer is put on the board. As soon as the mathematical authority (the teacher) has blessed one of the answers, the discourse is over. Instead, try saying “Thank you.” Pay attention to student response to others answers as well. There should not be comments made regarding surprising answers. A call for answers gives the teacher information about what students are thinking.

- Everyone has the same answer, students may have a good understanding
- All answers within a range, students are near the answer and need to work on accuracy
- Wide range of answers, this is a difficult problem for the students

With the list of answers on the board, it is often good to have the students reflect:

- What do you notice about the list of answers?
- Does the list center on a range of numbers?
• Are there any answers that are really high or really low (outliers)?

A purpose for doing this is to allow students separation from their own personal answer and rather focus on the answers the class is considering.

7. At this point, ask if someone wants to come up and support an answer from the list. Call on a student. Ask them to identify which answer they are supporting and record this with a check by the number (this becomes important). Students who share should try to follow Toulmin, Reike, & Janik’s (1979) framework for practical reasoning (see Figure 1). In this model, the student provides data or evidence for their answer, a warrant providing mathematical reasoning, and a claim that states why their answer is correct. The student in the diagram is arguing why 2/3 is place on a number line 1/3 away from 1.
8. After a student shares their work, have them leave their work displayed and sit down in their desk. We often clap for their presentation and then have them sit down. Ask the students, “If we as a class turned this answer in as our answer, what do you wonder about it?” It is important that the students discuss the math, not the person who shared the idea. Remind them to say, “I wonder why 2/3 doesn’t mean 2 wholes out of 3 wholes?” This allows students to question what is going on mathematically and hone in on possible errors or misconceptions.

*Figure 1.* Toulmin, Reike, & Janik’s (1979) framework for practical reasoning. Adapted from Rasmussen & Stephan (2008, p. 197).
9. Ask students, “Does anyone want to change their answer?” This is an important question. If someone does, ask them to share why they changed their mind. They will often clarify for the other students why one answer is correct over another. Cross their original answer off the list (if others still think it is that answer leave it on) and put a check by the answer that they changed it to (this often builds support for an answer).

10. Repeat steps 8 – 10 until the class has come to an agreement on an answer. Often, the class will be polarized between two or three answers. If you run out of time for the discourse, do not give away the answer. Table it for the next day. You can give the class information about the answer by:

- Asking for a show of hands on who is supporting which answer
- If you have more than two answers, you can remove one from the list. In particularly heated discourse, often children will get stuck on one incorrect answer and a few will keep arguing for the correct one. Before they leave for the day you can say, “I will tell you this. It is NOT this answer.” This provides disequilibrium both mathematically and socially. Begin the discussion again the next day.

11. Once the class has arrived on an answer, ask the class, “Does anyone need more convincing?” If someone does, invite them to call on peers for more explanations.*

*This step added to the routine as a result of the teaching experiment.
Appendix B. Levels of fraction conceptualization pre/posttest

(Adapted from Pantziara & Philippou, 2012)

Name: ______________________

A1. Circle the shapes that show $\frac{1}{4}$ shaded.

A2. Write in a fraction what part of the shapes are triangles.

A3. Circle $\frac{2}{3}$ of the whole.

A4. Place the fraction $\frac{3}{5}$ on the number line.

A5. Fill in the fraction with the correct numerator.

$$\frac{1}{3} = \underline{\quad}$$

A6. Circle the bigger fraction.

$$\frac{4}{7}, \frac{2}{7}$$

A7. Find the sum.

$$\frac{1}{6} + \frac{3}{6} = \underline{\quad}$$

A8. Place the following fractions on the number line.

$$\frac{1}{3}, \frac{5}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{2}$$

B1. Write in the fraction of the shaded part of each shape.

(a) _______   (b) _______
B2. The picture on the left represents a fraction. Use the picture on the right to represent the same fraction. The example will help you. Example:

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{image1}} \\
\text{\includegraphics[width=0.5\textwidth]{image2}}
\end{array}
\]

B3. If four A’s represent \(2/3\) of the whole, draw the whole set of As in the box below.

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{image3}} \\
\text{\includegraphics[width=0.5\textwidth]{image4}}
\end{array}
\]

B4. Place the fraction that represents the shaded part of shape B on the number line.

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{image5}}
\end{array}
\]

B5. Which of the five bars represents the same fraction as the one shown in the rectangle?

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{image6}}
\end{array}
\]

\[
\begin{array}{c}
A \\
B \\
C \\
D \\
E
\end{array}
\]

B6. Show a model to compare the fractions below. Which is bigger?

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{image7}}
\end{array}
\]

B7. Find the sum.

\[
\begin{array}{c}
2/4 + 1/8 =
\end{array}
\]
B8. Place these fractions on the number line.

\[ \frac{1}{4}, \frac{2}{3}, \frac{2}{8}, \frac{1}{10} \]

Write \( <, >, = \) to make the sentences true.

\[ \frac{1}{4} \quad \_\quad \frac{1}{10} \]
\[ \frac{1}{4} \quad \_\quad \frac{2}{8} \]
\[ \frac{2}{3} \quad \_\quad \frac{2}{8} \]

C1. Write the fraction that represents the shaded part of the shape.

C2. Circle the pictures that represent the same fraction.

C3. In the box, draw \( \frac{7}{6} \) of the objects below.

C4. Write the correct fraction in the box below.
C5. Two friends ordered pizzas the exact same size and ate some of their pizza several times during the day.
   • Harry ate \( \frac{1}{8} \) of his pizza 5 times.
   • Wendy ate \( \frac{1}{16} \) of her pizza 8 times.

Who ate more pizza? Draw a model. Write addition or multiplication sentences.

Model:

Equations:

Who ate more?

C6. Write a fraction bigger than \( \frac{1}{9} \) and smaller than \( \frac{1}{8} \).

D1. What equivalent fractions do you know for \( \frac{1}{4} \)? Please write some here.

C7. Find the sum. Make a model to show how you added the two fractions.

\[
\frac{2}{3} + \frac{1}{6} =
\]

Model:

D2. How many equivalent fractions for \( \frac{1}{3} \) do you think there are?

D3. How many fraction numbers are there between 0 and 1? Please give an estimate.
<table>
<thead>
<tr>
<th>Item</th>
<th>Correct?</th>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>A2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>A3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>A4</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>A5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>A6</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>A7</td>
<td>1, 2</td>
<td></td>
</tr>
<tr>
<td>A8</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>B1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>B2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>B3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>B4</td>
<td>4, 5</td>
<td></td>
</tr>
<tr>
<td>B5</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>B6</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>B7</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>B8</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>C1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>C2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>C3</td>
<td>4</td>
<td></td>
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<td>C4</td>
<td>5</td>
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<td>C5</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>C6</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>C7</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>C8</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>D1</td>
<td></td>
<td>Infinite divisibility</td>
</tr>
<tr>
<td>D2</td>
<td></td>
<td>Infinite divisibility</td>
</tr>
<tr>
<td>D3</td>
<td></td>
<td>Fraction Density</td>
</tr>
</tbody>
</table>

Adapted from Pantziara & Philippou (2012)

**Scoring**

**Overall Score:** _____ / 27 points

**Level of Fraction Conception:**

- Level 1: _____/ 1
- Level 2: _____/ 3
- Level 3: _____/ 8
- Level 4: _____/5
- Level 5: _____/4
- Level 6: _____/5

**Density/Divisibility Understanding** _____/ 3
Adjustments made to pre/posttest as a result of teaching experiment:

A1: “shaded” added. Confusing to students who see equally divided sections. All figures have a shaded section now.

B6: Only one model required and fractions changed to those that were put on rulers and races. This provides scaffolding for creating a model of the fractions that compare.

B7: Fractions changed to those that were placed on rulers and races to provide scaffolding for adding fractions with unlike denominators. This is a concept that is taught in later grades.

C1: The original figure had two triangles shaded which showed 2/16 or 1/8. The error in the problem would be to count the shaded figure out of the pieces all together which was also 1/8. It was unclear if students had simplified the fraction or committed an error. The answer is now 3/16.

C2: Picture of the people removed. Difficult to tell what the fraction was; ¼ or ¾.

C5: Added model and equations because the unit specifically teaches repeated addition equations and multiplication equations. This might increase accuracy.
Appendix C: Modified Fraction Magnitude Unit

Rulers, Races, and Regions: A Unit on Fraction Magnitude

This is a fraction measurement unit that uses the fraction measurement construct to deepen children’s understanding of fractional numbers overall. Lesson cycles in the unit will include three components, an initial activity, a problem solving task with mathematical discourse, and an independent activity.

Initial Activity: The initial activity will set the stage for the day’s lesson. It is intended to help the student understand the context of the lesson (rulers, races, lab measures, etc.) and to practice the type of knowledge needed for the problem solving task. Often, students will need to have a whole class mathematical discourse to work with this activity. This will help identify and work out misconceptions present in the community of learners.

Problem Solving Task: There will be a problem to solve based on the context set up in the initial activity. The problem solving task is an opportunity to use whole class mathematical discourse to develop a taken as shared (Cobb et al., 2001) understanding of the fraction measurement construct. The Whole Class Mathematical Discourse Routine as set up in this study will be followed for this part of the lesson.

Student Work:

Group Work: Students will be asked to work in small groups to solve tasks that will stretch their thinking. A whole class mathematical discourse can naturally arise from this type of work. Instead of correcting the paper as a class, students are encouraged to critique and argue the answers in order to solidify mathematical understanding.
Individual Work: Students will be asked to solve tasks that can be very similar to the problem solving tasks. Students are allowed to use models and equations from the problem solving task to assist in their thinking. Teacher intervention during this time and re-teaching of the lesson will be minimal, reserved for students with special needs (i.e. assistance in reading the problem). This means that students will be asked to work hard on their own to solve the problem in the independent task. This intentionally puts pressure on the value of being involved in the initial activity, group discussion, and whole class discourse. The independent work will assess the development of individual understanding from the experiences embedded in the initial activity, problem solving task, and mathematical discourse.

This cycle will circle through the eighteen lessons in this unit. Each lesson is intended to last from 45 minutes to an hour.
Lesson One: Pretest

Lesson one will consist of a pretest given to the students. The pretest consists of 30 questions. The questions focus on levels of conceptualization of fractions (Pantziara & Philippou, 2012; Sfard, 1991). The test will be scored and each student will receive an overall score out of 30 points possible as well as a designated level: basic interiorization, interiorization, transition to condensation, condensation, transition to reification, or reification. The questions are organized by level. Based on students’ ability to correctly answer the questions at each level, a determination will be made (see Table 1). The pretest will also include three questions regarding number density and the infinite property of fractions. These questions will be posed throughout the unit for discourse and developing the density property of fractions (DeWolf & Vosniadou, 2014; Smith, Solomon, & Carey, 2001).

Table 1. Levels of Fraction Conceptualization

<table>
<thead>
<tr>
<th>Difficulty Level of Fraction Conceptualization</th>
<th>Students in this phase are able to...</th>
<th>Corresponding Items on test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1</td>
<td>Perform a simple step-by-step procedure for computing sum of fractions with like denominators.</td>
<td>A7</td>
</tr>
<tr>
<td>Basic Interiorization</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Level 2</td>
<td>Perform simple fraction procedures</td>
<td>A5</td>
</tr>
<tr>
<td>Interiorization</td>
<td></td>
<td>A6</td>
</tr>
<tr>
<td></td>
<td>Fill in the missing numerators in two equivalent fractions</td>
<td>A5</td>
</tr>
<tr>
<td></td>
<td>Compare two fractions with common denominators</td>
<td>A6</td>
</tr>
<tr>
<td>Level 3</td>
<td>Find the fraction of a set of discrete objects</td>
<td>A2</td>
</tr>
<tr>
<td>Transition to Condensation</td>
<td></td>
<td>A3</td>
</tr>
<tr>
<td></td>
<td>Select the correct fraction representation as a part of equally divided whole</td>
<td>A1</td>
</tr>
<tr>
<td></td>
<td>Locate a fraction on a number line with iterations that match the fraction denominator</td>
<td>A4</td>
</tr>
<tr>
<td></td>
<td>Calculate the sum of two fractions with unlike denominators</td>
<td>B7</td>
</tr>
<tr>
<td>Level 4</td>
<td>Condensation</td>
<td>Task Description</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Divide a given representation of a fraction into equal parts and write fraction of the shaded area</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level 5</th>
<th>Transition to Reification</th>
<th>Task Description</th>
<th>B2</th>
<th>B4</th>
<th>B5</th>
<th>B3</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alternate between different representations of a fraction</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reconstruction of the whole from a given quantity (i.e. if 2/3 equals four objects, what is the value of 1/3? 3/3?)</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level 6</th>
<th>Reification</th>
<th>Task Description</th>
<th>B6</th>
<th>B4</th>
<th>C4</th>
<th>B3</th>
<th>B8*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compare two fractions in more than one way</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Find fraction represented as a continuous quantity (3/4) and place it on a number line divided into different parts (eighths) than the ones represented by fraction’s denominator (fourths).</td>
<td></td>
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<tr>
<td>Fill in a number line from 0 to ½, with a missing fraction that is divided into thirds.</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>Find an improper fraction from a given set of objects which constituted the whole</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>List several equivalent fractions to a given fraction.</td>
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<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level 6</th>
<th>Reification</th>
<th>Task Description</th>
<th>C6</th>
<th>C7</th>
<th>C8*</th>
<th>A8*</th>
<th>C5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find a fraction between two consecutive fractions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Draw process of fraction addition</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Place several fractions with unlike denominators on an un-iterated number line</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Place several fractions on a number line including negative fractions, improper fractions, and mixed numbers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use the variable x to represent a problem of fraction equivalence</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Pre/Post Test

A1. Circle the shapes that show ¼ shaded.

A2. Write in a fraction what part of the shapes are triangles.

A3. Circle 2/3 of the whole.

A4. Place the fraction 3/5 on the number line.

A5. Fill in the fraction with the correct numerator.

\[
\frac{1}{3} = \_\_\_
\]

A6. Circle the bigger fraction.

\[
\frac{4}{7} \quad \frac{2}{7}
\]

A7. Find the sum.

\[
\frac{1}{6} + \frac{3}{6} =
\]

A8. Place the following fractions on the number line.

\[
\frac{1}{3}, \frac{5}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{2}
\]

B1. Write in the fraction of the shaded part of each shape.

(a) _____  (b) _____
B2. The picture on the left represents a fraction. Use the picture on the right to represent the same fraction. The example will help you. Example:

\[
\begin{array}{c}
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\end{array} =
\begin{array}{c}
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\end{array}
\]

? =
\[
\begin{array}{c}
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\end{array}
\]

B3. If four A’s represent 2/3 of the whole, draw the whole set of A’s in the box below.

\[
\begin{array}{c}
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\end{array} = \frac{2}{3}
\]

\[
\begin{array}{c}
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\text{ } \text{ } \text{ } \\
\end{array} = 1
\]

B4. Place the fraction that represents the shaded part of shape B on the number line.

B5. Which of the five bars represents the same fraction as the one shown in the rectangle?

\[
\begin{array}{c}
\text{ } \text{ } \\
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\text{ } \text{ } \\
\text{ } \text{ } \\
\end{array}
\]

A

B

C

D

E

B6. Show a model to compare the fractions below. Which is bigger?

\[
\frac{7}{16} \quad \frac{5}{8}
\]

B7. Find the sum.

\[
\frac{2}{4} + \frac{1}{8} =
\]
C5. Two friends ordered pizzas the exact same size and ate some of their pizza several times during the day.
   • Harry ate 1/8 of his pizza 5 times.
   • Wendy ate 1/16 of her pizza 8 times.

Who ate more pizza? Draw a model. Write addition or multiplication sentences.

Model:

C6. Write a fraction bigger than 1/9 and smaller than 1/8.

Who ate more?

C7. Find the sum. Make a model to show how you added the two fractions.

\[
\frac{2}{3} + \frac{1}{6} =
\]

Model:

C8. Place these fractions on the number line.

1/5, 2/3, 5/8, 5/5, 6/5

D1. What equivalent fractions do you know for \( \frac{1}{2} \)? Please write some here.

D2. How many equivalent fractions for \( \frac{1}{2} \) do you think there are?

D3. How many fraction numbers are there between 0 and 1? Please give an estimate.
Scoring the Pre/Post Test

<table>
<thead>
<tr>
<th>Item</th>
<th>Correct?</th>
<th>Level</th>
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</thead>
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<tr>
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</tr>
<tr>
<td>D3</td>
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</table>

Adapted from Pantziara & Philippou (2012)

**Scoring**

**Overall Score:** _____ / 27 points

**Level of Fraction Conception:**

Level 1: _____ / 1

Level 2: _____ / 3

Level 3: _____ / 8

Level 4: _____ / 5

Level 5: _____ / 4

Level 6: _____ / 5

**Density/Divisibility Understanding:** _____ / 3
Lesson One: Part 1: The Rational Ruler Company

The Rational Ruler Company makes rulers that show fractions. Sometimes the whole ruler will equal 1. Sometimes the whole ruler will equal 2, and sometimes the whole thing will equal 1/3. The rulers do not measure a standard unit of measure like inches or cm. Some rulers might start at 0, and some might start at -1. They are made to show fractions of the unit. They are highly specialized math tools, and they would like our help to make some!

The Rational Ruler Company has rules we will follow to make the rulers (make a poster or flipchart of these rules to be posted while they work).

**Rational Ruler Regulations**

1. **Use fractions, improper fractions, and whole numbers for the rulers (no mixed numbers).**

2. **The spaces made on the ruler must be made up of equal sections.**

3. **The lines are labeled with the correct fractions.**

4. **Rational Rulers are neat and tidy.**

Today, we will be having on the job training. (Using paper strips, model for students how to create the following rulers. For each ruler, give them individual/pair/whole class discourse time to come up with the solutions):

1. Four equal sections, label the center 1. This ruler starts at 0 and ends at 2. Label the ends. What are the lines? How can we use improper instead of mixed numbers? Are there any equivalent fractions?

Discuss. Is the first fold a fraction of the whole bar or of the one in the middle of the bar?
Lesson 2

Six equal sections (demonstrate how to fold into thirds, then in half). Label the center 1. This ruler starts at 0 and ends at 2. Label the ends. How can we use improper instead of mixed numbers? Are there any equivalent fractions?

Lead a discussion on how to label the lines. Should they be 1/3 or 1/6? What is the whole for each fraction?

Lesson 3

Eight equal sections, label the middle fold 1. This ruler starts at 0, and ends at 2.

Move toward generalizations:

Solve these two problems:

1. Jose is running a race. He runs from the start to the first fold. The race is two miles. How much of the race has he finished when he stops at the first fold?

Looking at when would the answer be 1/8? When we have a 2 mile race, and we run to the first fold, how much OF THE RACE have we completed? 1/8 of the race. The whole = 2. The fraction is an operator, operating on the whole. Interestingly, 1/8 of 2 or 1/8 X 2 = 1/4. This is *Part of a Whole Thinking* as well because students are focusing on the whole ruler, even when the whole ruler is greater than one.

2. Talia is running the same race. She runs from the start to the first fold. How many miles has she run?

Students wonder:

When would the answer be 1/4? When we have a two mile race and I want to know how far I have run in miles when I get to the first fold. This would be 1/4 of a mile. This would be the
measurement concept, and it applies to placing fractions on a number line, which implies distance. This is *Measurement Thinking*.

How do we know which case is which?

Students each write on a sticky note the answer to the following:

When is the answer $1/8$? (Part/Whole Thinking)

When is the answer $\frac{1}{4}$? (Measurement Thinking)

Make class poster of student understanding of the two cases labeling each type of thinking.
Student Resource: Rational Ruler Company Rules

Have students cut and paste these rules in their journals to reference while working.

Rational Ruler Regulations

1. Use fractions, improper fractions, and whole numbers for the rulers (no mixed numbers).

2. The spaces made on the ruler must be made up of equal sections.

3. The lines are labeled with the correct fractions.

4. Rational Rulers are neat and tidy.
Lesson Four: Labeling Rulers using Measurement Thinking

In this lesson students will be working on labeling rulers that have been partitioned for them. Review the two types of thinking discussed: measurement thinking and part/whole thinking. The independent work will require measurement thinking.

Students will work with the idea of improper fractions and labeling whole numbers as fractions. They will also begin to investigate using the ruler to add and subtract fractions with the same denominator.

Initial Task:

Students will again look at both perspectives and the goal is to prompt their understanding that in making rulers (and making number lines). As they move to independent work, remind them it is MEASUREMENT THINKING.
Stuart was running a race that is 4 km long. He ran to the seventh line. How much of the race has he finished?

What is the whole in this case?

How many equal pieces in the whole? (Denominator)

How many equal pieces did Stuart run out of the whole race?

Write the fraction that tells how much of the race he has finished:
Stuart was running a race that is 4 km long. He ran to the seventh line. How many kilometers has Stuart run?

What is the whole in this case? (Measurement)

How many pieces are in each unit? (Denominator)

How many equal pieces did Stuart run? (Numerator)

Write the fraction that shows how many kilometers Stuart has run:
Student Work: Make some Rational Rulers

Employee Name: _______________________

You will be making two Rational Rulers on your own. For each ruler job, we will be using measurement thinking. Be a hard worker!

Rational Ruler 1: Independent, partner, class discussion.

This ruler shows two units. Use measurement thinking to label the lines. Use improper fractions. Label the whole numbers with fractions. (Example: 1 = 4/4)

1. Sam is running a race that is two miles long. She runs to the second line. How many miles has she run so far?

2. Lily is running the race to and runs to the seventh line. How many miles of the race has she run so far?

3. How many miles does Lily have left to run?
Rational Ruler 2: Independent Work

This ruler shows two units. Use measurement thinking to label the lines. Use improper fractions. Label the whole numbers with fractions. (Example: $1 = \frac{4}{4}$)

Harry is driving two miles. He stops at the fourth line.

1. How many miles has he driven?

2. How many miles does he have left to go to get to 2 miles?

Rational Ruler 3: Independent Challenge

This ruler shows three units. Use measurement thinking to label the lines. Use improper fractions. Label the whole numbers with fractions. (Example: $1 = \frac{4}{4}$)

Phil wants to ride his bike three miles. He rides to the fifth line. How many miles has he ridden so far?
Homework: Lesson Three

Name: _______________________________

Use measurement thinking to label the lines of the following Rational Rulers.

1.

2.

3.
Lesson Five: Making Combinations from Rational Rulers

Initial Task:
Each student needs access to strips of construction paper (2 ¼“ X 12”, use 9 X 12 light colored construction paper).

Task: To make a ruler for the “Rational Ruler Company”. This company makes lots of different kinds of rulers, they don’t match our inch and cm rulers. Each ruler has different specifications. Today we are going to make a ruler for this company.

Fold the strip in ½. Repeat 2 more times so there are 8 sections in the strip. Exact folding is important, monitor for accuracy. Model for students where to label the ruler, ½.

The task it to label the other lines of the ruler. Draw student’s attention to the following:
WHAT IS THE WHOLE?
HOW MANY EQUAL PIECES ARE IN THE WHOLE?

Students will work quietly for two minutes and then be able to discuss with partners and table mates. When students have answers, ask several to come up and show their labels for the ruler.

Use the discourse routine if there are many different understandings. What is the correct ruler labeling with ½ at the center? It is important that the students construct the answer to this problem through argument and reasoning. The correct labels are only provided as a guide. **Do not give the answers, resist the temptation to save the students from mathematical struggle-this is how they may learn new concepts.**
Use this ruler to find as many combinations as you can to make $\frac{8}{8}$. Number bonds are an excellent way to introduce this idea. Challenge students to break $\frac{8}{8}$ into 2, 3, 4, 5, 6, 7, and 8 pieces. For those who finish early, can they do 9 pieces? 10? 11? Students write the addition problem for each number bond as shown.

\[
\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{2}{8} + \frac{3}{8} = \frac{8}{8}
\]
\[
3\left(\frac{1}{8}\right) + \frac{2}{8} + \frac{3}{8} = \frac{8}{8}
\]

If possible, move to repeated addition of same unit = multiplication

\[
\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{8}{8} \quad (\frac{1}{8}) = \frac{8}{8} = 1
\]

Glue the ruler in their journals and have them list all decomposition equations. Use the ruler to count out equations for accuracy.
Lesson 6

Have students get another ruler paper, because the Rational Ruler Company has put in an order for a different ruler. Show them how to label the ruler in this way:

![Ruler diagram](image)

Compare this ruler to the one in Lesson 5. Ask students the following questions:

What is the whole in this case? (Clarify this is measurement thinking as we have several wholes in the ruler. Using a colored pencil or pen, have them clearly delineate each whole after they have agreed upon what constitutes a whole (4/4).

How many pieces are in each unit? (Denominator)

For this ruler, label with fractions and whole numbers, have students use improper fractions. Have them list equivalent fractions if they pertain to the structure of the ruler. For example, it would be good to include 2/4 under ½ because we are counting by fourths. Similarly, students may see 4/4 as also 1.

Give students time to independently label this new ruler. Then move to small groups and finally a whole class discourse. What is the correct way to label this ruler? (Let the students debate this understanding).
**Use this ruler to write ways to decompose 7/4.** Make sure to use “decompose” to indicate the action of breaking into smaller parts. Use number bonds as shown in Lesson 5. Link repeated addition to multiplication.

**Example:** ¼ + ¼ + ¼ + ¼+ ¼+ ¼+ ¼= 7/4

\[
7 \left(\frac{1}{4}\right) = \frac{7}{4}
\]

Have students glue the rulers into their journals under the title: Rational Rulers. Glue ½ of the ruler so it can fold and fit in their journal. Have them write the equations generated in the discussion under the rulers. Make a list of the equations to hang in the room. Model the multiplication as 4 (1/4) = 4/4 = 1.

Create a class stickperson that reminds them of important things to remember when designing rulers for the Rational Ruler Company. Share ideas if time, create a list to be posted.

Make sure to think about what the ruler is counting by. Don’t be fooled by the denominator of the fraction given on the ruler!
Lesson Seven: Rational Ruler Work with Addition and Multiplication Equations

This lesson continues with addressing the infinite quality of a ruler. Essentially, because it ends at 2, it can be interpreted as going on into infinity. Students will draw a number line under their rulers to show that a ruler implies that it goes to infinity.

Example:
Lesson Seven Whole Class Work: Make Some Rational Rulers

Employee Name: _______________________

You will be making two Rational Rulers. For each ruler job, create it out of paper and then glue it down. Draw a number line under your ruler to show that it continues beyond. Be a hard worker!

**Rational Ruler 1:**

Fold the ruler into 12 equal sections. Put 1/3 on the second line after the first section.

Label each line. Label where it starts and where it ends. Draw a labeled number line underneath your ruler.

Glue it here:

![Rational Ruler 1]

Create 2 number bonds showing how you can decompose 8/6. Break it up 4 ways and 8 ways.

Write an addition and multiplication sentence for each bond.
Lesson Seven Individual Work: Rational Ruler 2:

Fold the ruler into 8 equal sections. Put the number 1/3 on the second line from the start. Label each line. This ruler starts at 0. Where does it end?

Create 2 number bonds showing how you can decompose 7/6. Break it up 4 ways and 7 ways.

Write an addition and multiplication sentence for each bond.
Lesson Eight: Problems at the Rational Ruler Company!

In lesson 8, students will investigate equivalent values on a ruler by making a ruler with halves, fourths, eighths, and sixteenths. The creation of this ruler will be a teacher-led activity to assist students in accuracy and neatness as they make their rulers. While the ruler is being built, make a poster of values: \( \frac{1}{2} > \frac{1}{4}, \frac{1}{4} > \frac{1}{8}, \) etc. Ask students to create conjectures about the size of the denominator and the magnitude of the fraction. This work will help solidify the meaning of the denominator (equal pieces of the whole) and refute the tendency to see the denominator as a whole number with whole number value, i.e. \( \frac{1}{8} \) is greater than \( \frac{1}{4} \) because 8 is bigger than 4.

The list of equivalent fractions will be built under the ruler. Students will hopefully see the pattern of equivalent values and continue in their understanding of the density of fractions and their infinite property. Using the ruler, they will create number bonds for various fractions using 16ths. They will write addition and multiplication sentences to decompose each fraction.

Example of completed ruler:

```
Rational Ruler: halves, fourths, eighths and sixteenths
Write your fractions below the ruler.

\[
\begin{array}{cccccccccccccccc}
0 & \frac{1}{16} & \frac{1}{8} & \frac{3}{16} & \frac{1}{4} & \frac{5}{16} & \frac{3}{8} & \frac{7}{16} & \frac{5}{8} & \frac{9}{16} & \frac{11}{16} & \frac{3}{4} & \frac{13}{16} & \frac{7}{8} & \frac{15}{16} & 1
\end{array}
\]
```
Examples of number bonds with addition and multiplication sentences:
Problem Solving: Rational Ruler Order

A university in New York has ordered a very interesting ruler. They would like a ruler that starts at 0 and ends at 1. They need the ruler to show halves, fourths, eighths AND sixteenths. Build this ruler on the page you are given.

The company is interested in the following:

1. Use 16ths to decompose each fraction.
2. Make a number bond for each.
3. Write an addition sentence.
4. Write a multiplication sentence.

1/8          1/4

7/8          1
Rational Ruler: halves, fourths, eighths and sixteenths

Write your fractions below the ruler.
Lesson Nine: Work for the Rational Ruler Company

For lesson nine, students will work in groups to divide bars into a variety of rulers. They start out easy and become more difficult. Students will be drawing lines instead of folding them, they are still to pay attention to equal iterations (spaces). When finished, student teams may check with other teams for accuracy. When groups are done, have students share answers for the rulers. Each ruler may become a mathematical discourse as the students grapple with where they place the lines and what fraction the lines represent.

**Hints for Iterating thirds** (Demonstrate with students in their journals):

Visualize the center of the ruler, then draw lines on either side.

![Diagram of thirds]

**Hints for Iterating fifths** (Demonstrate with students in their journals):

Visualize the center of the ruler, then draw lines on either side, about 1/5 of the bar. Split the remaining sections in half.

![Diagram of fifths]

**Hints for Iterating tenths** (Demonstrate with students in their journals):

Draw a line down the center of the ruler, then follow directions for iterating fifths on EACH half of the bar.

![Diagram of tenths]
Rational Ruler Company

Student Work: Making Blueprints for Rational Rulers

Employee Name: ______________________________________

Rational Ruler Regulations

1. Use fractions, improper fractions, and whole numbers for the rulers (no mixed numbers).
2. The spaces made on the ruler must be made up of equal sections.
3. The lines are labeled with correct fractions.
4. Rational Rulers are neat and tidy.

1. Work with your team to divide this ruler into 4 equal parts. Write the correct fractions on the lines you make.

Decompose ¼ into four equal pieces.

Addition Sentence: ______________________________________

Multiplication Sentence: ____________________________________
Work with your team to divide this ruler into 9 equal parts. Write the correct fractions on the lines you make.

Decompose 3 into nine equal pieces.

Addition Sentence:  

Multiplication Sentence:
Work independently to divide this ruler into 6 equal parts. Write the correct fractions or whole numbers on the lines you make.

Decompose 4/3 into four equal pieces.

Addition Sentence: _____________________________________________

Multiplication Sentence: _____________________________________________
Lesson Ten: Student Assessment on Building Rational Rulers

Lesson ten is an opportunity for students to show what they have learned from the previous lessons and discussions. Teachers will avoid giving advice except in cases of special needs (i.e. student cannot read the text). Remind students that they can use their notes and other rulers they have built in their journal.
Student Assessment: Building Rational Rulers

EMPLOYEE NAME: ____________________________________________

Rational Ruler Regulations
Use fractions, improper fractions, and whole numbers for the rulers (no mixed numbers).
The spaces made on the ruler must be made up of equal sections.
The lines are labeled with correct fractions.
Rational Rulers are neat and tidy.

1. Work independently to divide this ruler into 8 equal parts. Write the correct fractions or whole numbers on the lines you make.

Decompose 5/4 into five equal pieces.

Addition Sentence: ____________________________________________

Multiplication Sentence: ________________________________________
2. Work independently to divide this ruler into 3 equal parts. Write the correct fractions or whole numbers on the lines you make.

Decompose \( \frac{3}{6} \) into three equal pieces.

Addition Sentence: ________________________________

Multiplication Sentence: ________________________________

3. Write several fractions equivalent to \( \frac{1}{2} \):

4. How many equivalent fractions are there for \( \frac{1}{2} \)?

5. How many fractions are there between 0 and 1?
Lesson Eleven: Running Races

For Running Races, students will use the iteration principles from making the rulers, and move this to lines which represent running races. The students will be asked to put different events at certain distances from the start (0) of the race (i.e. a water stand at ¾ of the race). Lesson eleven focuses on this concept. In order to scaffold this lesson, have the students create a race with all fractions labeled (halves, fourths, eighths). Each division should be a different color (colored pencils work well). For example, half in red, fourths in green, eighths in blue. Once this is complete, have them create the race plan on the final draft and work on the further questions.
Lesson Eleven Problem Solving: Running Races

Race Organizer: _________________________________

Our elementary school is planning a running race at school. The fourth graders are in charge of setting up spots along the race for water, and different snacks. The race organizers are very specific about where they want the water and snacks:

1. At 1/2 of the way through the race they want to have oranges.
2. At 1/4, 3/8, and 3/4 of the way through the race they want water stands.
3. At 7/8 of the way through the race they want to have pretzels.

Here is a line that represents the race. Label the line with each fraction and what the spot will offer.

**Working Draft:**

```
Start                        Finish
```

**Final Plan (AFTER discussion):**

```
Start                        Finish
```

Race Planning (Lesson 11)

Based on the race plan you made, answer the following questions.

1. In what order will the racers get to the stations? What comes first? List all the stations in order.

1.

2.

3.

4.

5.

2. Look at the number line with fractions that you have created. Write 6 number sentences about the numbers on the line, comparing them with <, >, or =. Prove your sentences to a partner with your number line.

1. ___________________________  
   Partner Check

2. ___________________________  

3. ___________________________  

4. ___________________________  

5. ___________________________  

6. ___________________________  

Lesson 12: Independent Work

Race Organizer: ________________________________

Planning a Race

The middle school thought the race was so awesome, they want to sponsor one too. The fourth graders are in charge (again) of setting up spots along the race for water, and different snacks. The middle school is very specific about where they want the water and snacks:

1. At 1/8 and 5/8 will be Gatorade
2. At 1/3 and 2/3 of the through the race they want to have potato chips.
3. At ½ and 3/4 of the way through the race they want to have people cheering on runners.

Race Ruler for thirds:

Race Ruler for halves, fourths, and eighths:

Final Race Plan:
Independent Race Planning (Lesson 12)

Based on the race plan you made, answer the following questions.

1. **In what order will the racers get to the stations? What comes first?**

<table>
<thead>
<tr>
<th>Station</th>
<th>Fraction</th>
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</thead>
<tbody>
<tr>
<td>1.</td>
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<td>5.</td>
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<tr>
<td>6.</td>
<td></td>
</tr>
</tbody>
</table>

2. Look at the number line with fractions that you have created. Write 6 number sentences about the numbers on the line, comparing them with <, >, or =. Prove your sentences to yourself with your number line.

**Self-Check**

1. ___________________  
   
2. ___________________  
   
3. ___________________  
   
4. ___________________  
   
5. ___________________  
   
6. ___________________  
   
   
   
   
   
   

Lesson 13: Independent Work

Race Organizer: ______________________________

Another race to plan!!!

1. At 1/8 and 3/8 there will be oranges.

2. At ¼ and ¾ there will be water.

3. At 1/3 and 5/6 there will be granola bars.

Race Ruler for thirds and sixths:

Race Ruler for halves, fourths, and eighths:

Final Race Plan:
Independent Race Planning (Lesson 13)

1. In what order will the racers get to the stations? What comes first? List the fraction and what will be at the station.

<table>
<thead>
<tr>
<th>Station</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
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<td>6.</td>
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</tbody>
</table>

2. Look at the number line with fractions that you have created. Write 6 number sentences about the numbers on the line, comparing them with <, >, or =. Prove your sentences to yourself with your number line.

Self-Check

1. ____________________________ ☐
2. ____________________________ ☐
3. ____________________________ ☐
4. ____________________________ ☐
5. ____________________________ ☐
6. ____________________________ ☐
**Color Run!!!**

1. Every $\frac{1}{4}$ of the race is blue (including the end).
2. At $\frac{1}{6}$, $\frac{3}{6}$ and $\frac{6}{6}$ there will be yellow.
3. At $\frac{1}{3}$ and $\frac{5}{6}$ there will be red.

Race Ruler for thirds and sixths:

Race Ruler for halves, fourths, and eighths:

*Final Race Plan:*
Independent Race Planning (Lesson 13)

Based on the race plan you made, answer the following questions.

1. In what order will the racers get colored? What comes first? List the fraction and what will be at the station.

<table>
<thead>
<tr>
<th>Station</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
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<tr>
<td>6.</td>
<td></td>
</tr>
</tbody>
</table>

2. Look at the number line with fractions that you have created. Write 6 addition or multiplication problems from the race rulers you have made.

Self-Check

1. ____________________________

2. ____________________________

3. ____________________________

4. ____________________________

5. ____________________________

6. ____________________________
Lesson 13 Planning a Race: Homework

Race Organizer: ___________________________

Carson City wants to plan a run for their families. This is what they would like to have:

1. At 1/3 and 4/6 there will be grapes.
2. At 3/8 and 7/8 there will be water.
3. At ¼ and 4/4 there will be people cheering.

USE A STRING FOR ACCURACY!

Race Ruler for thirds and sixths:

Race Ruler for halves, fourths and eighths:

Final Race Plan:
Based on your race plan, in what order will the racers get to the stations? What comes first? Fill out the table.

<table>
<thead>
<tr>
<th>Station</th>
<th>Fraction of the Race</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td></td>
</tr>
</tbody>
</table>

2. Look at your race rulers. Write 4 addition or multiplication sentences with fractions from the rulers.

1. ________________________________________________________________

2. ________________________________________________________________

3. ________________________________________________________________

4. ________________________________________________________________
Lesson 14: Planning a Race: Assessment

Race Organizer: ________________________________

San Francisco wants to plan a fun walk for the students. This is what they would like to have:

4. At 2/3 and 1/4 there will be fizzy water.
5. At 1/8 and 6/6 there will be burritos.
6. At 1/2 and 6/8 there will be dancers.

USE A STRING FOR ACCURACY!

Race Ruler for thirds and sixths:

---

Race Ruler for halves, fourths and eighths:

---

Final Race Plan:
Based on your race plan, in what order will the racers get to the stations? What comes first? Fill out the table.

<table>
<thead>
<tr>
<th>Station</th>
<th>Fraction of the Race</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td></td>
</tr>
</tbody>
</table>

2. Look at your race rulers. Write 2 sentences that use <, >, or = to compare two fractions.

1. ________________________________
2. ________________________________

2. Look at your race rulers. Write 2 addition or multiplication sentences.

1. ________________________________
2. ________________________________
Lesson Fifteen: Mapping a Bike Race

Problem Solving: Biking Races

Race Organizer: _________________________________

The local bike team wants to have a bike race near our school. The riders will be thirsty and hungry during their ride. Here is a list of what they need.

1. Every 1/6 of the race they want a water stand including the finish line.
2. At the ½ way point in the race they want to serve sandwiches.
3. At 2/3 of the race they want to have oranges.
4. Right in the middle between the start of the race and the sandwiches, they want to have bananas.
5. At 6/6 of the race, they want to hand out ribbons.
6. At the fourth water stand, they want to serve popsicles.

Race Ruler

START

FINISH

Final Race Plan:
Answer these questions based on your final plan:

What fraction of the race will each stop land on? Write all the fractions for the stops.

<table>
<thead>
<tr>
<th>Stop</th>
<th>Fraction(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td></td>
</tr>
<tr>
<td>Sandwiches</td>
<td></td>
</tr>
<tr>
<td>Bananas</td>
<td></td>
</tr>
<tr>
<td>Popsicles</td>
<td></td>
</tr>
<tr>
<td>Ribbons</td>
<td></td>
</tr>
<tr>
<td>Oranges</td>
<td></td>
</tr>
</tbody>
</table>

1. How many stops are there in the entire race (including the finish line)? _____

4. Which fraction of the race offers the most snacks? ___________

5. How far is it from the first water stand to the sandwich stop? ___________

6. How far is it from the banana stand to the finish of the race? ___________

7. Write equivalent fractions for the following. How many can you create?

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Fractions that are equivalent (equal)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>2/6</td>
<td></td>
</tr>
<tr>
<td>4/6</td>
<td></td>
</tr>
<tr>
<td>6/6</td>
<td></td>
</tr>
<tr>
<td>1/4</td>
<td></td>
</tr>
<tr>
<td>3/4</td>
<td></td>
</tr>
</tbody>
</table>
Lesson Fifteen: When will I get a Popsicle???

Before the students start their group work, review the final plan of the bike race. Discuss as a class the following questions to prepare for group work. Have students create a model drawing of the race using the student worksheet from lesson 15. Demonstrate how to analyze the race in this way. Make sure to have each question discussed in pairs or groups before asking someone to share. If only one or two students raise their hands, ask them to discuss it again. This helps students take ownership of who is supposed to be thinking.

- From the final bike race plan, do we know how many miles the bike race will be?
- From the final bike race plan, can you tell me how many miles the riders will go until they get to eat sandwiches?
- Let’s say the bike race is 12 miles long.
- How far will the riders go to get popsicles?
- How many miles are the sandwiches from the popsicles?
- How many miles will they ride until the last water stand before the finish?
- If a rider can ride a mile in 12 minutes, how long will it take her to finish the race?
Lesson 16 Student Worksheet

Name: ____________________________

START

FINISH

12 miles
Lesson 15 Group Work: When will I get a Popsicle?????

The total distance for the bike race is actually 36 miles. This is a long ways! The bike riders have questions about the race plan. Mostly they are concerned about snacks! Use your final plan for the bike race to help the riders with their questions:

How many miles will I have to go before I get to the first water stop?
_______ Miles.

How many miles will I have to go before I get a sandwich???
_______ Miles.

How many miles will I have to go to get to the banana stand????
_______ Miles.

How many miles to the popsicles?
_______ Miles.

If I can ride one mile in 2 minutes, how long will it take me to get a ribbon? _______ Miles. Write your number sentence:
__________________________________
Lesson 15: Group Work Model Drawing

START

FINISH

36 miles
Lesson Sixteen: Independent Race Planning

Before the independent work, set the stage by discussing relay races and Tough Mudder races.

**Relay Races:**

Demonstrate in the classroom how the race is split evenly and runners hand off a baton to the next runner. If time, taking the students outside to run short relay races along a distance where they hand off a baton would be excellent. This You Tube clip shows a 4 x 100 women’s Olympic relay. Describe the idea of the curbed start due to the curved track. Start it at 2:50 to see the race, end at 4:44. [https://www.youtube.com/watch?v=sAfhf_u_QBI](https://www.youtube.com/watch?v=sAfhf_u_QBI)

**Tough Mudder:**

Tough Mudder is a race with obstacles during the race. See this short video to help kids understand. At the website below, load the 2015 Redefined (Official Video). [https://toughmudder.com/events/what-is-tough-mudder](https://toughmudder.com/events/what-is-tough-mudder)

If time, taking kids outside to a Tough Mudder type run that you have set up would be beneficial.

Use PE equipment for obstacles like: 10 jump ropes, crawl for 15 feet, etc. This is an excellent way to get the students to understand the nature of the Tough Mudder.

Have students work with colored squares to do the warm up. This helps them practice using a fraction as an operator on a whole number. The denominator tells them how many equal group to make and the numerator tells them how many of those groups. For example, \(\frac{3}{4}\) of 20 means 4 equal groups of 5 in each group. The three tells us to add up three groups worth- or 15.
Student Examples of Number Bonds showing a fraction operating on a whole number.

**Divide 20 into 5 equal pieces.**

1/5 of 20 = 4

2/5 of 20 = 8

**Divide 20 into 10 equal pieces.**

1/10 of 20 = 2

9/10 of 20 = 18
Lesson Sixteen: Warm Up.

Using number bonds to operate with fractions. Use tiles to work out these problems.

Divide 10 into 2 equal groups.

\[ \frac{1}{2} \text{ of } 10 = \text{_______} \]

Divide 10 into 5 equal groups.

\[ \frac{1}{5} \text{ of } 10 = \text{_______} \]
\[ \frac{2}{5} \text{ of } 10 = \text{_______} \]

Divide 10 into 10 equal groups.

\[ \frac{1}{10} \text{ of } 10 = \text{_______} \]
\[ \frac{9}{10} \text{ of } 10 = \text{_______} \]
Using number bonds to operate with fractions.

**Divide 20 into 2 equal groups.**

\[ \frac{1}{2} \text{ of } 20 = \_\_\_\_\_ \]

**Divide 20 into 5 equal groups.**

\[ \frac{1}{5} \text{ of } 20 = \_\_\_\_\_ \]

\[ \frac{2}{5} \text{ of } 20 = \_\_\_\_\_ \]

**Divide 20 into 10 equal groups.**

\[ \frac{1}{10} \text{ of } 20 = \_\_\_\_\_ \]

\[ \frac{9}{10} \text{ of } 20 = \_\_\_\_\_ \]
Lesson 16: Planning a Tough Mudder

A “Tough Mudder” running race is being planned for a fund raiser. There are obstacles the runners need to pass during the race. Create a race plan for these obstacles. Label the fractions and the obstacle.

- At 1/10 of the race there is a MUD PUDDLE to go through.
- At 1/5 of the race there is a ROCK WALL to climb over.
- At ½ of the race, there is an ICE BATH runners have to swim through.
- At 9/10 of the race, there is a TUNNEL to crawl through.
- At 4/5 of the race, runners need to go on a ZIP LINE.
- At 5/5 of the race there is HOT SAUCE and CHILI PEPPERS.

Race Ruler

Final Race Plan
There are going to be several distances of the race. The events will stay at the same fraction of the race. How many miles will it take the runners to get to each event?

<table>
<thead>
<tr>
<th>10 mile mudder</th>
<th>20 mile mudder</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obstacle</td>
<td>Number of miles</td>
</tr>
<tr>
<td>Mud Puddle</td>
<td></td>
</tr>
<tr>
<td>Rock Wall</td>
<td></td>
</tr>
<tr>
<td>Ice Bath</td>
<td></td>
</tr>
<tr>
<td>Tunnel</td>
<td></td>
</tr>
<tr>
<td>Zip Line</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>30 mile mudder</th>
<th>50 mile mudder</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obstacle</td>
<td>Number of miles</td>
</tr>
<tr>
<td>Mud Puddle</td>
<td></td>
</tr>
<tr>
<td>Rock Wall</td>
<td></td>
</tr>
<tr>
<td>Ice Bath</td>
<td></td>
</tr>
<tr>
<td>Tunnel</td>
<td></td>
</tr>
<tr>
<td>Zip Line</td>
<td></td>
</tr>
</tbody>
</table>
### 100 mile mudder

<table>
<thead>
<tr>
<th>Obstacle</th>
<th>Number of miles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mud Puddle</td>
<td></td>
</tr>
<tr>
<td>Rock Wall</td>
<td></td>
</tr>
<tr>
<td>Ice Bath</td>
<td></td>
</tr>
<tr>
<td>Tunnel</td>
<td></td>
</tr>
<tr>
<td>Zip Line</td>
<td></td>
</tr>
</tbody>
</table>

### 200 mile mudder

<table>
<thead>
<tr>
<th>Obstacle</th>
<th>Number of miles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mud Puddle</td>
<td></td>
</tr>
<tr>
<td>Rock Wall</td>
<td></td>
</tr>
<tr>
<td>Ice Bath</td>
<td></td>
</tr>
<tr>
<td>Tunnel</td>
<td></td>
</tr>
<tr>
<td>Zip Line</td>
<td></td>
</tr>
</tbody>
</table>
Lesson Seventeen and Eighteen: Problem Solving with Regions

The third measurement construct in this unit will be area. In the next five lessons, students will work with finding fractional amounts of an area given. They will work with estimating shaded parts and naming fractions for those parts (Rational Number Project, 2009). They will make fractional partitions of a given space and solve problems based on this procedure. Similar to the rulers and the races, students will be creating their own partitions to develop flexible thinking around fractional amounts of figures.

The first task is to be done with partners or small groups. Move into mathematical discourse routine to clear up misconceptions and develop shared understandings of what is going on in the work.

Lesson twenty presents three problem solving situations.

**Part 1**: Students are asked to find the number of trees to fill $\frac{3}{4}$ of a park that is 48 square units.

**Part 2**: Students are asked to give the fraction of the park that 35 trees will fill of a 50 square unit park.

**Part 3**: Students are given a number of trees (12) and told this is $\frac{1}{5}$ of a park. They are expected to outline the whole park and determine the number of square units it covers (55).

Follow the mathematical discourse routine with each problem.
Lesson Seventeen: Warm Up

Name: _______________________________________

Using number bonds to operate with fractions.

**Break up 24 into 6 equal groups.**

- 3/6 of 24 = ______
- 4/6 of 24 = ______
- 5/6 of 24 = ______
- 7/6 of 24 = ______

**Break up 18 into 3 equal groups.**

- 1/3 of 18 = ______
- 2/3 of 18 = ______
- 3/3 of 18 = ______
- 4/3 of 18 = ______

**Divide 100 into 10 equal groups.**

- 1/10 of 100 = __________
- 5/10 of 100 = __________
- 9/10 of 100 = __________
- 11/10 of 100 = __________
Using number bonds to operate with fractions.

Divide 25 into 5 equal groups.  

Divide 32 into 8 equal groups.

\[ \frac{3}{5} \text{ of } 25 = \]  
\[ \frac{1}{8} \text{ of } 32 = \]  
\[ \frac{4}{5} \text{ of } 25 = \]  
\[ \frac{3}{8} \text{ of } 32 = \]  
\[ \frac{5}{5} \text{ of } 25 = \]  
\[ \frac{1}{2} \text{ of } 32 = \]  
\[ \frac{6}{5} \text{ of } 25 = \]  
\[ \frac{9}{8} \text{ of } 32 = \]

Divide 200 into 4 equal groups.

\[ \frac{1}{4} \text{ of } 200 = \]  
\[ \frac{3}{4} \text{ of } 200 = \]  
\[ \frac{1}{2} \text{ of } 200 = \]  
\[ \frac{5}{4} \text{ of } 200 = \]
Fred takes care of planting the parks with trees in the summer. His park looks like this:

Fred wants to plant one tree in \( \frac{3}{4} \) of the squares of his park. How many trees will he need?

Explain how you decided on your answer:

If Fred wants to plant 1/4 of the park with shrubs (1 per square), how many does he need?

______ How do you know?

Fred put 6 flower planters (1 per square) in his park. What fraction of the park has flower planters? ________ How do you know?
Park Planning Issues: Part 2

Henry and Jenny are working on a park south of where Fred is working. Their park looks like this:

Henry bought 35 trees to plant at the park, one in each square. What is the fraction of the park they will plant with trees?
Lesson Eighteen: Planning with Regions

Review breaking up a region by splitting an array 5 X 6 array (30 tiles) into different sections.

Have students share strategies for breaking up the group and identifying the amount.

This will be done at their seats. When everyone has built their model (i.e. 2/5 of 30), have the students walk around the class to see how the other students have shown this.

We will do four problems on the first side:

Create a 5X6 array.

Split the array into 6 equal groups. How many in each group? Create a number bond. Write as many multiplication sentences as you can.

Split the array into 3 equal groups. Create a model with the squares that shows 2/3 of 30.

Split the array into 5 equal groups. How can we find 4/5 of 30?

Split the array into 2 equal groups. What is 3/2 of 30?
Broken Number Bonds

Show the following Broken Bonds and have students work on fixing the bond. The first step is to create a row of 7, for example. The students can see that there are 3/3 in the bond from our previous work. They can build the whole (3 sets of 7) at their seats. Play Pass the Pen for each number bond so that the students see how it is constructed.

Pass the Pen:

This is done quietly. One student starts the work, when the teacher says “Pass the Pen”, they find another student who is raising their hand to come up and continue the work. This should be done in boy/girl order. The rule is that you cannot pick a student who is loudly begging to be chosen. You also have to pick within 3 seconds or the teacher picks. This helps it flow quickly.

Students may correct an error that they see by erasing and redoing.

\[
\begin{align*}
7 &- 1/3 \\
4 &- 1/10 \\
5 &- 1/7 \\
3 &- 1/9
\end{align*}
\]
(Example of a Broken Number Bond for students to complete)

(Example of a completed Broken Number Bond by students playing Pass the Pen).
Lesson Eighteen:  More Park Planning

Regional Planner: __________________________________________

A new park is being developed in the area. The people planning the park want only one thing in each square. For example, a tree and a rose bush would NOT be in a square together. The planning team wants to have the following in the park:

1/3 of the park will have a tree in each square.

1/12 of the park will have a drinking fountain in each square.

1/6 of the park will have a rose bush in each square.

Draw the park with trees, drinking fountains, and rose bushes.

The leftover squares will be covered with grass.

How many trees will you need? ___________ trees

How many drinking fountains will you need? ___________ drinking fountains

How many rose bushes will you need? ___________ rose bushes

How many squares will be covered in grass? ________ squares covered in grass
More Park Planning Issues: How big is the park?

The park ranger delivered this many trees for the park where Walter works:

The ranger told Walter that this number of trees (planted one in each square) should cover 1/5 of his park. How big is Walter’s park? Outline and shade Walter’s whole park on this grid:

How many unit squares make up Walter’s park? ________ unit square
Plan your own park.

The park planners want a new plan for a park. Here is what they would like:

The park needs to be made up of 40 squares in a 4X10 array. Decide how many squares each item will use and then tell the fraction of the park. Each item MUST be an equal fraction of the park.

<table>
<thead>
<tr>
<th>Park Item</th>
<th>Number of squares</th>
<th>Fraction of the park</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trees</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shrubbery</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Play Ground</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exercise Stations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Drinking Fountains</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bathrooms</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Outline and label your park:
Lesson Nineteen: A Fourth Grade Race

As a culminating event, have the students plan a race at your school. Complete the race plan the day before the event. It is helpful to have them see the outside area designated for the race, so they can accurately estimate fractions of the race. Provide snacks (if possible) and obstacle equipment (hula hoops, etc.) for groups to set up on the race plan. The next day, students will set up the race and run it. Fun!!
A Fourth Grade Race Plan

Race Planner: ________________________________________

The fourth graders are planning a race to take place at their school. The race map is provided below. The class wants to have snacks and events at certain places in the race. Put the events on the map at the accurate spot: In this race, students need to go around 2 times.
At 1/3, 2/3, and 5/6 of the race, they will have obstacles.

At ½ they want small water bottles.

At 1/6 of the race, they want gummy worms.
At 3/4 of the race, they want apple slices and grapes.

At 12/6 of the race, they want popsicles.

1. In what order will the students get to the race events or treats? List them here.

<table>
<thead>
<tr>
<th>Race Event</th>
<th>Fraction of the Race</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td></td>
</tr>
<tr>
<td>2)</td>
<td></td>
</tr>
<tr>
<td>3)</td>
<td></td>
</tr>
<tr>
<td>4)</td>
<td></td>
</tr>
<tr>
<td>5)</td>
<td></td>
</tr>
<tr>
<td>6)</td>
<td></td>
</tr>
<tr>
<td>7)</td>
<td></td>
</tr>
</tbody>
</table>

2. One student says that he will bet to an obstacle BEFORE he gets to the gummy worms.

Another student says that she will get to the gummy worms first. Who is correct? Can you prove it?
Lesson Twenty: Post Test

In lesson twenty, students will take the same test as they took for the pre-test, to see if gains have been made in the conceptualization of fractions.
Extra Independent Work: Planning Races and Relays

Race Planner: _______________________________________

1. There is going to be a running race that is done with a relay team. Each member of the team will run a different fraction of the race. The race is 1 mile long. Here is their plan:

<table>
<thead>
<tr>
<th>Runners (in order)</th>
<th>Fraction of the mile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Sammy</td>
<td>1/8</td>
</tr>
<tr>
<td>2. Gail</td>
<td>2/8</td>
</tr>
<tr>
<td>3. Linda</td>
<td>1/4</td>
</tr>
<tr>
<td>4. Liam</td>
<td>1/16</td>
</tr>
<tr>
<td>5. Andrew</td>
<td>3/16</td>
</tr>
</tbody>
</table>

Is this a good race plan? Give advice to the running team. Should they make changes? What do you recommend?

______________________________________________________________________________
______________________________________________________________________________
______________________________________________________________________________
Extra: Problem Solving with Races

Trouble with the Tough Mudder

A Tough Mudder race is being planned for kids. The race planners want the obstacles to be every $\frac{4}{5}$ of a mile. They have the following obstacles planned and know what order they want the runners to do them.

First: Mud Bath (MB)

Second: Rope Climb (RC)

Third: Tunnel Crawl through ice (TC)

Fourth: Mud Slip and Slide down a hill (MSS)

Fifth: Zip Line over ice-filled pond (ZL)

Make a race plan that shows the fractions along with the event to help. Write each fraction in fifths.

<table>
<thead>
<tr>
<th>Obstacles</th>
<th>How many fifths?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mud Bath</td>
<td></td>
</tr>
<tr>
<td>Rope Climb</td>
<td></td>
</tr>
<tr>
<td>Tunnel Crawl</td>
<td></td>
</tr>
<tr>
<td>Mud Slip and Slide</td>
<td></td>
</tr>
<tr>
<td>Zip Line</td>
<td></td>
</tr>
</tbody>
</table>

How many miles will the race be altogether? _______
Extra: Creating a Line Plot with Fractions

Park workers clean the city parks for a summer job. They clean several parks each day. They need to report to their boss how much of the parks they have cleaned at the end of the day. The workers reported the following after Monday’s cleaning:

<table>
<thead>
<tr>
<th>Worker</th>
<th>Parks Cleaned</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joe</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>Sammy</td>
<td>$\frac{3}{4}$</td>
</tr>
<tr>
<td>Ariana</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>Megan</td>
<td>$\frac{5}{8}$</td>
</tr>
<tr>
<td>Jack</td>
<td>$\frac{5}{8}$</td>
</tr>
<tr>
<td>Todd</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Anna</td>
<td>$\frac{2}{4}$</td>
</tr>
<tr>
<td>Zach</td>
<td>$\frac{5}{8}$</td>
</tr>
<tr>
<td>Tammy</td>
<td>$\frac{2}{8}$</td>
</tr>
<tr>
<td>Chandra</td>
<td>$\frac{4}{8}$</td>
</tr>
</tbody>
</table>

1. Create a line plot to display the data in the table on another page.

2. Answer the following questions.
   
   a. Who cleaned more parks than Ariana?

   b. Who cleaned less parks than Joe?

   c. What is the difference between the most amount of parks cleaned and the least amount cleaned?

   d. Compare the amount that Anna cleaned to the amount that Tammy cleaned. Use $<$, $>$, or $=$ to describe your comparison.

   e. Who cleaned one whole park more than Joe?

(Adapted from Eureka Math, 2015)
Line Plot Page

Remember, a line plot looks like this:

Create a line plot for the amount of parks cleaned on Monday. Your line plot will have different fractions:
**Extra: Independent Assessment of Race Planning**

This is an assessment to evaluate students’ understanding of placing fractions on the number line, iterating by a fractional amount (4/5) and applying this to addition and multiplication properties. Students may have a strategy for placing fractions related to the principle that \( \frac{p}{q} \) is equal to \( p \) of \( \frac{1}{q} \) sized pieces. In this assessment they will be using the number line to help them solve problems of the type:

Order the fractions in the set from smallest to largest: 6/7, 2/20, 6/10.

Before starting the assessment, have students make a number line in their journals and place these fractions on the line. Have a mathematical discourse about the correct answer. Then, develop a list (stickperson) about “How to Place the Fractions on the Line”. Let students share strategies. Post in classroom. Students also copy list in journals.

The list might include:

- Break up the number line into the number of parts on the bottom of the fraction, count how many of these for the top number.
- Make sure you create equal parts.
Extra: Race Planning Assessment

Race planner: _____________________________

For each race below, place the fractions in the accurate place on the race path and label what is happening at that spot.

1) A running race will have snacks every 2/3 of the race. The order for the snacks are:

   First: Water
   Second: Pretzels
   Third: Gatorade
   Fourth: Gummy Bears
   Fifth: Water
   Sixth: Trophies

Fill in the table.

<table>
<thead>
<tr>
<th>Stops</th>
<th>How many thirds?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td></td>
</tr>
<tr>
<td>Pretzels</td>
<td></td>
</tr>
<tr>
<td>Gatorade</td>
<td></td>
</tr>
<tr>
<td>Gummy Bears</td>
<td></td>
</tr>
<tr>
<td>Water</td>
<td></td>
</tr>
<tr>
<td>Trophies</td>
<td></td>
</tr>
</tbody>
</table>
2. How long is the entire race? ____________ miles

3. A bike race is planned for the summer. The race planners want the following spots on their race:

   At ¾ of the race there will be a crowd cheering.
   At 1/6 and 4/6 of the race there will be a water stop.
   At 4/12 of the race there will be potato chips.
   At 3/6 of the race there will be sprinklers to cool off the riders.

4. If the entire race is 12 miles long, fill in the table for the number of miles the bikers will go in order to get to each stop.

<table>
<thead>
<tr>
<th>Event</th>
<th>Number of miles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sprinklers</td>
<td></td>
</tr>
<tr>
<td>Water Stops</td>
<td>First stop</td>
</tr>
<tr>
<td>Potato Chips</td>
<td></td>
</tr>
<tr>
<td>Cheering crowds</td>
<td></td>
</tr>
</tbody>
</table>

5. If a biker can ride one mile in 10 minutes, how long will it take her to get to the sprinklers?
Extra: Problem Solving with Regions

Park Problems

Regional Planner: ________________________________

Three workers work at different parks that are all the same size, but each worker divided their park up into a different amount of equal pieces.

George created 16 equal pieces and planted \( \frac{1}{4} \) of the park with one tree in each piece. Draw George’s park (don’t forget the trees):

Sam divided his park into 12 equal pieces and also planted trees in \( \frac{1}{4} \) of the park. Draw Sam’s park (don’t forget the trees):

Laura divided her park into equal pieces and needed 2 trees to plant \( \frac{1}{4} \) of the park. Draw Laura’s park (don’t forget the trees):
The park manager wonders why George, Sam and Laura all planted \( \frac{1}{4} \) of their park, but ended up planting a different amount of trees. He thinks they should all have planted the same amount of trees. What do you think? Prove your thinking with a model.
Appendix D: Daily Teacher Lesson Reflection Protocol

**Daily Teacher Lesson Reflection Protocol**

**Lesson # _____**

1. Did you see evidence of student understanding of fraction magnitude? How do you know?

2. What went well? Explain why you think it went well.

3. If there was a whole class mathematical discourse, did it support student understanding of fraction magnitude? What would you change about the routine?

4. Are there any sociomathematical norms that need to be addressed?

5. If you taught this lesson again, what would you change about the lesson?

6. What can be changed about tomorrow’s lesson based on today’s lesson?

7. Other comments:
Appendix E: Student Reflection Protocol

**Student Reflection Protocol**

1. Describe what you know about putting fractions on a number line? How do you do it?

2. How confident do you feel about working with fractions? Do you feel like you understand them well?

3. Do you participate in the mathematical discourse of the class? Do you feel safe getting up in front of the class? Why or why not?

4. Does the mathematical discourse help you in any way? Give examples.

5. Other comments:
Discuss with your group how you solved the following problems on the Fraction Conceptualization Test.

Write the correct number in the box below.

B part of shape B on the number line.

C8. Place these fractions on the number line.

\[ \frac{1}{5}, \frac{2}{3}, \frac{5}{8}, \frac{5}{5}, \frac{6}{5} \]
Appendix F: Final Teacher Debriefing Protocol

Final Teacher Debriefing Protocol

1. Do you think the students learned about fraction magnitude from this unit? How do you know?

2. Do you think the students have a different understanding of fractions after this unit? Is this understanding valuable?

3. What went well in this unit?

4. If you taught this unit again, would you do it differently?

5. Based on your experience as a teacher, was the mathematical discourse effective in supporting students in learning?

6. What was it like using the mathematical discourse routine? Was it helpful? Why or why not?

7. Describe anything else you would like to add about this unit or the experience using mathematical discourse.