Approximate Techniques for the Solution of Unconfined Groundwater Flow
Equations and a Derivation of a Two-Sided Fractional Conservation of Mass
Equation

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Abstract

This work consists of a study of particular aspects of nonlinear flow equations and a non-traditional mass conservation equation. First, methods of constructing approximate polynomial solutions to nonlinear groundwater flow equations are presented. In addition to this work, another researcher’s solution of the Boussinesq equation is discussed. Second, a derivation of a non-traditional conservation of mass equation, involving left and right Caputo fractional derivatives, is presented. The resulting equation can be used to derive fractional governing equations for fluid flow and contaminant transport.

In Chapter 1, the Introduction, the two nonlinear unconfined groundwater flow equations considered in this study are described. Next, a brief review is given of other works in which fractional differential equations are used for flow and contaminant transport modeling. The later chapters consist of three papers that were previously published and one unpublished chapter.

In Chapter 2, published in *Water Resources Research* 49(5), polynomial approximate solutions are constructed for infiltration into an initially dry, horizontal unconfined aquifer. The flows are described by a porous medium equation with a power-law head condition at the inlet of the aquifer. These approximate solutions can be used to validate numerical solutions to nonlinear equations when exact solutions do not exist. The approximate solutions are compared to numerical solutions computed using a modification of a method of Shampine. The polynomial approximate solutions reproduce known exact solutions and closely match the numerical solutions.

Chapter 3, published in *Water Resources Research* 50(9), is a comment on another researcher’s solution of the Boussinesq equation. This researcher used a traveling wave transformation and a perturbation series to generate a solution of the Boussinesq equation.
He assumed a constant wave speed in the derivation, but later noted that the derivation implies that the wave speed is time dependent. The comment on this work outlines how his derivation will change if the wave speed is assumed to be time dependent from the beginning of the derivation.

In Chapter 4, published in *Advances in Water Resources* 91, May 2016, a two-sided fractional conservation of mass equation is derived. The equation uses left and right Caputo fractional derivatives and extends the work of other researchers who derived a one-sided fractional conservation of mass equation. The mass conservation equation is based on fractional mean-value theorems. A case is also presented in which a fractional Taylor series is used to derive the mass conservation equation.

In Chapter 5, the techniques described in Chapter 2 are applied to a different class of unconfined groundwater flow equation. This equation is derived using a form of the Forchheimer equation in place of Darcy’s equation. The resulting groundwater flow equation can be used to model turbulent flow in coarse and fractured porous media. The techniques described in Chapter 2 are used to construct polynomial approximate solutions to the governing equation for power-law head, exponential head, power-law flux and exponential flux conditions at the inlet of the aquifer. The constructed approximate solutions closely match numerical solutions generated using a modification of the method of Shampine.

Chapter 6 concludes this work with a summary, conclusions and recommendations for further research.
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Chapter 1

Introduction

This work presents the results of two different lines of research: one on groundwater flow equations and the other on a mass transport equation. First, methods of generating approximate polynomial solutions to unconfined groundwater flow equations are investigated. Second, a fractional conservation of mass equation, involving both left and right Caputo spatial fractional derivatives, is derived.

Nonlinear groundwater flow equations, such as the Boussinesq equation, can be used to model unconfined groundwater flow. The Boussinesq equation is derived under the assumption that Darcy’s Law is applicable. Unconfined flow equations based on generalizations of Darcy’s law and on non-Darcy relationships are considered in this work and are described in Section 1.1. Methods of deriving polynomial approximate solutions for these nonlinear flow equations are presented in Chapters 2 and 5. In Chapter 3, another researcher's analytical solution of the Boussinesq equation is discussed.

Flow and transport equations involving fractional derivatives have been successfully used to model anomalous phenomena in surface water and groundwater. In Section 1.2, some applications and derivations of these equations are discussed. In Chapter 4, a derivation of a fractional conservation of mass equation involving both left and right Caputo spatial fractional derivatives is presented. The presented derivation is from first principles and involves the analysis of mass transport through a control volume.
1.1 Unconfined Groundwater Flow Equations

The groundwater flow equations of this study describe one-dimensional, unconfined flow per unit width in homogeneous aquifers with horizontal impervious bottom layers. The Dupuit-Forchheimer assumptions, which are assumed to describe the flows, state that (1) “the hydraulic gradient is equal to the slope of the water table” and (2) “for small water-table gradients, the streamlines are horizontal and the equipotential lines are vertical” (Fetter, 1994). The equations can alternatively be derived by vertically averaging the three-dimensional equation for flows in fully-saturated porous media (Yeh et al., 2015).

A common equation used to describe unconfined flow is the Boussinesq equation. When the mass conservation equation

\[
S_y \frac{\partial h}{\partial t} = -\frac{\partial (hq)}{\partial x},
\]

in terms of flow per unit aquifer width \(hq\) (Moutsopoulos, 2009; Rupp and Selker, 2005), is combined with Darcy’s Law,

\[
q = -K \frac{\partial h}{\partial x},
\]

the Boussinesq equation is obtained:

\[
S_y \frac{\partial h}{\partial t} = K \frac{\partial}{\partial x} \left[ h \frac{\partial h}{\partial x} \right].
\]

If another constitutive relationship is used in place of Darcy’s Law, then unconfined flow equations of different forms can be derived. Two such equations are described in Sections 1.1.1 and 1.1.2.

For nonlinear unconfined flow equations, exact solutions rarely exist and numerical methods are used to generate approximate solutions. In cases where exact solutions are
not known, some method is needed to validate these numerical solutions. Approximate analytical solutions can be used to validate numerical solutions of such equations. In this study, methods are presented for constructing polynomial approximate solutions to these equations.

1.1.1 The Porous Medium Equation and its Applications

A porous medium equation (Vásquez, 2007) for one-dimensional flow in a homogeneous medium is an equation of the form

\[
\frac{\partial h}{\partial t} = a \frac{\partial^2 (h^k)}{\partial x^2}, \quad k > 1,
\]

(1.4)

where \(a\) represents properties of the fluid and medium. The Boussinesq equation is a special case of the porous medium equation for which \(k = 2\).

For groundwater flow, (1.4) can be derived in a manner similar to the derivation of the Boussinesq equation, except that in Darcy’s Law the hydraulic conductivity is assumed to be power-law function of head:

\[
K = K_0 h^n.
\]

(1.5)

The generalized Darcy Law then has the form

\[
q = -K \frac{\partial h}{\partial x} = -K_0 h^n \frac{\partial h}{\partial x}.
\]

(1.6)

When (1.6) is combined with the conservation of mass equation,

\[
S_y \frac{\partial h}{\partial t} = -\frac{\partial (h q)}{\partial x},
\]

(1.7)

the porous medium equation
\[
S_y \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left[ K_0 h^{n+1} \frac{\partial h}{\partial x} \right],
\]

or

\[
\frac{\partial h}{\partial t} = a \frac{\partial^2 (h^k)}{\partial x^2},
\]

is obtained. Here, \(S_y\) is the specific yield, \(a\) involves \(K_0\) and \(S_y\), and \(k = n + 2\).

Cases in which hydraulic conductivity is not constant have been studied by various researchers. Rupp and Selker [2005] considered the situation in which hydraulic conductivity is a power-law function of elevation. Specifically, the conductivity was assumed to decrease with depth. By vertically averaging the conductivity function over the depth of an aquifer, Rupp and Selker obtained a generalized Darcy law similar to (1.5). Some possible values of \(n\) for (1.5) are shown in Table 1.1.

<table>
<thead>
<tr>
<th>(n)</th>
<th>application</th>
<th>source</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Boussinesq equation</td>
<td></td>
</tr>
<tr>
<td>1.2 - 7.9</td>
<td>forest soils</td>
<td>Beven [1982]</td>
</tr>
<tr>
<td>4</td>
<td>concretes</td>
<td>Lockington et al. [1999]</td>
</tr>
<tr>
<td>1.405</td>
<td>air flow at normal temperature</td>
<td>Vásquez [2007]</td>
</tr>
</tbody>
</table>

Polynomial approximate solutions were constructed for a porous medium equation with a power-law head condition at the inlet of an initially-dry aquifer. This work with the porous medium equation appeared in Olsen and Telyakovskiy [2013] and is described in Chapter 2.

1.1.2 The Forchheimer Equation and its Applications

The quadratic Forchheimer equation (Hassanizadeh and Gray, 1987) is a generalization of Darcy’s Law of the form
\[- \frac{\partial h}{\partial x} = aq + bq|q|, \quad \text{(1.10)}\]

where \(a\) and \(b\) are constants and \(q\) is the volumetric flux. The parameter \(a\) is the reciprocal of the hydraulic conductivity. When \(b = 0\), (1.10) simplifies to Darcy’s Law.

The Forchheimer equation has been found to describe turbulent flows in coarse porous media (Bordier and Zimmer, 2000). Hassanizadeh and Gray [1987] explain that the dependence of microscopic drag forces on fluid velocity may cause the deviations of flows from Darcy’s Law.

If \(q > 0\), the solution for \(q\) in (1.10) has the form

\[q = \frac{a \pm \sqrt{a^2 - 4b \frac{\partial h}{\partial x}}}{2b}. \quad \text{(1.11)}\]

When \(-4b \frac{\partial h}{\partial x} \gg a^2\), and consequently \(\sqrt{-4b \frac{\partial h}{\partial x}} \gg a\), (1.11) can be simplified (Moutsopolous, 2009) to

\[q \approx \frac{a \pm \sqrt{-4b \frac{\partial h}{\partial x}}}{2b} \approx \frac{\sqrt{-4b \frac{\partial h}{\partial x}}}{2b} = \sqrt{-\frac{1}{b} \frac{\partial h}{\partial x}}. \quad \text{(1.12)}\]

The negative sign appears inside the square root because \(\frac{\partial h}{\partial x} < 0\) when \(q > 0\). When this form of Forchheimer’s equation (1.12) is combined with the conservation of mass equation for unconfined flow per unit width of aquifer, the following groundwater flow equation is obtained

\[S_y \frac{\partial h}{\partial t} = -\frac{\partial}{\partial x} \left[ h \sqrt{-\frac{1}{b} \frac{\partial h}{\partial x}} \right]. \quad \text{(1.13)}\]

Polynomial approximate solutions were constructed for (1.13) with power-law head, exponential head, power-law flux and exponential flux conditions at the inlet of the initially-dry unconfined aquifer. This work with the Forchheimer equation is described in Chapter
1.2 Fractional Conservation of Mass

Fractional differential equations have been used to model contaminant transport in groundwater Benson et al. [2000] and surface water Deng et al. [2004]. More recently, fractional governing equations have been derived for fluid flows. Su [2017] and Mehdinejadiani et al. [2013] derived fractional Boussinesq equations. Kavvas and Ercan [2017] derived a time-fractional governing equation for open-channel flows. Telyakovskiy et al. [to appear] comment on the nature of the fractional derivatives appearing in Kavvas and Ercan [2017]. Such fractional derivatives also appear in Mehdinejadiani et al. [2013].

Left fractional spatial derivatives of order between 1 and 2 can model superdiffusive or rapid transport of particles while temporal left derivatives of order between 0 and 1 can be used to model subdiffusive delays in particle transport Schumer et al. [2009]. Kelley and Meerschaert [2017] showed that left temporal fractional derivatives and right spatial fractional derivatives can both model delays in particle transport. Zhang et al. [2009] investigated four types of fractional advection-dispersion equations containing various fractional spatial and temporal derivatives. The nonlocal phenomena modeled by each equation type is discussed in detail.

Wheatcraft and Meerschaert [2008] derived a fractional conservation of mass equation involving left Caputo spatial fractional derivatives. Such equations could be used to derive fractional governing equations for nonlocal flow and transport phenomena. One of the main objectives of this work was to derive a fractional conservation of mass equation, containing both left and right Caputo spatial fractional derivatives, from first principles. This was done using fractional mean value theorems for Caputo fractional derivatives. The derivation of this two-sided fractional mass conservation equation appeared in Olsen et al. [2016] and is described in Chapter 4.
1.3 Outline

In Chapter 2, polynomial approximate solutions are constructed for an initially dry, unconfined horizontal aquifer with a power-law head condition at the inlet when the flow is described by a porous medium equation. The approximate solutions are compared to numerical solutions computed using a modification of Shampine’s method. Chapter 3 consists of a comment on another researcher’s traveling wave solution of the Boussinesq equation. Chapter 4 presents a derivation of a two-sided fractional conservation of mass equation involving Caputo spatial fractional derivatives. Chapter 5 presents polynomial approximate solutions to a nonlinear groundwater flow equation based on a Forchheimer representation of flux instead of Darcy’s Law. Power-law and exponential head and flux conditions at the inlet are considered. The results are again compared to numerical solutions generated using a modification of Shampine’s method. Finally, conclusions and recommendations for further research are outlined in Chapter 6.
Bibliography


Chapter 2

Polynomial Approximate Solutions of a Generalized Boussinesq Equation

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Abstract

A generalized Boussinesq equation, the porous medium equation, was analyzed for a semi-infinite initially dry unconfined aquifer. The boundary conditions were a power-law head condition at the inlet boundary and a zero-head condition at infinity. Quadratic and cubic polynomial approximate solutions to the equation were derived. These approximate solutions replicate known exact solutions of the Boussinesq and porous medium equations. The approximate solutions were also compared to numerical solutions of the generalized Boussinesq equation computed using a method of Shampine. It was found that the solutions are easy to use and they have sufficient accuracy to be useful in practical applications, such as when the hydraulic conductivity is a power-law function of elevation.
2.1 Introduction

The Boussinesq equation is used to model groundwater flow in unconfined aquifers when the Dupuit-Forchheimer assumptions are valid. The porous medium equation is a generalization of the Boussinesq equation that can be derived when the hydraulic conductivity is a power-law function of elevation Rupp and Selker [2005]. For example, such situations appear during the filtration of water in concretes Lockington et al. [1999] and in flows in forest soils Beven [1982]. The porous medium equation also models the flow of polytropic gases, gases where the pressure in the gas is a power-law function of density, in porous media Barenblatt [1952]. Moreover the porous medium equation is a special case of a more general nonlinear equation appearing in mathematical hydraulics Daly and Porporato [2004].

Lockington et al. [2000] analyzed the problem of flow in a one-dimensional unconfined semi-infinite horizontal aquifer. The aquifer was assumed to be initially dry with head as a power-law function of time at the inlet. The head at infinity was zero. The flow was modeled using the Boussinesq equation. Dimensional analysis was used to convert the problem to a boundary value problem for an ordinary differential equation. A quadratic approximate solution to this problem was derived. The approximate solution was in agreement with the numerical solution obtained using the method of Shampine [1973]. Furthermore, the quadratic approximate solution reproduced two known exact solutions of the Boussinesq equation Barenblatt [1952].

Telyakovskiy et al. [2002] extended the work of Lockington et al. [2000] by constructing a cubic approximate solution to the same boundary value problem. This solution improved the quadratic solution of Lockington et al. [2000] and reproduced the two known exact solutions. The method was later applied to other types of boundary conditions in Telyakovskiy and Allen [2006] and Telyakovskiy and Kurita [2007].

In this note, we apply the methods of Lockington et al. [2000] and Telyakovskiy et
al. [2002] to model one-dimensional, unconfined groundwater flow in a horizontal aquifer using the porous medium equation

$$\partial_t h = a\partial_x^2 h^k, \quad (x,t) \in (0,\infty) \times (0,\infty) \quad \text{and} \quad k > 1. \quad (2.1)$$

Such equations occur when hydraulic conductivity is a power-law function of water head, $K(h) = K_0 h^m$.

In Equation (2.1), $a$ depends on properties of the porous medium and fluid, $h$ is the head, $k$ is related to the exponent in the conductivity power law by $k = m + 1$, $x$ is the horizontal distance from the left boundary and $t$ is time.

We assume that the aquifer is initially dry,

$$h(x,0) = 0 \quad \text{for} \quad x > 0, \quad (2.2)$$

and that a power-law describes the head at the inlet boundary,

$$h(0,t) = \sigma t^\alpha, \quad \sigma > 0 \quad \text{and} \quad -\frac{1}{k+1} \leq \alpha < \infty. \quad (2.3)$$

The parameters $\alpha$ and $\sigma$ are constants. The value of $\alpha$ cannot be less than $-(k+1)^{-1}$, otherwise the water level in the inlet will be decreasing faster than the water level in the aquifer. In this case, the water would be discharged from the aquifer into the body of water, which is not the recharge situation considered here. For example, for flow in concretes $k = 5$ can be taken (Lockington et al., 1999), for flow in forest soils $k$ varies between 2.2 and 8.9 (Beven, 1982), and for air flow at normal temperature $k = 2.405$ (Vásquez, 2007).

Finally, the head at infinity is assumed to be zero,

$$\lim_{x \to \infty} h(x,t) = 0 \quad \text{for} \quad t > 0. \quad (2.4)$$

In Figure 1, an example of a flow modeled by Equations (2.1)–(2.4) when $\alpha = 0$ is shown.
Figure 2.1: Propagation of water into an empty aquifer, \( \alpha = 0 \), i.e. the case of constant head at the boundary.

This is the case of a constant-head boundary condition at the inlet \( x = 0 \). Similarly in Figure 2, the case when \( \alpha > 0 \) is shown. This represents a power-law decay in head at the inlet.

The situation modeled by Equations (2.1)–(2.4) can appear during flooding when a water level increases over time or when it suddenly increases to a certain value and stays there for some time. Similarly, Equation (2.3) can be replaced with other types of boundary conditions, such as power-law condition on flux, exponential head or exponential flux boundary conditions (Barenblatt, 1952; Barenblatt et al., 1990).

In Section 2, a similarity transformation is used to convert the problem into a boundary value problem. In Section 3, quadratic and cubic approximate solutions to this problem are constructed. In Section 4, a numerical solution to the boundary value problem is generated using the method of Shampine [1973] and the polynomial approximate solutions are compared with the numerical solution and known exact solutions.
Figure 2.2: Propagation of water into an empty aquifer, $\alpha > 0$, water head increases at the boundary.

### 2.2 Similarity Transformation

We use the following similarity transformation of Barenblatt [1952],

$$
\xi = \frac{x(1 + \alpha(k-1))^{1/2}}{(\sigma k^{-1} t^{1+\alpha(k-1)})^{1/2}} \quad \text{and} \quad H(\xi) = \frac{h(x,t)}{\sigma t^{\alpha}},
$$

(2.5)

to convert (2.1)-(2.4) into the equivalent boundary value problem

$$
\frac{d^2 H^k}{d\xi^2} + \frac{\xi}{2} \frac{dH}{d\xi} - \lambda H = 0, \quad \lambda = \frac{\alpha}{1 + \alpha(k-1)}, \quad -\frac{1}{2} \leq \lambda < \frac{1}{k-1},
$$

(2.6)

$$
H(0) = 1 \quad \text{and} \quad H(\xi_0) = 0,
$$

(2.7)

where $\xi_0$ is the position of the wetting front to be found in the process of solution. In (2.5) $\xi$ is a similarity variable and $H$ is a scaling function. These dimensionless quantities can be obtained using dimensional analysis and the Pi-Theorem. The parameter $\lambda$ depends on the behavior of the boundary and the nonlinearity of the equation. Fixing $k$ in (2.1) and $\alpha$
Figure 2.3: Scaling function $H$ versus similarity variable $\xi$; $\xi_0$ is the wetting front position.

in (2.3) will fix $\lambda$ in (2.6) and below. For each value of $\lambda$, we need to solve (2.6)–(2.7) only once. Later through the rescaling given by (2.5) we obtain solutions in the physical $(x, t)$ plane for any $x$ and $t$. In variables $(\xi, H)$, the solution looks like that shown in Figure 3.

By making the substitution $u = H^{k-1}$ (see e.g. Aronson, 1986; Vásquez, 2007), (2.6) and (2.7) are transformed into

$$\frac{k}{k-1} u \frac{d^2 u}{d \xi^2} + \frac{k}{k-1} \left( \frac{k}{k-1} - 1 \right) \left( \frac{du}{d\xi} \right)^2 + \frac{\xi}{2} \frac{1}{k-1} \frac{du}{d\xi} - \lambda u = 0, \quad (2.8)$$

$$u(0) = 1, \quad u(\xi_0) = 0. \quad (2.9)$$

In this new formulation, Equation (2.8) resembles the Boussinesq equation since $u$ and its derivatives are now raised to integer powers. As a result, one might expect that methods of approximating solutions of Boussinesq equation could also be used to approximate solutions to (2.8).
2.3 Polynomial Approximate Solutions

Lockington et al. [2000] used the following solution procedure with the Boussinesq equation \((k = 2 \text{ in } (2.1))\): instead of solving the equations analogous to (2.6) and (2.7), construct a quadratic polynomial based on certain relations that the exact solution satisfies. This approach was extended to a cubic approximation by Telyakovskiy et al. [2002]. We use the same methods to construct quadratic and cubic approximate solutions to (2.8) and (2.9). These approximate solutions satisfy the following conditions that are also satisfied by the exact solution of (2.8) and (2.9):

\[
\begin{align*}
\text{C.1} & \quad u(0) = 1, \\
\text{C.2} & \quad u(\xi_0) = 0, \\
\text{C.3} & \quad \frac{du}{d\xi}(\xi_0) = -\frac{\xi_0 k - 1}{2k}, \\
\text{C.4} & \quad \frac{k}{k-1} \left( \frac{k}{k-1} - 2 \right) \int_0^{\xi_0} \left( \frac{du}{d\xi} \right)^2 \xi d\xi + \frac{k}{k-1} \frac{1}{2} - \left( \frac{1}{k-1} + \lambda \right) \int_0^{\xi_0} u \xi d\xi = 0, \\
\text{C.5} & \quad \frac{k}{k-1} \frac{d^2 u}{d\xi^2}(0) + \frac{k}{k-1} \left( \frac{k}{k-1} - 1 \right) \left( \frac{du}{d\xi} \right)^2(0) - \lambda = 0.
\end{align*}
\]

(2.10)

Conditions C.1 and C.2 come from the boundary conditions in (2.9). Condition C.3 follows from (2.8) when \(\xi \to \xi_0\) and C.4 is obtained when (2.8) multiplied by \(\xi\) and integrated over \(\xi \in [0, \xi_0]\). Finally, Condition C.5 is obtained from (2.8) in the limit as \(\xi \to 0\).
2.3.1 Quadratic Approximate Solution

The exact solution of (2.8) satisfies all conditions in (2.10). The quadratic polynomial is based on conditions C.1 - C.4 only. The polynomial is of the form

\[ U_q(\xi) = a(\xi_0 - \xi) + b(\xi_0 - \xi)^2. \]  

The quadratic polynomial (2.11) has three coefficients, plus \( \xi_0 \), for a total of four unknowns. The constant term of the quadratic polynomial is zero due to C.2. Using C.3 and C.1, respectively, the coefficients \( a \) and \( b \) are found to be

\[ a = \frac{\xi_0}{2} \frac{k - 1}{k} \quad \text{and} \quad b = \frac{1}{\xi_0^2} - \frac{a}{\xi_0}. \]  

The quadratic polynomial can now be written as

\[ U_q(z) = \frac{\xi_0^2}{2} \frac{k - 1}{k} (1 - z) + \left(1 - \frac{\xi_0^2}{2} \frac{k - 1}{k}\right) (1 - z)^2, \]  

where \( z = \xi/\xi_0 \).

Condition C.4 can be written in terms of \( z \) as

\[ \frac{k}{k - 1} \left( \frac{k}{k - 1} - 2 \right) \int_0^1 \left( \frac{du}{dz} \right)^2 z dz + \frac{1}{2} \frac{k}{k - 1} - \left(\frac{1}{k - 1} + \lambda\right) \xi_0^2 \int_0^1 u zdz = 0. \]  

We now introduce \( y \) defined by

\[ y = \frac{\xi_0^2}{2} \frac{k - 1}{k}. \]  

Equations (2.13) and (2.14) become

\[ U_q(z) = y(1 - z) + (1 - y)(1 - z)^2 \]  

(2.16)
and

\[
\frac{k}{k-1} \left( \frac{k}{k-1} - 2 \right) \int_0^1 \left( \frac{du}{dz} \right)^2 zdz + \frac{1}{2} \frac{k}{k-1} - \left( \frac{1}{k-1} + \lambda \right) \frac{k}{k-1} y \int_0^1 uzdz = 0. \tag{2.17}
\]

Substituting (2.16) into (2.17) yields a quadratic equation in \( y \):

\[
(\lambda + 1)y^2 + \left( \frac{1}{k-1} + \lambda \right) y - \frac{k+1}{k-1} = 0. \tag{2.18}
\]

We take the solution with the positive square root since it replicates earlier work and the exact solutions. Using (2.15), the position of the wetting front \( \xi_0 \) is given by

\[
\frac{\xi_0^2}{2} \frac{k-1}{k} = -\left( \frac{1}{k-1} + \lambda \right) + \sqrt{\left( \frac{1}{k-1} + \lambda \right)^2 + 4(\lambda + 1) \frac{k+1}{k-1}}.
\tag{2.19}
\]

When \( \lambda = -1/2 \) and \( \lambda = 1/2 \) for \( k = 2 \), \( \xi_0 \) is \( \sqrt{8} \) and 2, respectively.

### 2.3.2 Cubic Approximate Solution

The cubic polynomial is based on conditions C.1 - C.5. The polynomial is of the form:

\[
U_c(\xi) = a(\xi - \xi_0) + b(\xi_0 - \xi)^2 + c(\xi_0 - \xi)^3. \tag{2.20}
\]

There are four coefficients, plus \( \xi_0 \), so there are four unknown parameters in the cubic polynomial. The constant term is zero due to C.2. Using C.3, the coefficients \( a \) and \( b \) are found to be

\[
a = \frac{\xi_0}{2} \frac{k-1}{k} \quad \text{and} \quad b = \frac{1}{\xi_0} - \frac{a}{\xi_0} - \frac{\alpha_0}{\xi_0^2},
\tag{2.21}
\]

where \( \alpha_0 = c\xi_0^2 \).
Using a technique similar to that which was used to solve for $\xi_0$ in the quadratic approximation, conditions C.4 and C.5 can be used to derive a system of nonlinear equations for $\alpha_0$ and $y$:

$$2(1 - y + 2\alpha_0) + \left(\frac{k}{k - 1} - 1\right)(y - 2 - \alpha_0)^2 - 2\lambda y = 0 \quad (2.22)$$

and

$$\left(\frac{k}{k - 1} - 2\right)(-2\alpha_0 + 5y^2 - 3y\alpha_0 + \alpha_0^2 + 10) + 15 - \left(\frac{1}{k - 1} + \lambda\right)y(5 + 5y - 2\alpha_0) = 0,$$

where $y$ is again given by Equation (2.15).

The constructed quadratic and cubic polynomials in (2.11) and (2.20) in general do not satisfy (2.8), but again they reproduce known exact solutions. Also, when $k = 2$ we obtain the earlier constructed quadratic and cubic polynomials in Lockington et al. [2000] and Telyakovskiy et al. [2002].

### 2.4 Numerical Comparisons

A method of Shampine [1973] was used to numerically solve (2.8) and (2.9) for $u(\xi)$. Shampine’s method uses a Runge-Kutta method and rescaling to solve differential equations of diffusion type. For details about the implementation, see Telyakovskiy et al. [2002].

Below, we show comparisons of the predicted wetting front position, $\xi_0$, approximated by the numerical solution of Shampine and the polynomial solutions described above. As previously stated, when $k = 2$, this problem has known exact solutions for $\xi_0$ when $\lambda = -1/2$ and $\lambda = 1/2$. The numerical solution and the polynomial approximate solutions reproduced these exact results.
In Table (2.1), the parameter $\lambda$ was fixed at a value of 0 while $k$ was varied. By the definition of $\lambda$ in (2.6), this case represents a constant head condition at the inlet since $\lambda = 0$ implies $\alpha = 0$. If the Shampine results are taken as a benchmark, the table shows that the cubic approximation produced more accurate results than the quadratic approximation, but both polynomial approximations produced reasonable results.

In Table (2.2), the parameter $k$ was fixed at a value of 3.0 while $\lambda$ was varied. Again, if the Shampine results are taken as a benchmark, the table shows that the cubic approximation produced more accurate results than the quadratic approximation.

### 2.5 Conclusions

Quadratic and cubic approximate solutions of the porous medium equation were derived. For each of them the scaling function has the form $H = u^{1/(k-1)}$, where $u = U_q$ in case of the quadratic approximation (2.11) and $u = U_c$ for the cubic approximation (2.20). The position of the wetting front predicted by these solutions agrees with that predicted numerically using Shampine’s method, although the cubic approximation, given by the solution of the algebraic system (2.22)–(2.23), was more accurate than the quadratic approximation.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Shampine</th>
<th>$H_q$</th>
<th>$H_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>4.4338</td>
<td>4.4542</td>
<td>4.4276</td>
</tr>
<tr>
<td>1.8</td>
<td>2.4648</td>
<td>2.4624</td>
<td>2.4651</td>
</tr>
<tr>
<td>2.0</td>
<td>2.2855</td>
<td>2.2828</td>
<td>2.2859</td>
</tr>
<tr>
<td>2.8</td>
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<td>1.9358</td>
</tr>
<tr>
<td>4.0</td>
<td>1.7411</td>
<td>1.7398</td>
<td>1.7412</td>
</tr>
<tr>
<td>4.8</td>
<td>1.6762</td>
<td>1.6752</td>
<td>1.6763</td>
</tr>
</tbody>
</table>
Table 2.2: Comparison of the wetting front position $\xi_0$ using Shampine’s method and quadratic and cubic approximations for $k = 3.0$ and different values of $\lambda$. Here, $U_q = H_q^{k-1}$ and $U_c = H_c^{k-1}$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Shampine</th>
<th>$H_q$</th>
<th>$H_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5</td>
<td>2.4495</td>
<td>2.4495</td>
<td>2.4495</td>
</tr>
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<td>-0.4</td>
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<td>2.2876</td>
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</tr>
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</tr>
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<td>2.0562</td>
<td>2.0528</td>
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</tr>
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<td>1.9662</td>
<td>1.9634</td>
<td>1.9666</td>
</tr>
<tr>
<td>0.0</td>
<td>1.8885</td>
<td>1.8864</td>
<td>1.8886</td>
</tr>
<tr>
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<tr>
<td>0.3</td>
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<td>1.7061</td>
<td>1.7057</td>
</tr>
<tr>
<td>0.4</td>
<td>1.6567</td>
<td>1.6579</td>
<td>1.6569</td>
</tr>
</tbody>
</table>

(2.19). When $k = 2$, the polynomial approximate solutions of this work reproduce the polynomial approximate solutions of Lockington et al. [2000] and Telyakovskiy et al. [2002] for the Boussinesq equation. Finally, the polynomial approximate solutions reproduce known exact solutions of the Boussinesq equation when $k = 2$.

The previous work of Lockington et al. [2000], Telyakovskiy et al. [2002], Telyakovskiy and Allen [2006], and Telyakovskiy and Kurita [2007] dealt with the construction of approximate analytical solutions to the Boussinesq equation. In the current work, approximate analytical solutions to the generalized Boussinesq equation were obtained. The traditional Boussinesq equation is a special case of this more general equation that allows for the modeling of more classes of problems that appear in practice, such as flows through concretes, through forest soils, and flows of gases through porous media. Before for the traditional Boussinesq equation, the hydraulic conductivity was a linear function of water head, now the technique was extended to the case when hydraulic conductivity is a power-law function of water head. The presented solution gives a simple way to get an accurate representation of modeled flows without solving numerically this nonlinear differential equation.
Acknowledgments

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Bibliography


Chapter 3

Comment on “Traveling wave solution of the Boussinesq equation for groundwater flow in horizontal aquifers” by H.A. Basha

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Using a traveling wave transformation, Basha [2013] analyzed a problem for the Boussinesq equation with a nonzero initial condition in an aquifer. The employed traveling wave transformation was $\xi = X - \rho T$, where $\rho$ is the propagation speed that was assumed to be a constant in the derivation process, but as the author pointed out “comes out to be a time-dependent function rather than a constant value.” Equations (32) and (35) in Basha [2013] illustrate this. So the original assumption that $\rho$ is constant is not valid. Here we will show how the derivation will change if from the beginning we assume that $\rho = \rho(T)$, i.e. $\xi = X - \rho(T)T$ and $\tau = \varphi T$.

Equation (5) will become

$$\frac{\partial}{\partial \xi} \left( H \frac{\partial H}{\partial \xi} \right) = \varphi \frac{\partial H}{\partial \tau} - \left( \rho + \tau \frac{\partial \rho}{\partial \tau} \right) \frac{\partial H}{\partial \xi}.$$  \hspace{1cm} (5)

The zeroth-order equation (7) and the first-order equation (8) will be

$$\frac{\partial}{\partial \xi} \left( H_0 \frac{\partial H_0}{\partial \xi} \right) + \left( \rho + \tau \frac{\partial \rho}{\partial \tau} \right) \frac{\partial H_0}{\partial \xi} = 0,$$  \hspace{1cm} (7)

$$\frac{\partial^2 (H_0 H_1)}{\partial \xi^2} + \left( \rho + \tau \frac{\partial \rho}{\partial \tau} \right) \frac{\partial H_1}{\partial \xi} = \frac{\partial H_0}{\partial \tau}.$$  \hspace{1cm} (8)

From equation (7) we see that $H_0$ is not only a function of $\xi$, but of $\tau$ too. So the perturbation series will be

$$H(\xi, \tau) = H_0(\xi, \tau) + \varphi H_1(\xi, \tau) + \ldots,$$  \hspace{1cm} (6)

while in Basha [2013] $H_0$ was a function of $\xi$ only. This means that in equation (8) we have a nontrivial forcing function on the right-hand side. So in general $H_1 \neq 0$ although it will satisfy the trivial boundary conditions. However in Basha [2013] it was deduced that $H_1$ is identically equal to zero, which is not the case.
The new form of equation (12) will be

\[ \left( \rho + \tau \frac{\partial \rho}{\partial \tau} \right) X = H_b - H_0 - H_i \ln \left( \frac{H_0 - H_i}{H_b - H_i} \right). \] (12)

Similarly in (13) \( \rho \) will be replaced by \( \rho + \tau \frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial \tau}(\tau \rho) \).

When \( H \approx H_0 \) in Basha [2013], \( \rho \) was found first using a zeroth moment estimate in section 4.1.1 and then again using a first moment estimate in section 4.1.2. Since the left hand side of equation (12) has changed, equation (32) actually yields a formula for \( \rho_0 + \tau \frac{\partial \rho_0}{\partial \tau} \) and not for \( \rho_0 \). The revised equation is

\[ \rho_0 + \tau \frac{\partial \rho_0}{\partial \tau} = \frac{H_b^2 - H_i^2}{2V_0}. \] (32)

This means that the actual expression for the traveling wave velocity will be

\[ \rho_0 = \frac{1}{T} \int_0^T \frac{H_b^2 - H_i^2}{2V_0} d\hat{T}. \]

So the new estimate for \( \rho_0 \) is the average of the old estimate for \( \rho_0 \) in Basha [2013]. Similarly, equation (35) for the traveling wave velocity from the first moment estimate will have to be modified

\[ \left( \rho_1 + \tau \frac{\partial \rho_1}{\partial \tau} \right)^2 = \frac{A_1}{P_1}. \] (35)

We note that the results predicted by equation (13) will not change if from the beginning we assume that \( \rho = \rho(T) \), since in (13) instead of \( \rho \) we will have \( \rho + \tau \frac{\partial \rho}{\partial \tau} \), and the new forms of equations (32) and (35) will give an estimate for the same quantity \( \rho + \tau \frac{\partial \rho}{\partial \tau} \).

We note that the results from Basha [2013] discussed above were based on assumptions that resulted in \( H_i \equiv 0, i = 1, 2, \ldots \). See for example section 5.2 paragraph [99] in Basha [2013] where differences between the TW model (the traveling wave model) and the numerical method in Figure 3 are discussed: it is observed that “the TW model fails to capture
the curvature of the groundwater level because of the inherent assumption of the traveling wave approach \( \partial H / \partial T = 0 \).” The result of this assumption was that equation (8) had only the trivial solution. Effectively, the perturbation series was truncated after the zeroth order term. This explains the differences in Figure 3 Basha [2013] between the analytical and numerical solutions.

If one considers \( H \approx H_0 + \varphi H_1 \), then equation (8) should be resolved and the equations for the zeroth and the first moment should be resolved to get the new values of \( \rho \) which is a significantly more challenging task.
Bibliography

Chapter 4

A Two-Sided Fractional Conservation of Mass Equation

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Abstract

A two-sided fractional conservation of mass equation is derived by using left and right fractional Mean Value Theorems. This equation extends the one-sided fractional conservation of mass equation of Wheatcraft and Meerschaert. Also, a two-sided fractional advection-dispersion equation is derived. The derivations are based on Caputo fractional derivatives.
4.1 Introduction

Spatial and temporal fractional-order differential equations have practical application in the modeling of hydrologic processes such as solute transport in surface water (Schumer et al., 2003) and groundwater (Benson et al., 2001). Spatially fractional equations can model rapid solute transport while temporally fractional equations can model delays in transport (Meerschaert and Sikorskii, 2012).

Various approaches have been taken for selecting the fractional governing equations for hydrologic processes. Fractional constitutive laws (Benson et al., 2004), probabilistic derivations (Schumer et al., 2009) and fractional conservation laws (Wheatcraft and Meerschaert, 2008) have been used. Zhang et al. [2007] discuss the various forms that the fractional advection-dispersion equation can take. Definitions and properties of fractional derivatives can be found in the works of Kilbas et al. [2006] and Podlubny [1999].

Wheatcraft and Meerschaert [2008] derived a fractional conservation of mass equation using the left fractional Taylor series of Odibat and Shawagfeh [2007]. This conservation equation involves left local Caputo fractional derivatives and the equation is derived in a manner analogous to the derivation of the differential form of the classical conservation of mass equation. More recently, Mehdinejadiani et al. [2013] used the same approach to derive a fractional Boussinesq equation.

A derivation of a fractional advection-dispersion equation using a different fractional Taylor series was presented in Schumer et al. [2003]. The series was based on a Riemann-Liouville fractional derivative. The series was used to derive a fractional Fick’s Law. This fractional Fick’s Law was used with a classical conservation of mass equation to derive a fractional advection-dispersion equation.

In this work, we use fractional mean value theorems to derive a two-sided fractional conservation of mass equation involving both left and right Caputo fractional derivatives. In Section 2 we discuss the fractional mean value theorems. In Section 3 we derive a two-
sided conservation of mass equation. We show how the result obtained in Wheatcraft and Meerschaert [2008] is related to the construction presented here. In Section 4 we obtain the corresponding advection-dispersion equation. We finish the paper with conclusions in Section 5.

4.2 Fractional Mean Value Theorems

The left and right Caputo fractional derivatives (Kilbas et al., 2006) of orders $\alpha > 0$ and $\beta > 0$ of a function $f$ can be defined by

\begin{equation}
(LD^\alpha_a f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(u)}{(x-u)^{\alpha-m+1}} du
\end{equation}

and

\begin{equation}
(RD^\beta_b f)(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \int_x^b \frac{f^{(m)}(u)}{(u-x)^{\beta-m+1}} du,
\end{equation}

where $m-1 < \alpha \leq m$ and $m-1 < \beta \leq m$ for some positive integer $m$. The positions of the point of evaluation, $x$, and the endpoints of the interval are shown in Figure 4.1.

Figure 4.1: The left fractional derivative of a function depends on points from a left endpoint $a$ up to the point of evaluation $x$. The right fractional derivative depends on points from $x$ up to a right endpoint $b$.

Diethelm [2012] derived the following Mean Value Theorem for left Caputo fractional derivatives. For $0 < \alpha \leq 1$, $f \in C[a,b]$ and $LD^\alpha_a f \in C[a,b]$, there exists $\xi \in (a,b)$ such that

\begin{equation}
f(b) = f(a) + \frac{(LD^\alpha_a f)(\xi)}{\Gamma(\alpha+1)} (b-a)^\alpha.
\end{equation}
Figure 4.2 shows, by using the left Mean Value Theorem, that a function value at a point \( b \) depends on the function value at a point \( a \) and the left-fractional derivative of the function at some point \( \xi \in (a, b) \). The left fractional derivative of \( f \) at \( \xi \) is computed using points from \( a \) to \( \xi \).

![Diagram](a \xi b)

**Figure 4.2:** The left fractional Mean Value Theorem shows that a function value at the right endpoint of an interval can be written in terms of the function value at the left endpoint \( a \) and \( ^L D^\alpha_a f \) at some unspecified point \( \xi \) in \( (a, b) \).

By analogy with the results for the left Caputo fractional derivative given in Diethelm [2012], similar results can be obtained for right Caputo fractional derivatives. This Mean Value Theorem for right Caputo fractional derivatives states that when \( 0 < \beta \leq 1 \), \( f \in C[a, b] \) and \( ^R D^\beta_b f \in C[a, b] \), there exists \( \theta \in (a, b) \) such that

\[
f(a) = f(b) + \left( ^R D^\beta_b f \right)(\theta) \frac{(b-a)^\beta}{\Gamma(\beta+1)}.
\]  (4.4)

Figure 4.3 shows, by using the right Mean Value Theorem, that a function value at a point \( a \) depends on the function value at a point \( b \) and the right-fractional derivative of the function at some point \( \theta \in (a, b) \). The right fractional derivative of \( f \) at \( \theta \) is computed using points from \( \theta \) to \( b \).

![Diagram](a \theta b)

**Figure 4.3:** The right fractional Mean Value Theorem shows that the function value at the left endpoint of an interval can be written in terms of the function value at the right endpoint \( b \) and \( ^R D^\beta_b f \) at an unspecified point \( \theta \) in \( (a, b) \).

Now consider the interval \([x - \Delta x/2, x + \Delta x/2]\). If we take \( a = x \) and \( b = x + \Delta x/2 \) in (4.3) and \( a = x - \Delta x/2 \) and \( b = x \) in (4.4), we obtain
\[ f(x + \Delta x/2) = f(x) + \left( \frac{L^{-\alpha} x f(\xi)}{\Gamma(\alpha + 1)} \right) \left( \Delta \frac{x}{2} \right)^{\alpha} \] (4.5)

and

\[ f(x - \Delta x/2) = f(x) + \left( \frac{R^{-\beta} x f(\theta)}{\Gamma(\beta + 1)} \right) \left( \Delta \frac{x}{2} \right)^{\beta}. \] (4.6)

When the left and right Mean Value Theorems are used, Figures 4.4 and 4.5 show the points on which \( f(x + \Delta x/2) \) and \( f(x - \Delta x/2) \) depend.

**Figure 4.4:** The left fractional Mean Value Theorem allows \( f(x + \Delta x/2) \) to be written in terms of \( f(x) \) and the left fractional derivative of \( f \) at \( \xi \).

**Figure 4.5:** The right fractional Mean Value Theorem allows \( f(x - \Delta x/2) \) to be written in terms of \( f(x) \) and the right fractional derivative of \( f \) at \( \theta \).

Equations (4.5) and (4.6) are exact. We will also use approximate forms of these equations. Indeed, these fractional mean-value equations can be written in approximate form by letting \( \xi \) and \( \theta \) equal the right and left endpoints of the interval, respectively:

\[ f(x + \Delta x/2) \approx f(x) + \left( \frac{L^{-\alpha} x f(x + \Delta x/2)}{\Gamma(\alpha + 1)} \right) \left( \Delta \frac{x}{2} \right)^{\alpha} \] (4.7)

and

\[ f(x - \Delta x/2) \approx f(x) + \left( \frac{R^{-\beta} x f(x - \Delta x/2)}{\Gamma(\beta + 1)} \right) \left( \Delta \frac{x}{2} \right)^{\beta}. \] (4.8)

The fractional mean-value equations (4.5) and (4.6) can also be written in approximate
form by letting $\xi$ and $\theta$ approach the center, $x$, of the interval if fewer conditions on the smoothness of $f$ are assumed. In this case, the derivatives in these equations will be local fractional derivatives:

$$f(x + \Delta x/2) \approx f(x) + \frac{(LD_\xi^\alpha f)(x+)}{\Gamma(\alpha + 1)}(\Delta x/2)^\alpha$$

and

$$f(x - \Delta x/2) \approx f(x) + \frac{(RD_\xi^\beta f)(x-)}{\Gamma(\beta + 1)}(\Delta x/2)^\beta.$$  

The $+$ and $-$ in (4.9) and (4.10) denote the limits as the points at which the fractional derivatives are evaluated approach the center of the interval, $x$, from the right and left. In the Appendix, local fractional Taylor series are discussed since Equations (4.9) and (4.10) can also be obtained by truncating the fractional Taylor series after the second terms. Local left Caputo fractional derivatives appear in the fractional conservation of mass equation in Wheatcraft and Meerschaert [2008] and in the derivation of the fractional Boussinesq equation in Mehdinejadiani et al. [2013].

In Section 4.3 we use the fractional mean value theorems (4.5) and (4.6) and the approximations (4.7) and (4.8) to derive two-sided fractional conservation of mass equations.

### 4.3 Derivation of a Two-sided Conservation of Mass Equation

Now that the fractional mean value theorems have been described, we use them to derive a two-sided fractional conservation of mass equation. Consider in Figure 4.6 the control volume $\Delta V$, not necessarily infinitesimal in volume, with center at $(x_1,x_2,x_3)$. We first determine the temporal rate of change of mass in $\Delta V$ in the $x_1$ direction. The resulting
expression is later extended to the $x_2$ and $x_3$ directions.

![Diagram of the control volume $\Delta V$](image)

**Figure 4.6:** Diagram of the control volume $\Delta V$

Let $F_1(x_1 - \Delta x_1/2, x_2, x_3, t)$ and $F_1(x_1 + \Delta x_1/2, x_2, x_3, t)$ denote the components of the mass flux passing through the faces of $\Delta V$ at locations $x_1 - \Delta x_1/2$ and $x_1 + \Delta x_1/2$, respectively, in the $x_1$ direction. The temporal rate of change of mass in $\Delta V$ in the $x_1$ direction is then

$$F_1(x_1 - \Delta x_1/2, x_2, x_3, t)A_{x_1-\Delta x_1/2} - F_1(x_1 + \Delta x_1/2, x_2, x_3, t)A_{x_1+\Delta x_1/2}, \quad (4.11)$$

where $A_{x_1-\Delta x_1/2}$ and $A_{x_1+\Delta x_1/2}$ are the areas of the control volume faces perpendicular to the flux at locations $x_1 - \Delta x_1/2$ and $x_1 + \Delta x_1/2$.

We now use the exact fractional mean value theorems (4.5) and (4.6) to rewrite the flux factors $F_1$ at points $(x_1 - \Delta x_1/2, x_2, x_3)$ and $(x_1 + \Delta x_1/2, x_2, x_3)$. The values of $F_1$ at the faces depend on the flux in the $x_1$ direction at the center, $(x_1, x_2, x_3)$, of the control volume and on fractional derivatives evaluated at the points shown below:
\[
F_1(x_1 - \Delta x_1/2, x_2, x_3, t) = F_1(x_1, x_2, x_3, t) + \frac{(R D_{\beta_1} F_1)(\theta_1, x_2, x_3, t)}{\Gamma(\beta_1 + 1)} \left( \frac{\Delta x_1}{2} \right)^{\beta_1}
\]

and

\[
F_1(x_1 + \Delta x_1/2, x_2, x_3, t) = F_1(x_1, x_2, x_3, t) + \frac{(L D_{\alpha_1} F_1)(\xi_1, x_2, x_3, t)}{\Gamma(\alpha_1 + 1)} \left( \frac{\Delta x_1}{2} \right)^{\alpha_1}.
\]

The right sides of equations (4.12) and (4.13) can be substituted into (4.11) to obtain

\[
\left( \frac{(R D_{\beta_1} F_1)(\theta_1, x_2, x_3, t)}{\Gamma(\beta_1 + 1)} \left( \frac{\Delta x_1}{2} \right)^{\beta_1} \right) A_{x_1 - \Delta x_1/2} - \left( \frac{(L D_{\alpha_1} F_1)(\xi_1, x_2, x_3, t)}{\Gamma(\alpha_1 + 1)} \left( \frac{\Delta x_1}{2} \right)^{\alpha_1} \right) A_{x_1 + \Delta x_1/2}.
\]

For the volume \(\Delta V\), the area \(A_{x_1 - \Delta x_1/2} = A_{x_1 + \Delta x_1/2} = \Delta x_2 \Delta x_3\) and (4.14) becomes

\[
\left( \frac{(R D_{\beta_1} F_1)(\theta_1, x_2, x_3, t)}{\Gamma(\beta_1 + 1)} \left( \frac{\Delta x_1}{2} \right)^{\beta_1} \right) \Delta x_2 \Delta x_3
\]

\[
- \left( \frac{(L D_{\alpha_1} F_1)(\xi_1, x_2, x_3, t)}{\Gamma(\alpha_1 + 1)} \left( \frac{\Delta x_1}{2} \right)^{\alpha_1} \right) \Delta x_2 \Delta x_3.
\]

Similar substitutions can be made for \(F_2\) and \(F_3\) to describe the temporal rate of change of mass in the \(x_2\) and \(x_3\) directions. The sum of the three resulting expressions equals the temporal rate of change of mass in \(\Delta V\). This temporal rate of change can also be represented by the integral shown in (4.17) below to form a two-sided fractional conservation of mass.
In (4.15), the flux terms are evaluated at unknown points which make the fractional mean value theorems exact. The derivation given above can also be done in approximate form by using (4.7) and (4.8) or (4.9) and (4.10). The latter case involves local fractional derivatives.

Using (4.7) and (4.8) leads to an approximate form of (4.15):

\[
\begin{align*}
\frac{RD_{x_i}^{\beta_i}(F_1)}{\Gamma(\beta_i + 1)} (x_1 - \Delta x_1/2, x_2, x_3, t) \left( \frac{\Delta x_1}{2} \right)^{\beta_i} \Delta x_2 \Delta x_3 \\
- \frac{LD_{x_i}^{\alpha_i}(F_1)}{\Gamma(\alpha_i + 1)} (x_1 + \Delta x_1/2, x_2, x_3, t) \left( \frac{\Delta x_1}{2} \right)^{\alpha_i} \Delta x_2 \Delta x_3.
\end{align*}
\]

(4.16)

For groundwater flow, we substitute \( \rho q_i \) for the flux \( F_i \) and use Einstein’s summation notation to represent the sum of the net influxes to \( \Delta V \) in each direction. We suppress the arguments of the \( \rho q_i \) in the derivation that follows. The reader should keep in mind that the equations that follow are exact or approximate depending on where the \( \rho q_i \) terms are evaluated. We now assume that the control volume is a cube with sides of length \( \Delta x \):

\( \Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x \).

\[
\frac{RD_{x_i}^{\beta_i}(\rho q_i)}{\Gamma(\beta_i + 1)} (\Delta x)^{\beta_i+2} \frac{\partial}{\partial t} \int_{\Delta V} n_0 \rho dV - \frac{LD_{x_i}^{\alpha_i}(\rho q_i)}{\Gamma(\alpha_i + 1)} (\Delta x)^{\alpha_i+2} = \frac{\partial}{\partial t} \int_{\Delta V} n_0 \rho dV.
\]

(4.17)

Here, \( n_0 \) is the porosity of the medium, \( q_i \) is the volumetric flux in the \( x_i \) direction and \( \rho \) is the fluid density. This equation contains different derivative orders in each direction to allow for the modeling of anisotropic flow. If we now assume that these derivatives are all of the same order, \( \alpha \), (4.17) becomes

\[
RD_{x_i}^{\alpha_i}(\rho q_i) - LD_{x_i}^{\alpha_i}(\rho q_i) = \frac{2\alpha_i \Gamma(\alpha + 1)}{(\Delta x)^{\alpha_i+2}} \frac{\partial}{\partial t} \int_{\Delta V} n_0 \rho dV.
\]

(4.18)

Finally, if the porosity and the fluid density are spatially constant in \( \Delta V \), which is the
case if $\Delta V$ is infinitesimal in volume, we obtain

$$RD_{x_i}^\alpha(\rho q_i) - LD_{x_i}^\alpha(\rho q_i) = \frac{2^\alpha \Gamma(\alpha + 1)}{(\Delta x)^{\alpha+2}} \frac{\partial}{\partial t}(n_0\rho \Delta V), \quad (4.19)$$

where $\Delta V = (\Delta x)^3$ or $(\Delta x)^2 \Delta x_3$. The vertical direction $x_3$ can be distinguished from the other directions so that (4.19) can be written for a vertically compressible medium. Equation (4.19) is now written with $\Delta x_3$ separated from the other $\Delta x$ factors as follows:

$$RD_{x_i}^\alpha(\rho q_i) - LD_{x_i}^\alpha(\rho q_i) = \frac{2^\alpha \Gamma(\alpha + 1)}{(\Delta x)^{\alpha}} \frac{\partial}{\partial t}(\Delta x_3 n_0 \rho). \quad (4.20)$$

If the $\rho q_i$ in (4.20) are evaluated at the center of the control volume, the derivatives would be local fractional derivatives. Written explicitly in terms of integrals, this approximate form of (4.20) is

$$\lim_{x \to x^{-i}} \frac{-1}{\Gamma(1 - \alpha)} \int_x^{x_i} \frac{(\rho q_i)'}{(u-x)^{\alpha}} du - \lim_{x \to x^{+i}} \frac{1}{\Gamma(1 - \alpha)} \int_{x_i}^x \frac{(\rho q_i)'}{(x-u)^{\alpha}} du = \frac{2^\alpha \Gamma(\alpha + 1)}{(\Delta x)^{\alpha}} \frac{\partial}{\partial t}(\Delta x_3 n_0 \rho). \quad (4.21)$$

For comparison, the fractional conservation of mass equation of *Meerschaert and Wheatcraft* (Equation (28) of [16]), which involves local left Caputo fractional derivatives, written using the notation of (4.20) is

$$-LD_{x_i}^\alpha(\rho q_i) = \frac{\Gamma(\alpha + 1)}{(\Delta x)^{\alpha}} \frac{\partial}{\partial t}(\Delta x_3 n_0 \rho). \quad (4.22)$$

For $\alpha = 1$, we note that the equations (4.20) and (4.22) reduce to the classical mass conservation equation involving traditional derivatives.

In the derivation of the two-sided conservation of mass equation we used mean value theorems based on Caputo fractional derivatives (*Diethelm*, 2012). We note that there are
other types of fractional derivatives, such as the Riemann-Liouville fractional derivative, and a Taylor series based on this fractional derivative has been derived (Trujillo et al., 1999). We did not derive the corresponding conservation of mass equation using this Taylor series since in hydrologic applications Caputo fractional derivatives are more commonly used.

### 4.4 Derivation of the advection-dispersion equation

Using the conservation of mass equation derived in the previous section, we obtain a corresponding fractional advection-dispersion equation for the concentration of a solute. The traditional assumptions are used in the derivation of this advection-dispersion equation. Namely that the pores are filled with fluid and that Darcy’s law can be used. For the derivation of the classical advection-dispersion equation using integer-order derivatives, see Fetter [1999].

The flux function in direction $x_i$ can be written as

$$F_i = v_i n_0 C - n_0 D_i \frac{\partial C}{\partial x_i}, \quad (4.23)$$

where the respective terms represent the advective and dispersive flux. We denote the concentration of the solute by $C$, which is the mass of solute per unit volume of the solution; the linear velocity in the direction $x_i$ by $v_i$, the hydrodynamic dispersion coefficient in the direction $x_i$ by $D_i$; and as before $n_0$ represents the porosity.

Substituting the expression for the flux (4.23) into the mass balance equation (4.18) we obtain

$$RD_{x_i}^\alpha (v_i n_0 C) - LD_{x_i}^\alpha (v_i n_0 C) - RD_{x_i}^\alpha \left( n_0 D_i \frac{\partial C}{\partial x_i} \right) + LD_{x_i}^\alpha \left( n_0 D_i \frac{\partial C}{\partial x_i} \right) = \frac{2^{\alpha} \Gamma(\alpha + 1)}{(\Delta x)^{\alpha+2}} \frac{\partial}{\partial t} \int_{\Delta V} n_0 C dV. \quad (4.24)$$
Equation (4.24) can represent transport in more than one dimension when the left side of equation (4.24) is summed over the index $i$.

For purely advective transport, equation (4.24) takes the following form:

$$R^\alpha D^\alpha_{x_i} (v_i n_0 C) - L^\alpha D^\alpha_{x_i} (v_i n_0 C) = \frac{2^\alpha \Gamma (\alpha + 1)}{(\Delta x)^{\alpha+2}} \frac{\partial}{\partial t} \int_{\Delta V} n_0 C dV. \quad (4.25)$$

If the velocity field is uniform and the porosity is constant or if these quantities can be represented by average values in a heterogeneous medium, then (4.25) can be reduced to

$$v_i R^\alpha D^\alpha_{x_i} (C) - v_i L^\alpha D^\alpha_{x_i} (C) = \frac{2^\alpha \Gamma (\alpha + 1)}{(\Delta x)^{\alpha+2}} \frac{\partial}{\partial t} \int_{\Delta V} C dV. \quad (4.26)$$

Unlike the classical advection-dispersion equation that involves integer-order derivatives, we keep the integral over $\Delta V$ on the right-hand side of (4.24) and (4.26) since $C$ does not have to be spatially constant and $\Delta V$ does not have to be an infinitesimal volume.

Often one of the coordinate axes is taken in the direction of the flow. Assume that the flow is uniform and two-dimensional in the direction $x_1$. If the porous medium is homogeneous and isotropic or if the dispersion coefficients can be replaced by average values in a heterogeneous medium, we can rewrite equation (4.24) as follows:

$$v \left[ R^\alpha D^\alpha_{x_1} (C) - L^\alpha D^\alpha_{x_1} (C) \right] + D_L \left[ - R^\alpha D^\alpha_{x_1} \left( \frac{\partial C}{\partial x_1} \right) + L^\alpha D^\alpha_{x_1} \left( \frac{\partial C}{\partial x_1} \right) \right]$$

$$+ D_T \left[ - R^\alpha D^\alpha_{x_2} \left( \frac{\partial C}{\partial x_2} \right) + L^\alpha D^\alpha_{x_2} \left( \frac{\partial C}{\partial x_2} \right) \right] = \frac{2^\alpha \Gamma (\alpha + 1)}{(\Delta x)^{\alpha+2}} \frac{\partial}{\partial t} \int_{\Delta V} C dV. \quad (4.27)$$

Here, $D_L$ is the longitudinal hydrodynamic dispersion, and $D_T$ is the transverse hydrodynamic dispersion.

Equation (4.27) with local fractional derivatives or with fractional derivatives evaluated at the edges of the control volume can be viewed as an intermediate model between the classical advection-dispersion equation that uses integer-order derivatives and the advection-
dispersion equation that uses conventional fractional derivatives. The classical advection-dispersion equation with integer-order derivatives has the rates of change evaluated at a point in the spatial domain, while the equation with fractional derivatives involves integration over the spatial domain. In Equation (4.27), unlike with the infinitesimal control volume that is considered in the derivation of the equation involving integer-order derivatives, we have a finite control volume and some global properties of the function are captured by the fractional derivatives appearing on the left side of (4.27), over the control volume or in the limiting sense. Furthermore, the control volume appears explicitly in the equation. So Equation (4.27) is an intermediate model between the advection-dispersion equation with integer order derivatives and the equation involving conventional fractional derivatives in space, where the integration is done over the whole spatial domain.

4.5 Conclusions

We introduced a local right fractional Taylor series and derived a fractional conservation of mass equation based on left and right Caputo fractional derivatives. This derivation extends the earlier fractional conservation of mass equation of Wheatcraft and Meerschaert [2008] that was based on the left fractional Taylor series of Odibat and Shawagfeh [2007] and the fractional derivatives were local fractional derivatives. The fractional Boussinesq equation derived in Mehdinejadiani et al. [2013] used the same local fractional derivatives. The conservation of mass equations, (4.17) and (4.18), incorporate nonlocality by including the dimension $\Delta x$ of the control volume, by taking the integral over the control volume $\Delta V$, and by the evaluation of fractional derivatives at the edges of the control volume $\Delta V$. Also, we derived the corresponding advection-dispersion equation.

By adding a right fractional derivative to the mass conservation law, we can potentially model more realistic particle motions. Also, by using different orders of left and right fractional derivatives, we can model behavior that differs depending on geometric direc-
tion. Our equation can be viewed as an intermediate model between local-in-nature models involving integer-order derivatives and global models with conventional fractional derivatives. As a result, some global effects are included in the equations that we present in the paper.

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4.6 Appendix A: Derivation of a local, right fractional Taylor series

Odibat and Shawagfeh [2007] derived the following generalized Taylor series using local left Caputo fractional derivatives. For $0 < \alpha < 1$,

$$
f(x) = \sum_{k=0}^{n} \frac{(L D_a^{(k\alpha)} f)(a^+) (x-a)^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{(L D_a^{((n+1)\alpha)} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\alpha}, \tag{4.28}
$$

for some $\xi$ such that $a \leq \xi \leq x$.

The $a^+$ appearing in the series represents the limit of the function as the argument approaches $a$ from the right. The presence of this limit makes each of the coefficients in this Taylor series a local fractional derivative since the upper limit of integration is approaching the lower limit. For example, the $k = 1$ term contains the derivative
\[ (L D_a^{(\alpha)} f)(a+) = \lim_{x \to a^+} \frac{1}{\Gamma(1 - \alpha)} \int_a^x \frac{f'(u)}{(x-u)^\alpha} du. \] (4.29)

In this fractional Taylor series, derivative orders greater than one represent sequential fractional derivatives. The order of differentiation appears in parentheses to denote this fact. For example, \( L D_a^{(2\alpha)} f \) represents \( L D_a^{\alpha}(L D_a^{\alpha} f) \), which is not the same as \( L D_a^{2\alpha} f \) (Li and Deng, 2007). In the derivation of fractional conservation of mass, we will use this series truncated to two terms:

\[ f(x) \approx f(a+) + \frac{(L D_a^{\alpha} f)(a+)}{\Gamma(\alpha + 1)} (x - a)^\alpha. \] (4.30)

We note that there are other types of local fractional derivatives, where again the upper limit of integration approaches the lower limit, see e.g. Chen et al. [2010].

Using the methods of Odibat and Shawagfeh [2007] and analogous assumptions about properties of the function \( f \), for \( 0 < \beta < 1 \) a generalized Taylor series involving right Caputo fractional derivatives can be derived.

The derivation of the Taylor series involving right fractional Caputo derivatives is similar to the derivation of the Taylor series that uses left fractional Caputo derivatives (Odibat and Shawagfeh, 2007). The derivatives used in Odibat and Shawagfeh [2007] are local left Caputo fractional derivatives. In our derivation the right fractional derivatives are also of a local nature.

We note that Caputo derivatives on a finite interval \([a, b]\) are, by definition, Riemann-Liouville fractional integrals composed with classical derivatives (Kilbas et al., 2006).

\[ (R D_b^{\beta} f)(x) = (-1)^m (R J_b^{-\beta} D_b^m f)(x) \] (4.31)
where the Riemann-Liouville fractional integral is

\[
(R^\beta J_b) f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (u-x)^{\beta-1} f(u) \, du
\]  

(4.32)

with \(x \leq b, m = \lceil \beta \rceil\).

We can establish the following right-fractional mean value theorem:

**Theorem 1.** If \(f \in C[a, b]\) and \(RD_b^\beta f \in C[a, b]\) for \(0 < \beta \leq 1\), then

\[
f(x) = f(b) + \frac{1}{\Gamma(\beta)} (RD_b^\beta f)(\xi)(b-x)^\beta
\]  

(4.33)

where \(x \leq \xi \leq b\).

The proof is analogous to the proof for left fractional derivatives (Diethelm, 2012; Odibat and Shawagfeh, 2007).

To obtain the expression for the right-fractional Taylor series we need the following technical statement.

**Lemma 2.** Suppose that \(RD_b^{(n \beta)} f(x) \in C[a, b]\) and \(RD_b^{((n+1) \beta)} f(x) \in C[a, b]\) for \(0 < \beta \leq 1\), then

\[
(R^\beta J_b RD_{b}^{(n \beta)} f(x)) - (R^\beta J_b RD_{b}^{((n+1) \beta)} f(x)) = \frac{(b-x)^{n \beta}}{(n\beta + 1)} (RD_{b}^{(n \beta)} f)(b-). \tag{4.34}
\]

The proof is similar to the proof in Odibat and Shawagfeh [2007].

Now we can write the expression for the right-fractional Taylor series.

**Theorem 3.** Suppose that \(RD_b^{(k \beta)} f(x) \in C[a, b]\) for \(k = 0, 1, 2, \ldots, n+1\) and \(0 < \beta \leq 1\), then

\[
f(x) = \sum_{k=0}^{n} \frac{(RD_{b}^{(k \beta)} f)(b-)}{\Gamma(k\beta + 1)} (b-x)^{k \beta} + \frac{(RD_{b}^{((n+1) \beta)} f)(\xi)}{\Gamma((n+1)\beta + 1)} (b-x)^{(n+1)\beta}
\]  

(4.35)

where \(x \leq \xi \leq b\).
In the proof we used the previous technical lemma and the key steps of the proof are similar to the steps in *Odibat and Shawagfeh* [2007].
Bibliography


Chapter 5

Polynomial Approximate Solutions of an Unconfined Forchheimer Groundwater Flow Equation

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Abstract

We consider a one-dimensional, unconfined groundwater flow equation derived using a simplified form of the Forchheimer equation in place of Darcy’s Law. Such equations can model turbulent flows in coarse and fractured porous media. For power-law head, exponential head, power-law flux and exponential flux boundary conditions at the inlet, the problem can be reduced to a boundary-value problem for a nonlinear ordinary differential equation. We construct quadratic and cubic approximate solutions of the equation. We also numerically compute solutions using a modification of a method of Shampine, which exploits scaling properties of the governing equation. The polynomial approximate solutions replicate well the numerical solutions. The work demonstrates the value of polynomial approximate solutions for validating numerical solutions and a new application of Shampine’s method for this class of groundwater flow equation.
5.1 Introduction

Unconfined groundwater flow can be modeled using the Boussinesq equation, the derivation of which relies on vertical averaging and an assumption of Darcian flow. When flows are non-Darcian, other groundwater flow equations can be derived. Generalizations of Darcy’s Law, such as the Forchheimer and Izbash equations, can be used to derive flow equations for non-Darcy conditions (Moutsopoulos, 2009).

Non-Darcy flows have been studied in a variety of contexts. Bordier and Zimmer [2000] note that the Forchheimer and Izbash equations can be used to model flows in coarse gravel and geosynthetic media. Chin et al. [2009] found that flows through karstic limestone samples can be described by the Forchheimer equation. Xia et al. [2017] constructed artificial rock joints and found that the Forchheimer and Izbash equations both accurately describe fluid flows in a single joint. Ghane et al. [2014] observed that the Forchheimer equation describes flows through a woodchip denitrification bed. Other researchers have used the Forchheimer equation to model flows to wells (Wen et al., 2011; Mathias and Moutsopoulos, 2016).

The two-term Forchheimer equation (Hassanizadeh and Gray, 1987) is a generalization of Darcy’s Law for flows that satisfy

\[-\frac{\partial h}{\partial x} = aq + bq|q|.\]  

(5.1)

The Forchheimer coefficient $a$ depends on properties of the fluid and porous medium while $b$ depends only on properties of the porous medium (Chin et al., 2009). A detailed discussion of the physical meaning of the parameters is given by Venkataraman and Rao [1998]. The parameter $a$ is the reciprocal of the hydraulic conductivity. When inertial effects, represented by the quadratic term of (5.1), are negligible, $b = 0$ and Darcy’s Law is recovered (Xia et al., 2017). For fully turbulent flows, the viscous effects, represented by the linear
term of (5.1), are dominated by the inertial effects and the linear term is negligible \( (\text{Xia et al., 2017}) \). In the current work, we utilize a simplified form of the Forchheimer equation in which the linear term of the right side of (5.1) can be neglected.

Researchers have studied the conditions for which non-Darcy flows occur. \( \text{Xiong et al. [2017]} \) note that non-Darcy flows can occur at small and large flow rates. \( \text{Mathias et al. [2008]} \) classify these flows as pre-linear and post-linear. Some investigators classify non-Darcian flows by Reynolds number. \( \text{Irmay [1958]} \) and \( \text{Burcharth and Anderson [1995]} \) discuss these classifications. \( \text{Barenblatt et al. [1990]} \) caution that non-Darcy flows can occur at low, pre-turbulence Reynolds numbers due to the tortuous flow paths taken by fluids in porous media. \( \text{Zeng and Grigg [2006]} \) argue for using a Forchheimer number, instead of the Reynolds number, to identify non-Darcy flows. \( \text{Hassanizadeh and Gray [1987]} \) summarize various potential causes of nonlinear effects on flows. These nonlinear effects have been attributed to turbulence, microscopic inertial forces and microscopic drag forces that occur at high flow velocities.

Numerical methods are used to approximate solutions to nonlinear flow problems since exact solutions cannot be constructed, in general. To justify the validity of numerical codes and to interpret experimental results, approximate analytic solutions are required. These and other uses of analytical solutions are discussed by \( \text{Hayek [2016]} \). Approximate analytic methods such as the Adomian \( \text{(Serrano, 1997)} \) and homotopy perturbation \( \text{(Munusamy and Dhar, 2016)} \) methods have been used to generate solutions to groundwater flow problems. The polynomial approximate solutions constructed in the current work are based on the methods introduced in \( \text{Lockington et al [2000]} \) for the Boussinesq equation. The method was extended in \( \text{Telyakovskiy and Allen [2006]} \) to more classes of boundary conditions and in \( \text{Olsen and Telyakovskiy [2013]} \) it was applied to the porous medium equation.

\( \text{Moutsopoulos [2009]} \) modeled unconfined, one-dimensional turbulent flow in porous media using a nonlinear governing equation based on a form of the Forchheimer equa-
tion in place of Darcy’s Law. The governing equation was derived using the Dupuit-Forchheimer assumptions. Moutsopoulos [2009] notes that turbulent flows can exist for which the Dupuit-Forchheimer assumptions hold. Moutsopolous considered the recharge of an initially-dry aquifer with power-law head and constant flux conditions at the inlet and a zero-head condition at infinity. The author used a similarity transformation to convert the problem into a dimensionless ordinary differential equation and derived approximate quintic polynomial solutions of the equation using the Adomian method. For certain recharge regimes, the problem has exact solutions, which the Adomian procedure reproduced. Moutsopolous also used a shooting type of procedure based on the Runge-Kutta method to compute numerical solutions of the governing equation.

In the current work, we consider the problem described in Moutsopoulos [2009], but we use different methods and extend his work to include a broader set of boundary conditions. In addition to a power-law head condition at the inlet, we consider exponential head, power-law flux and exponential flux conditions. The exponential boundary conditions are shown to be limiting cases of the corresponding power-law conditions. We introduce dimensionless variables that transform the problems into boundary-value problems involving an ordinary differential equation. We then construct quadratic and cubic approximate solutions to these problems. We compare our polynomial approximate solutions to numerical solutions obtained using a modification of a method of Shampine [1973].

In the sections below, we present the four flow problems for each of the boundary conditions, introduce dimensionless variables and convert the problems into boundary-value problems involving ordinary differential equations. Next, we describe conditions that the actual solutions satisfy. These conditions are used to determine the coefficients of the polynomial approximate solutions and the unknown position of the wetting front. One set of properties is used for the head conditions at the inlet while a different set is used for flux conditions. Finally, we describe our modification of Shampine’s method and numerically
compute solutions of the dimensionless equations. We compare the polynomial approximate solutions to the numerical solutions.

### 5.2 Problem Formulation

We consider unconfined flow in an aquifer with a horizontal impermeable lower boundary, as shown in Figure 5.1. For each of four boundary conditions at the inlet, we present the initial and boundary conditions, the dimensionless variables and the reduced dimensionless form of each problem. The derivation of the governing equation, the formulation of the power-law head problem and the exact solutions to this problem are described by Moutsopoulos [2009].

![Figure 5.1: Propagation of water into an empty aquifer with an increase in head at the inlet for $t_1 < t_2 < t_3$.](image)

For $q > 0$ and when viscous forces can be neglected under turbulent flow conditions (Moutsopoulos, 2009), (5.1) can be simplified to

$$ q = \sqrt{-\frac{1}{b} \frac{\partial h}{\partial x}}. $$

(5.2)

By using (5.2) in place of Darcy’s Law in the conservation of mass equation, the following groundwater flow equation can be derived:
\[
S \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left( h \sqrt{-\frac{1}{b} \frac{\partial h}{\partial x}} \right) = 0, \quad x > 0 \text{ and } t > 0. \tag{5.3}
\]

Here \( S \) is the specific yield and \( b \) is the quadratic coefficient from (5.1).

In the current work, we construct approximate solutions to (5.3) with four boundary conditions at the inlet: power-law head, exponential head, power-law flux and exponential flux.

### 5.2.1 Problem I: Power-Law Head boundary condition at the inlet

The power-law head problem consists of (5.3) with initial and boundary conditions

\[
h(x, 0) = 0 \text{ for } x > 0, \quad h(0, t) = \sigma \tau^\alpha \quad \text{and} \quad \lim_{x \to \infty} h(x, t) = 0 \text{ for } t > 0. \tag{5.4}
\]

Here, \( \sigma \) is a non-negative constant. The case of \( \alpha = -1/2 \) corresponds to a finite redistribution of water into the aquifer (Moutsopoulos, 2009). We consider values of \( \alpha \geq -1/2 \).

For this problem, the following dimensionless variables and parameters can be used (Moutsopoulos, 2009):

\[
\xi = x \left( \frac{\beta}{\tau^{\alpha+2}} \right)^{1/3}, \tag{5.5}
\]

\[
\beta = \frac{bS^2}{\sigma} \left( \frac{\alpha + 2}{3} \right)^2, \tag{5.6}
\]

and

\[
H(\xi) = \frac{h(x,t)}{\sigma \tau^\alpha}. \tag{5.7}
\]

Here, \( \xi \) is the similarity variable and \( H(\xi) \) is the scaling function. This choice of similarity
variable converts (5.3) into

\[ \frac{d}{d\xi} \left( H \sqrt{- \frac{dH}{d\xi}} \right) - \xi \frac{dH}{d\xi} + \lambda H = 0, \]  

(5.8)

or equivalently,

\[ \frac{H d^2 H}{2 d\xi^2} + \left( \frac{dH}{d\xi} \right)^2 + \xi \frac{dH}{d\xi} \sqrt{- \frac{dH}{d\xi}} - \lambda \sqrt{- \frac{dH}{d\xi}} = 0, \]  

(5.9)

where

\[ \lambda = \frac{3\alpha}{\alpha + 2}. \]  

(5.10)

The case of a finite redistribution of water into the aquifer corresponds to \( \alpha = -1/2 \) and \( \lambda = -1 \). If the water level rises at the inlet, \( \alpha \geq 0 \), and correspondingly, \( \lambda \geq 0 \).

The initial and boundary conditions become

\[ H(0,\lambda) = 1 \quad \text{and} \quad \lim_{\xi \to \infty} H(\xi,\lambda) = 0. \]  

(5.11)

Two exact solutions exist for this problem. When \( \lambda = -1 \), \( H(\xi) = -\xi^3/3 + 1 \) for \( 0 \leq \xi \leq \sqrt{3} \) and \( H(\xi) = 0 \) for \( \xi > \sqrt{3} \). Second, when \( \lambda = 1 \), \( H(\xi) = -\xi + 1 \) for \( 0 \leq \xi \leq 1 \) and \( H(\xi) = 0 \) for \( \xi > 1 \).

### 5.2.2 Problem II: Exponential Head boundary condition at the inlet

The exponential head problem consists of (5.3) with initial and boundary conditions

\[ h(x,0) = 0 \quad \text{for} \quad x > 0, \quad h(0,t) = h_0 e^{kt} \quad \text{and} \quad \lim_{x \to \infty} h(x,t) = 0 \quad \text{for} \quad t > 0. \]  

(5.12)

The parameter \( h_0 \) is a positive quantity representing the initial head at the inlet and \( k \) is the
continuous rate of change of head at the inlet.

We choose the following dimensionless variables:

\[ \xi = \left( \frac{bk^2S^2}{9h_0e^{kt}} \right)^{1/3} x \]  \hspace{1cm} (5.13)

and

\[ H(\xi) = \frac{h(x,t)}{h_0e^{kt}}. \]  \hspace{1cm} (5.14)

With this choice of variables, (5.3) is transformed into

\[ \frac{d}{d\xi} \left( H \sqrt{\frac{dh}{d\xi}} \right) - \xi \frac{dh}{d\xi} + 3H = 0, \]  \hspace{1cm} (5.15)

or equivalently,

\[ \frac{H d^2H}{2 d\xi^2} + \left( \frac{dH}{d\xi} \right)^2 + \xi \frac{dH}{d\xi} \sqrt{\frac{dh}{d\xi}} - 3H \sqrt{\frac{dh}{d\xi}} = 0 \]  \hspace{1cm} (5.16)

with boundary conditions

\[ H(0) = 1 \quad \text{and} \quad \lim_{\xi \to \infty} H(\xi) = 0. \]  \hspace{1cm} (5.17)

We note that (5.16) and (5.17) can be considered a limiting case of the power-law head problem since as \(\alpha \to \infty, \lambda \to 3\) in (5.10).

5.2.3 Problem III: Power-Law Flux boundary condition at the inlet

The power-law flux problem consists of (5.3) with initial and boundary conditions
\( h(x,0) = 0 \) for \( x > 0 \), 
\( h(0,t) \sqrt{-\frac{1}{b} \frac{\partial h}{\partial x}} \bigg|_{x=0} = \tau t^{\gamma} \) and

\[
\lim_{x \to 0} h(x,t) = 0 \quad \text{for} \quad t > 0. \tag{5.18}
\]

The second condition in (5.18) represents a power-law flux condition at the inlet based on the Forchheimer equation (5.2). We consider the cases of \( \gamma > -1/3 \) and \( \tau > 0 \).

By selecting

\[
\xi = \left( \frac{\gamma + 3}{4} \right)^{3/4} \left( \frac{bS^3}{\tau^{\gamma+3}} \right)^{1/4} x
\]

and

\[
H(\xi) = \left( \frac{3\gamma + 1}{4} \right)^{1/4} \left( \frac{S}{b\tau^{3\gamma+1}} \right)^{1/4} h,
\]

(5.3) is converted into

\[
\frac{d}{d\xi} \left( H \sqrt{-\frac{dH}{d\xi}} \right) - \xi \frac{dH}{d\xi} + \lambda H = 0,
\]

or equivalently,

\[
\frac{H d^2 H}{2 \frac{d\xi^2}{d\xi}^2} + \left( \frac{dH}{d\xi} \right)^2 + \xi \frac{dH}{d\xi} \sqrt{-\frac{dH}{d\xi}} - \lambda H \sqrt{-\frac{dH}{d\xi}} = 0,
\]

with boundary conditions

\[
H(0,\lambda) \sqrt{-\frac{dH}{d\xi}} \bigg|_{\xi=0} = 1 \quad \text{and} \quad \lim_{\xi \to \infty} H(\xi,\lambda) = 0,
\]

where
\[ \lambda = \frac{3\gamma + 1}{\gamma + 3}. \]  

(5.24)

Since we consider the cases corresponding to \( \gamma > -1/3 \), by (5.24), \( \lambda \) must be greater than 0. We note that when \( \lambda = 1 \) this problem has an exact solution: \( H(\xi) = -\xi + 1 \) for \( \xi \leq 1 \) and \( H(\xi) = 0 \) otherwise.

This problem given by (5.21)-(5.24) is very similar to the problem for the power-law head boundary condition given by (5.8)-(5.11). The only differences are in the condition at the inlet and the way in which \( \lambda \) is defined.

### 5.2.4 Problem IV: Exponential Flux boundary condition at the inlet

The exponential flux problem consists of (5.3) with initial and boundary conditions

\[
\begin{align*}
    h(x, 0) &= 0 \quad \text{for} \quad x > 0, \quad h(0, t) = \sqrt{-1} \frac{1}{b} \frac{\partial h}{\partial x} |_{x=0} = h_1 e^{kt} \quad \text{and} \quad \\
    \lim_{x \to \infty} h(x, t) &= 0 \quad \text{for} \quad t > 0. \quad (5.25)
\end{align*}
\]

The parameters \( h_1 \) and \( k \) are positive quantities representing the initial flux at the inlet and the continuous rate of increase of flux at the inlet, respectively.

By selecting

\[
    \xi = \left( \frac{bk^3S^3}{64h_1e^{kt}} \right)^{1/4} x
\]

(5.26) and

\[
    H(\xi) = \left( \frac{kS}{4b(h_1e^{kt})^3} \right)^{1/4} h,
\]

(5.3) is converted into
\[
\frac{d}{d\xi} \left( H \sqrt{-\frac{dH}{d\xi}} \right) - \xi \frac{dH}{d\xi} + 3H = 0, \quad (5.28)
\]
or equivalently,
\[
\frac{H d^2H}{2 \frac{d\xi^2}{}^2} + \left( \frac{dH}{d\xi} \right)^2 + \xi \frac{dH}{d\xi} \sqrt{-\frac{dH}{d\xi}} - 3H \sqrt{-\frac{dH}{d\xi}} = 0, \quad (5.29)
\]
with boundary conditions
\[
H(0) \sqrt{-\frac{dH}{d\xi} \bigg|_{\xi=0}} = 1 \quad \text{and} \quad \lim_{\xi \to \infty} H(\xi) = 0. \quad (5.30)
\]

This problem is a limiting case of the power-law flux problem since as \( \gamma \to \infty \) in (5.24), \( \lambda \to 3 \).

### 5.3 Conditions Used for Determining Polynomial Approximate Solutions

For the four problems that we introduced in Section 5.2, we construct approximate analytical solutions in polynomial form. For special values of the parameter \( \lambda \), some of the problems have exact polynomial solutions. This makes polynomials a natural choice of function for the approximations.

To construct polynomial approximate solutions, we state a number of conditions satisfied by the exact solutions. The approximating polynomials are found by requiring that they satisfy these conditions even though they may not satisfy the governing differential equation itself. This approach was introduced by Lockington et al [2000] and extended by Telyakovskiy and Allen [2006].

For the power-law head and power-law flux problems, we construct quadratic and cubic approximate solutions of (5.8) and (5.9) of the forms
\[ H_q(\xi) = a(\xi_0 - \xi) + b(\xi_0 - \xi)^2 + c \] (5.31)

and

\[ H_c(\xi) = a(\xi_0 - \xi) + b(\xi_0 - \xi)^2 + c(\xi_0 - \xi)^3 + d. \] (5.32)

The polynomial coefficients \( a \) through \( d \) depend on the parameter \( \lambda \) and on \( \xi_0 \), which in turn depends on \( \lambda \). By setting \( \lambda = 3 \) when solving for the polynomial coefficients, we will also generate approximate solutions to the exponential head and exponential flux problems.

We will construct the quadratic and cubic approximate solutions using the conditions listed below. However, instead of using the zero head condition at infinity, we assume in Condition 2, which follows, that the dimensionless head is zero at a finite location, \( \xi_0 \), which is not known a priori. This \( \xi_0 \) exists because for an initially dry aquifer, (5.8) and (5.9) produce a wetting front that propagates at finite speed (Barenblatt, 1952). In addition to the unknown polynomial coefficients, this unknown \( \xi_0 \) must be estimated using the conditions listed below. These conditions represent properties of the solution function at the boundaries of the problem domain and at points between the boundaries. This approach differs from using a Taylor polynomial approximation in which the Taylor coefficients depend on the solution properties at only one point.
C1  \( a \) \( H(0, \lambda) = 1, \)

\( b \) \( H(0, \lambda) \sqrt{-\frac{dH}{d\xi}}\bigg|_{\xi=0} = 1, \)

C2 \( H(\xi_0, \lambda) = 0, \)

C3 \( \frac{dH}{d\xi} \bigg|_{\xi=\xi_0} = -\xi_0^2, \)

C4  \( a \) \[ \frac{1}{2} \int_0^{\xi_0} \xi H \frac{d^2H}{d\xi^2} d\xi + \int_0^{\xi_0} \xi \left( \frac{dH}{d\xi} \right)^2 d\xi + \int_0^{\xi_0} \xi^2 \sqrt{-\frac{dH}{d\xi} \frac{dH}{d\xi} d\xi} - \lambda \int_0^{\xi_0} H \sqrt{-\frac{dH}{d\xi} d\xi} d\xi = 0, \] (5.33)

\( b \) \[ \frac{1}{2} \int_0^{\xi_0} \frac{H}{d\xi} d\xi \frac{d^2H}{d\xi^2} d\xi + \int_0^{\xi_0} \left( \frac{dH}{d\xi} \right)^2 d\xi + \int_0^{\xi_0} \xi \sqrt{-\frac{dH}{d\xi} \frac{dH}{d\xi} d\xi} - \lambda \int_0^{\xi_0} H \sqrt{-\frac{dH}{d\xi} d\xi} d\xi = 0, \]

C5  \( a \) \[ \frac{1}{2} \left. \frac{d^2H}{d\xi^2} \right|_{\xi=0} + \left. \left( \frac{dH}{d\xi} \right)^2 \right|_{\xi=0} - \lambda \left. \sqrt{-\frac{dH}{d\xi}} \right|_{\xi=0} = 0. \]

\( b \) \[ \left. \frac{H(0, \lambda)}{2} \frac{d^2H}{d\xi^2} \right|_{\xi=0} + \left. \left( \frac{dH}{d\xi} \right)^2 \right|_{\xi=0} - \lambda = 0. \]

The conditions capture properties of the solutions to the dimensionless problems. Conditions 1 represent the boundary conditions at the inlet: \( a \) is for the power-law head condition and \( b \) is for the power-law flux condition. Conditions 2 represents the head being zero at the wetting front location. C3 follows from (5.9) when \( \xi \to \xi_0 \). It represents the
water-table slope near the wetting front. C4a is obtained when (5.9) multiplied by ξ and integrated over ξ ∈ [0, ξ₀]. C4b is obtained by integrating (5.9) over [0, ξ₀]. Conditions 4 capture global behavior of the solution over the problem domain. Finally, Conditions 5 are obtained from (5.9) in the limit as ξ → 0. These conditions capture the limiting behavior of the solution near the inlet and incorporate Conditions 1. Again, C5a corresponds to the power-law head condition and C5b corresponds to the power-law flux condition. In the Appendix, we present an alternative formulation of Conditions 4 based on (5.8).

For the power-law head problems, we use four conditions; C1a, C2, C3 and C4a; to determine the quadratic coefficients a, b and c and the wetting front position ξ₀. We use five conditions; C1a, C2, C3, C4a and C5a; to determine ξ₀ and the cubic coefficients a, b, c and d. The constant terms in H_q and H_c are zero due to C2. For the power-law flux problems, we use C1b, C4b and C5b in place of C1a, C4a and C5a.

We note that exact solutions of the governing equations (5.8) and (5.9) must satisfy the conditions described in Section 5.2. We emphasize, however, that the constructed quadratic and cubic approximate solutions do not, in general, satisfy (5.8) and (5.9).

### 5.3.1 Polynomial-Approximate Solutions to the Power-Law Head Boundary Problem

#### Quadratic Approximate Solution

To construct the quadratic polynomials H_q (5.31), we must find four unknowns a, b, c and ξ₀. We use C1a, C2, C3 and C4a. By applying C2, H_q can be written in the form

\[
H_q(ξ) = a(ξ₀ − ξ) + b(ξ₀ − ξ)^2.
\]  

(5.34)

Conditions C1a and C3 are used to determine a and b and C4a is used to find ξ₀. From C1a and C3, the coefficients a and b are
\[ a = \xi_0^2 \quad \text{and} \quad b = \frac{1 - \xi_0^3}{\xi_0^2}. \quad (5.35) \]

The polynomial \( H_q \) can be rewritten in normalized form as

\[ H_q(z) = y(1 - z) + (1 - y)(1 - z)^2, \quad (5.36) \]

where \( y = \xi_0^3 \) and \( z = \xi / \xi_0 \). Condition C4a, written in terms of \( y \) and \( z \), is

\[
\frac{1}{4} + \frac{1}{2} \int_0^1 z \left( \frac{dH_q}{dz} \right)^2 dz + \sqrt{y} \int_0^1 z^2 \frac{dH_q}{dz} \sqrt{-\frac{dH_q}{dz}} dz - \lambda \sqrt{y} \int_0^1 z H_q \sqrt{-\frac{dH_q}{dz}} dz = 0. \quad (5.37)
\]

Equation (5.37) is a nonlinear equation in \( y \). The wetting front position \( \xi_0 \) can be determined by numerically solving this algebraic equation for \( y \). The wetting front position \( \xi_0 \) is found using \( \xi_0 = y^{1/3} \) and \( \xi_0 \) yields the values of coefficients \( a \) and \( b \). Results comparing these, and the other polynomial approximate solutions that follow, to the numerical solutions are given in Section 5.5.

**Cubic Approximate Solution**

For the cubic approximation \( H_c \) (5.32), we use Conditions C1a, C2, C3, C4a and C5a to find \( a, b, c, d \) and \( \xi_0 \). Application of Condition C2 yields

\[ H_c(\xi) = a(\xi_0 - \xi) + b(\xi_0 - \xi)^2 + c(\xi_0 - \xi)^3. \quad (5.38) \]

Using C1a and C3, the coefficients \( a \) and \( b \) are found to be
\[ a = \xi_0^2 \quad \text{and} \quad b = \frac{1}{\xi_0^2} - \frac{a}{\xi_0} - \frac{a_0}{\xi_0^2}, \tag{5.39} \]

where \( a_0 = c \xi_0^3 \).

In normalized form,

\[ H_c(z) = y(1 - z) + (1 - y - a_0)(1 - z)^2 + a_0(1 - z)^3, \tag{5.40} \]

where again \( y = \xi_3/\xi_0 \) and \( z = \xi_0/\xi_0 \).

Conditions C4a and C5a, shown below in terms of \( y \) and \( z \), form a system of nonlinear algebraic equations that should be solved numerically to find \( a_0 \) and \( y \):

\[
\frac{1}{4} + \frac{1}{2} \int_0^1 z \left( \frac{dH_c}{dz} \right)^2 dz + \sqrt{y} \int_0^1 z^2 \frac{dH_c}{dz} \sqrt{-\frac{dH_c}{dz}} dz - \lambda \sqrt{y} \int_0^1 z H_c \sqrt{-\frac{dH_c}{dz}} dz = 0. \tag{5.41} \]

and

\[
\left. \frac{d^2H_c}{dz^2} \right|_{z=0} + 2 \left. \left( \frac{dH_c}{dz} \right)^2 \right|_{z=0} - 2 \lambda y^{1/6} \sqrt{-\frac{dH_c}{dz}} \bigg|_{z=0} = 0. \tag{5.42} \]

The \( a_0 \) and \( y \) solutions allow one to determine \( \xi_0 \) and coefficients \( a \), \( b \) and \( c \) in \( H_c (5.38) \).

The results are again presented in Section 5.5.

5.3.2 Polynomial-Approximate Solutions to the Power-Law Flux Boundary Problem

The case of the power-law flux boundary condition is treated similarly to the case of the power-law head boundary condition. We only need to use different conditions; C1b, C2,
C3, C4b and C5b; to construct the approximating quadratic and cubic polynomials.

**Quadratic Approximate Solution**

To find the unknown coefficients $a$, $b$, $c$ and $\xi_0$ in $H_q$ (5.31), we use conditions C1b, C2, C3 and C4b. When C2 is used, (5.31) can be written as

$$H_q(\xi) = a(\xi_0 - \xi) + b(\xi_0 - \xi)^2. \quad (5.43)$$

Now the remaining conditions C1b, C3 and C4b will be used to find the rest of the unknowns. Using C3, we find that

$$a = \xi_0^2. \quad (5.44)$$

C1b leads to the following equation in $b$ and $\xi_0$:

$$(\xi_0^3 + \xi_0^2 b)(\xi_0^2 + 2\xi_0 b)^{1/2} = 1 \quad (5.45)$$

The polynomial $H_q$ can be written in normalized form as

$$H_q(z) = \xi_0^3(1 - z) + \xi_0^2 b(1 - z)^2, \quad (5.46)$$

where $z = \xi/\xi_0$. Condition C4b, written in terms of $z$, is

$$H_q(0)\left.\frac{dH_q}{dz}\right|_{z=0} - \xi_0^{-1/2} \int_0^1 \left(\frac{dH_q}{dz}\right)^2 dz - 2\xi_0 \int_0^1 z \frac{dH_q}{dz} \sqrt{-\frac{dH_q}{dz}} dz + 2\lambda \xi_0 \int_0^1 H_q(z) \sqrt{-\frac{dH_q}{dz}} dz = 0. \quad (5.47)$$

By substituting $H_q$ (5.46) into equation (5.47), we obtain a nonlinear algebraic equation...
in \( b \) and \( \xi_0 \). The values of \( b \) and \( \xi_0 \) can be determined by numerically solving the nonlinear system of algebraic equations formed by (5.45) and (5.47). The results are shown in Section 5.5.

Cubic Approximate Solution

To construct the cubic polynomial \( H_c \) (5.32), in addition to the conditions used in the previous section, we use condition C5b. When C2 is used, \( H_c \) will be of the form

\[
H_c(\xi) = a(\xi - \xi_0) + b(\xi - \xi_0)^2 + c(\xi - \xi_0)^3.
\] (5.48)

We next use C1b, C3, C4b and C5b to determine \( a, b, c \) and \( \xi_0 \) for \( H_c \) (5.48). Using C3, we find that

\[
a = \frac{\xi_0^2}{\xi_0^2}.
\] (5.49)

Condition C1b leads to the following equation in \( b, c \) and \( \xi_0 \):

\[
(\xi_0^3 + \xi_0^2 b + \xi_0 c)(\xi_0^2 + 2\xi_0 b + 3\xi_0^2 c)^{1/2} = 1.
\] (5.50)

The polynomial \( H_c \) can now be written in normalized form as

\[
H_c(z) = \frac{\xi_0^3}{\xi_0^3}(1-z) + \frac{\xi_0^2}{\xi_0^2}b(1-z)^2 + \frac{\xi_0^3}{\xi_0^3}c(1-z)^3,
\] (5.51)

where \( z = \frac{\xi}{\xi_0} \). Condition C4b, written in terms of \( z \), is

\[
H_c(0) \frac{dH_c}{dz} \bigg|_{z=0} - \frac{\xi_0^{-1/2}}{\xi_0} \int_0^1 \left( \frac{dH_c}{dz} \right)^2 dz - 2\xi_0 \int_0^1 z \frac{dH_c}{dz} \sqrt{-\frac{dH_c}{dz}} dz + 2\lambda \xi_0 \int_0^1 H_c \sqrt{-\frac{dH_c}{dz}} dz = 0.
\] (5.52)
Finally, Condition 5b, shown below in terms of $z$, is:

$$H_c(0) \frac{d^2 H_c}{d z^2} \bigg|_{z=0} + 2 \left( \frac{d H_c}{d z} \bigg|_{z=0} \right)^2 - 2 \lambda \xi_0^2 = 0. \quad (5.53)$$

After we substitute $H_c$ (5.51) into (5.52) and (5.53), we obtain the system of nonlinear algebraic equations given by (5.50), (5.52) and (5.53) and we solve it for $b$, $c$ and $\xi_0$. The value of $a$ is obtained from $\xi_0$. The results for this approximation are presented in Section 5.5.

### 5.4 Numerical Solutions

We solve the power-law head problem, (5.9) and (5.11), and the power-law flux problem, (5.22) and (5.23), using a modification of Shampine’s method (Shampine, 1973; Telyakovskiy and Allen, 2006). In this procedure, the governing equation is numerically solved as an initial-value problem by assuming that $\xi_0 = 1$. Starting at $\xi_0$, the equation is solved back towards the inlet to yield a water surface profile. To start the calculations, we use a Taylor series expansion of the solution function about $\xi_0 = 1$ to begin the step from $\xi_0$ towards $\xi = 0$. The Taylor series expansion uses information about the derivatives of the solution function. This derivative information is obtained from analysis of the governing equation. Equation (5.9) is solved by writing it as a first-order system of differential equations and a Runge-Kutta method is then used.

Since $\xi_0$ need not be one, the intermediate numerical solution described above does not, in general, match the boundary condition at $\xi = 0$. Shampine’s method exploits scale-invariance properties of the governing equation to produce a solution that matches the inlet boundary condition. The intermediate solution is rescaled to force it to satisfy this boundary condition. The rescaling process also allows the true value of $\xi_0$ to be found. The details of the rescaling process are discussed below.
In the power-law head problem, equation (5.9) is invariant with respect to the transformations $\xi \rightarrow L^{1/3} \xi$ and $H \rightarrow L \cdot H$ for any positive constant $L$. If $H_0(\xi, \lambda)$ is a solution of (5.9), then $H_1(\xi, \lambda) = L_0H_0(L_0^{1/3} \xi, \lambda)$ is also a solution. In particular, if $H_0$ is an intermediate solution such that $H_0(1, \lambda) = 0$, we can choose $L_0$ so that $H_1$ will satisfy the inlet boundary condition C1a for the power-law head boundary problem. In this case, $L_0 = (H_0(0, \lambda))^{-1}$. The wetting front position for this problem is obtained by solving for $\xi_{0,PLH}$ in $L_0^{1/3} \xi_{0,PLH} = 1$. Therefore, $\xi_{0,PLH} = (H_0(0, \lambda))^{1/3}$.

A similar rescaling process can be conducted to generate solutions to the power-law flux boundary problem. An intermediate solution $H_0$ to the power-law flux problem, such that $H_0(1, \lambda) = 0$, can be used. Alternatively, the solution $H_1$ to the power-law head boundary problem can be used. For the latter approach, let $H_2$ denote a solution of the power-law flux problem. We rescale $H_1$ to produce a solution $H_2$: $H_2(\xi, \lambda) = L_1H_1(L_1^{1/3} \xi, \lambda)$. The scaling factor $L_1$ is selected so that $H_2$ satisfies the flux boundary condition C1b at the inlet. In this case, $L_1 = (-dH_1/d\xi|_{\xi = 0})^{-3/10}$. The wetting front position is obtained by solving for $\xi_{0,PLF}$ in $L_1^{1/3} \xi_{0,PLF} = \xi_{0,PLH}$. Therefore, $\xi_{0,PLF} = L_1^{-1/3} \xi_{0,PLH}$.

Numerical results for the power-law head and power-law flux problems are shown in tabular and graphical forms in Section 5.5. These numerical results are compared with the polynomial approximate solutions.

### 5.5 Comparison of the Polynomial Approximate and Numerical Solutions

For each of the four problems described above, we compare the estimated positions of the dimensionless wetting front, $\xi_0$, with the numerically calculated values. These results are shown in Tables 5.1 and 5.2. We also compare the graphs of the polynomial approximate and numerical solutions in Figures 5.2 and 5.3. The results for the exponential bound-
ary conditions are special cases of the power-law results for $\lambda = 3$. We also show in the Appendix how the form of the approximate solutions will change if we use (5.8) as the governing equation instead of (5.9), on which the derivations of Sections 5.2 and 5.3 were based.

5.5.1 Power-Law Head Boundary Problem

Table 5.1 shows, for various values of $\lambda$, the positions of the dimensionless wetting fronts $\xi_0$ as predicted by Shampine’s method and by the quadratic and cubic approximate solutions. The case of $\lambda = 3$ corresponds to the exponential head condition at the inlet. This is the limiting case of the power-law head boundary condition as $\alpha \to \infty$. Exact cubic and linear solutions exist for the cases of $\lambda = -1$ and $\lambda = 1$, respectively. For all of the cases shown, the polynomial approximate solutions provide reasonable predictions of the wetting front positions.

The dimensionless solution curves are shown in Figure 5.2. In each case the cubic approximate solution closely agrees with the numerical solution. The cubic approximations also reproduce the exact solutions for $\lambda = \pm 1$. The quadratic approximate solutions approximate reasonably well the numerical solutions except in the case of $\lambda = -1$ in which the exact solution is a cubic polynomial. Overall, the cubic approximate solutions are more accurate than the quadratic approximate solutions when the numerical solutions are used as the basis of comparison.

5.5.2 Power-Law Flux Boundary Problem

In Table 5.2, the positions of the dimensionless wetting fronts $\xi_0$ for various values of $\lambda$ are shown. The case of $\lambda = 3$ corresponds to the exponential flux condition at the inlet. This is the limiting case of the power-law flux boundary condition as $\gamma \to \infty$. An exact linear
Table 5.1: Comparison of the wetting front position $\xi_0$ for the power-law head boundary condition using the numerical (Shampine’s) method and quadratic $H_q$ and cubic $H_c$ approximations. The parameter $\alpha$ is the exponent of the power-law head condition at the inlet. The case of $\lambda = 3$ corresponds to the exponential head boundary condition at the inlet.

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solution exists for the case of $\lambda = 1$. For the cases considered, the polynomial approximate solutions provide reasonable predictions of the wetting front positions. The cubic approximate solutions provide better estimates of the wetting front positions than the quadratic solutions.

Figure 5.3 shows the dimensionless solution curves. In each case the cubic approximate solution closely agrees with the numerical solution and the quadratic approximate solution gives a slightly worse estimate. The quadratic and cubic approximations also reproduce the exact solution for $\lambda = 1$.  


Figure 5.2: $H$ versus $\xi$ for, left to right, $\lambda = 3, 5/2, 2, 3/2, 1, 1/3, 0$ and -1 for the power-law head boundary condition at the inlet. Exact cubic and linear solutions exist for $\lambda = -1$ and $\lambda = 1$, respectively. The case of $\lambda = 3$ corresponds to the exponential head boundary condition at the inlet.

5.6 Summary and Conclusions

We derived polynomial approximate solutions of a Forchheimer groundwater flow equation for four types of boundary conditions at the inlet of a one-dimensional, initially-dry unconfined aquifer. The particular form of the Forchheimer equation that we utilized represents turbulent flow in the aquifer. For power-law head and flux conditions at the inlet, quadratic and cubic polynomial approximate solutions replicated accurately the numerical solutions
Table 5.2: Comparison of the wetting front position $\xi_0$ for the Power-Law-Flux boundary condition using numerical (Shampine’s) method, quadratic $H_q$ and cubic $H_c$ approximations for various values of $\lambda$. The parameter $\gamma$ is the exponent of the power-law flux condition at the inlet. The case of $\lambda = 3$ corresponds to the exponential head boundary condition at the inlet.

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of the equation. The cubic approximate solutions provide excellent estimates of the numerical solutions for all cases considered. The cubic approximate solutions also replicated the three exact solutions. The quadratic approximate solutions approximated well the numerical solutions, except in the case of a finite redistribution of water ($\lambda = -1$). In this case, however, the problem has an exact cubic solution. The other two exact solutions were replicated by the quadratic approximate solutions. The problems with exponential boundary conditions at the inlet were shown to be limiting cases of the corresponding power-law problems as the power-law exponents increase without bound.

Using the procedures presented in this work, the process of constructing approximate solutions to the nonlinear ordinary differential equation is reduced to solving a nonlinear algebraic equation or system of nonlinear algebraic equations for the wetting front position and polynomial coefficients. After these approximations have been constructed, they can be used for different values of the constants appearing in the boundary conditions. Using a modification of a method of Shampine, we also numerically solved the equation using
Figure 5.3: $H$ versus $\xi$ for, left to right, $\lambda = 3, 5/2, 2, 3/2, 1, 1/2$ and 1/3 for the power-law flux boundary condition at the inlet. An exact linear solution exists for $\lambda = 1$. The case of $\lambda = 3$ corresponds to the exponential flux boundary condition at the inlet.

A procedure which exploited its scaling properties and showed that these numerical solutions can be validated using low-degree polynomials. This demonstrates the utility of the approximate solutions for validating numerical solutions when exact solutions do not exist.

Variations of Shampine’s method and of the methods used for the construction of the polynomial approximate solutions have been previously used with the Boussinesq and porous medium equations. In the current work, the methods were extended to a different class of nonlinear groundwater flow equation. The current work and work done for
the Boussinesq and porous medium equation show that approximate solutions can be constructed using only values of the solution function and its derivatives near the problem boundaries and one integral measure of the global behavior of the solution. This information was sufficient for the polynomials to closely match the numerical solutions, as evidenced by the polynomials' close agreement to numerical solutions and their replication of the exact solutions.

**Appendix**

Here we present an alternative set of Conditions 4 that can be used to determine the coefficients of the polynomial approximate solutions and the wetting front positions. These alternative conditions are derived in the same manner as the original Conditions 4, but (5.8) is used in place of (5.9). Again, C4a was obtained by multiplying (5.8) by $\xi$ and integrating it over the problem domain. Condition C4b is obtained by integrating (5.8) over the problem domain. Conditions C1a and C1b were used, respectively, to simplify the forms of C4a and C4b that appear below. The other conditions, listed in (5.33), were not changed.

\[
C4 \quad (a) \int_0^{\xi_0} H \sqrt{-\frac{dH}{d\xi}} d\xi - (\lambda + 2) \int_0^{\xi_0} \xi H d\xi = 0
\]

\[
(b) \int_0^{\xi_0} Hd\xi = \frac{1}{1+\lambda}
\]

(5.54)

### 5.6.1 Alternative Power-Law Head formulation

For the power-law head condition at the inlet, Conditions C1a, C2 and C3 are again used to write the quadratic and cubic polynomials in normalized form as
\[ H_q(z) = y(1 - z) + (1 - y)(1 - z)^2 \]  
\hspace{1cm} (5.55)

and

\[ H_c(z) = y(1 - z) + (1 - y - \alpha_0)(1 - z)^2 + \alpha_0(1 - z)^3, \]  
\hspace{1cm} (5.56)

where \( y = \xi_3^3, z = \xi/\xi_0 \) and \( \alpha_0 = c\xi_0^3 \).

The alternative Condition 4a can be written in terms of \( y \) and the normalized variable \( z \) as

\[ \int_0^1 H \sqrt{-\frac{dH}{dz}} \, dz - (\lambda + 2) \sqrt{y} \int_0^1 z H \, dz = 0. \]  
\hspace{1cm} (5.57)

Solving (5.57) with \( H_q \) yields a solution for \( y \) and then \( \xi_0 \). Solving the system consisting of (5.57) and C5a with \( H_c \), allows one to solve for \( y \) and \( \alpha_0 \). The values of \( \xi_0 \) and the coefficients \( a, b \) and \( c \) can then be found.

For the power-law head problem, the alternative quadratic approximate solution better approximates the wetting front position when \( \lambda > 1 \) than the previous quadratic solution. However, the new solutions yield worse approximations when \( \lambda < 1 \). The alternative quadratic approximate solution does not yield the correct value of \( \xi_0 \) when \( \lambda = -1 \). When (5.9) was used as the governing equation, the quadratic approximate solution did give the correct value of \( \xi_0 \) for \( \lambda = -1 \). The alternative cubic solutions produce wetting front positions close to those of the original cubic approximate solutions for the power-law head problem. The comparison is shown in Table 5.3.
Table 5.3: Comparison of the wetting front position $\xi_0$ for the Power-Law-Head boundary problems using the numerical (Shampine’s) method and quadratic $H_q$ and cubic $H_c$ approximations for various values of $\lambda$. The alternative form of Condition 4a was used in the columns titled $H_qalt$ and $H_calt$.

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5.6.2 Alternative Power-Law Flux formulation

For the power-law flux problem, Conditions C1b, C2 and C3 are again used to write the quadratic and cubic polynomials in normalized form as

$$H_q(z, \lambda) = y(1 - z) + y^{2/3}b(1 - z)^2$$

(5.58)

and

$$H_c(z, \lambda) = y(1 - z) + y^{2/3}b(1 - z)^2 + \alpha_0(1 - z)^3,$$

(5.59)

where $y = \frac{\xi^3}{\xi_0}$, $z = \frac{\xi}{\xi_0}$ and $\alpha_0 = c_0\xi_0^3$.

The alternative Condition 4b can be written in terms of $y$ and the normalized variable $z$ as
Solving (5.60) and the nonlinear equation (5.45) for C1b with \( H_q \) yields solutions for \( y \) and \( b \) and then \( \xi_0 \). Solving the system consisting of (5.60) and Conditions C1b and C5b with \( H_c \), (5.50) and (5.53), allows one to solve for \( y \), \( b \) and \( \alpha_0 \). The coefficients \( a \) and \( c \) and the wetting front position \( \xi_0 \) can then be found. The alternative formulation of C4b produces similar results to the original formulation.

**Table 5.4:** Comparison of the wetting front position \( \xi_0 \) for the Power-Law-Flux boundary condition using numerical (Shampine’s) method, quadratic \( H_q \) and cubic \( H_c \) approximations for various values of \( \lambda \). The alternative form of Condition 4b was used in the columns titled \( H_{q,alt} \) and \( H_{c,alt} \).

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Bibliography


Chapter 6

Summary and Conclusions

This work consists of two main areas of research: the construction of polynomial approximate solutions for nonlinear groundwater flow problems and a derivation of a fractional conservation of mass equation involving left and right Caputo fractional derivatives. A comment on another researcher’s traveling-wave solution of the Boussinesq equation was also presented. The solution procedure is related to the construction of approximate solutions.

In Chapters 2 and 5, two classes of nonlinear partial differential equations describing unconfined groundwater flow were studied. For an initially dry aquifer with a horizontal impervious lower boundary, polynomial approximate solutions were constructed for these two classes of nonlinear groundwater flow equations. The first was a porous medium equation with a power-law head condition at the inlet. The second was an equation based on the Forchheimer generalization of Darcy’s Law. For this Forchheimer equation, power-law head, exponential head, power-law flux and exponential flux conditions at the inlet were considered. The problems involving exponential head and flux conditions were shown to be limiting cases of the corresponding power-law head conditions as the power-law exponents increase without bound. For each boundary condition, the governing flow equations were converted to free boundary-value problems involving nonlinear ordinary differential equations.
Numerical solutions were generated for the ordinary differential equations using modifications of a method of Shampine’s [1973] method, which exploits scaling properties of the equations. Quadratic and cubic approximate solutions were constructed for the ordinary differential equations. The polynomial approximate solutions closely matched the numerical solutions. The cubic approximate solutions replicated known exact solutions and provided better approximations of the numerical solutions than the quadratic approximate solutions, while the quadratic approximations replicate only some of the exact solutions.

In conclusion, Chapters 2 and 5 extended similar work with polynomial approximate solutions that had been previously done by Lockington et al. [2000] and Telyakovskiy and Allen [2006] for the Boussinesq equation. In addition, the procedure for generating numerical solutions demonstrated the use of the modified Shampine’s method with new classes of nonlinear groundwater flow equations. The methods for constructing approximate and numerical solutions were successfully applied to equations containing two different types of nonlinearities. In the porous medium equation the hydraulic conductivity is a power-law function of head, while the flow equation based on Forchheimer’s law contains a more general type of nonlinearity.

In Chapter 4, the fractional mean-value theorems were used to derive the fractional conservation of mass equation. It was noted that in the control volume there exist locations where the approximations will be exact. One-sided and two-sided conservation laws can be obtained in this way. The one-sided fractional conservation of mass equation by Wheatcraft and Meerschaert [2008], under additional assumptions, was obtained as a special case of the derivation based on the mean value theorems. The two-sided fractional conservation of mass equation allows one to derive special types of fractional groundwater and surface water flow and contaminant transport equations containing left and right Caputo fractional derivatives.

The work of Chapter 4 shows that a traditional control volume analysis can be used to
derive a mass conservation equation containing left and right Caputo fractional derivatives. However, the derivation leads to fractional governing equations that only model nonlocal effects within the control volume. The derived equations do not model nonlocal effects over the entire problem domain.

6.1 Recommendations

Outlined below are additional open questions and lines of pursuit for further research related to the preceding chapters.

6.1.1 Polynomial Approximate Solutions

In Chapters 2 and 5, polynomial approximate solutions were constructed for two classes of nonlinear groundwater flow equations. These approximations were constructed using conditions satisfied by the solution function and its derivatives. Why and when do certain combinations of conditions give better approximations than others? Under what circumstances do the different forms of the conditions produce better results than others? When solving the nonlinear systems of equations for the parameter coefficients, sometimes there exist multiple solutions. Can the process of selecting the correct solution be automated?

6.1.2 Use of the Traveling Wave Solution and Perturbation Series

The implications of Basha’s work, discussed in Chapter 3, should be investigated in greater depth. The solution from the perturbation series should be found under the explicit assumption that the wave speed is a function of time. The new solution will differ from the solution presented by Basha.
6.1.3 Fractional Conservation of Mass

Finally, in Chapter 4, a fractional conservation of mass equation involving left and right Caputo fractional derivatives was derived. Numerical solutions of groundwater flow or contaminant transport equations based on this fractional conservation of mass equation should be found. Such numerical solutions should be compared to numerical solutions of other types of fractional flow or transport equations.
Bibliography


