University of Nevada, Reno

A Bivariate Distribution Connected with Poissonian Maxima of Exponential Variables

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

by

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Abstract

In this work we present a bivariate distribution of X and N, where N has a Poisson distribution and X is the maximum of N independent, identically distributed exponential variables. The joint bivariate distribution is developed, along with some useful representations of the model. The results also include univariate marginal and conditional distributions, moments and the covariance matrix, simulation, and estimation of the parameters.

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Chapter 1

Introduction

Extreme values are a topic of interest in many different industries and various aspects of life. Businesses hope to obtain maximum profits and minimize costs. Companies may wonder, "What is the greatest number of orders we will have to fill in a given day?" or "What is the longest wait time a customer will experience today?" Insurance companies hope to predict and be able to cover most of their largest claims. Casinos wish to know the maximum they will be liable for if a player wins big. Water resource engineers want to know the probability of water storage falling below an unsafe value. A builder desires to know the largest magnitude of an earthquake to expect, maximum wind speeds, maximum snow load, and other extreme conditions to consider. Often the maximum, rather than the minimum, is of most interest. Frequently the maximum is the hardest value to predict and it is commonly unbounded.

When the distribution of a non-negative continuous random variable is known, we can use the probability density function, $f_X(x)$, to obtain the probability that the variable will be less than or equal to a value α, which is equivalent to the probability the maximum value is no more than α. This relationship is stated below.

$$P(X \leq \alpha) = \int_0^\alpha f_X(x)\,dx. \tag{1.1}$$

Alternatively we can use the cumulative distribution function,

$$P(X \leq \alpha) = F_X(\alpha). \tag{1.2}$$

These equations are useful when the big question of "What is the probability my variable of interest will be no more than α?" is asked. However, they only apply to a single random variable. When we are concerned about the maximum value of N random variables, which are mutually independent, but share a common distribution function, it is well-known to calculate this as follows:

$$\begin{aligned} P\left(\max_{1\leq i\leq N} X_i \leq \alpha \middle| N = n\right) &= P(X_i \leq \alpha \text{ for each i} = 1, 2, ..., n) \\ &= [P(X_i \leq \alpha)]^n \\ &= [F_X(\alpha)]^n. \end{aligned} \tag{1.3}$$

Note the above function is a genuine cumulative distribution function (c.d.f.) itself and is valid even when n is not an integer, as long as $n > 0$. In particular when F is exponential, this distribution coincides with the generalized exponential distribution studied in Gupta and Kundu (1999).

Beyond the expansion to a c.d.f. for N random variables, we can study a bivariate distribution of both the maximum of N variables and the distribution of N, where N itself is a random variable. When the discrete distribution of N is known, further properties of the maximum of the random variables can be derived. It is useful, however difficult, to consider both the number of terms and the maximum of those terms concurrently.

The maximum of N independent, identically distributed (i.i.d.) exponential variables where N is a geometric random variable ($N \sim \mathcal{GEO}(\text{p})$) was studied by Kozubowski and Panorska (2008). Geometric random variables can be found in nature in such areas as water resources and climate. It can also appear in the subject of finance, particularly in the area of exchange rates (See [5]). The bivariate model of the sum of the geometric number of exponential variables was discussed in Kozubowski and Panorska (2005). For a comprehensive treatment of bivariate models for sums and maxima of a geometric number of exponential random variables and their applications please see Kozubowski, Panorska, and Biondi (2008).

In this thesis we study the maximum of i.i.d. exponential variables that arrive at the rate of the Poisson distribution. To accomplish this we let N be a Poisson random variable with parameter $\lambda > 0$ and probability mass function (p.m.f.)

$$f_N(n) = P(N = n) = \frac{e^{-\lambda}\lambda^n}{n!},\ n \in \{0, 1, 2, ...\}\,. \tag{1.4}$$

We shall denote this distribution by $\mathcal{POISSON}(\lambda)$. Also, we let $X_1, X_2, ..., X_N$ be

N i.i.d. exponential variables with parameter $\beta > 0$, given by the probability density function (p.d.f.)

$$f_{X_i}(x) = \beta e^{-\beta x},\ x \geq 0, \tag{1.5}$$

and c.d.f.

$$F_{X_i}(x) = 1 - e^{-\beta x},\ x \geq 0. \tag{1.6}$$

We shall denote this distribution by $\mathcal{EXP}(\beta)$. We then consider

$$X = \max_{1 \leq i \leq N} X_i, \tag{1.7}$$

and in this research we study the bivariate distribution of the random vector (X, N). By construction of our model we have a bivariate vector in which the distribution is neither continuous nor discrete. When considering applications of our bivariate model it is advantageous that the Poisson distribution is closely related to the exponential distribution. It is common to consider N exponential variables which arrive at a Poissonian rate. For example, the number of customer arrivals can be modeled with the Poisson distribution, where the length of time between two consecutive arrivals has an exponential distribution. The memoryless property of exponential distribution is useful in many applications, which further supports why our model is important. For instance, regarding the customer example, the memoryless property states that the probability a minute will pass before the next customer arrives is equal to the probability that another minute will pass before the next customer will arrive given that three minutes have already passed with no customer arrivals.

In equation (1.3) we had a deterministic number of terms, n. It is straightforward to expand this idea to fit our model with a random number of terms. By independence of the X_i's and N we have

$$P\left(\max_{1\leq i\leq N} X_i \leq \alpha, N = n\right) = [P(X_i \leq \alpha)]^n \cdot P(N = n) = [F_X(\alpha)]^n \cdot f_N(n). \quad (1.8)$$

A related bivariate distribution that has been studied in the past is the sum of N i.i.d. discrete random variables. Most closely related to this research is the work of Park (1970) who found properties of the bivariate distribution of the total time for the emission of particles emitted from a radioactive substance and the number of these particles, where the waiting times of the emissions are exponentially distributed and the number of particles has a Poisson distribution.

This is closely related to a Poisson Process, where the waiting time between observations is exponentially distributed, but the total time to be studied is not set. The results from this thesis can be applied to Park's radioactive particles example, where our results would apply to the *maximum* of the radioactive particle decay times in conjunction with the number of emitted particles. The choice for us to study a bivariate model, rather than a univariate model with constant N is clear here. One would expect the maximum time for decay to be largely affected by the number of particles. Another application of our model is closely related to queuing theory, which has many applications to businesses concerned with maximum customer wait times.

Sarabia and Guillén (2008) emphasize the value of studying a bivariate model rather than two univariate models. They use various models for the bivariate vector (S, N), where S is the sum of random variables, and N is the number. Their application of interest is from the insurance industry where N represents the number of claims and S represents the total claim amount.

Kozubowski and Panorska (2005) studied the bivariate distribution of the sum of N exponential variables, where $N \sim \mathcal{GEO}(\text{p})$. Biondi, Kozubowski, and Panorska (2005) applied this model to stochastic hydrology. This bivariate distribution can be expanded upon by letting N be a negative binomial process, dependent on time. This expansion was studied by Kozubowski, Panorska, and Podgórski (2008). An idea for further study in conjunction to this thesis is to study properties of the random process

$$(X(t), N(t)) = \left(\max_{1 \le i \le N(t)} X_i, N(t) \right), \, t \ge 0. \tag{1.9}$$

This work is organized as follows. The construction of the bivariate p.d.f. and c.d.f. is presented in Chapter 2, useful representations of the bivariate model appear in Chapter 3, univariate marginal and conditional distributions are treated in Chapter 4, derivation of moments and the covariance matrix appear in Chapter 5, and finally estimation and simulation are discussed in Chapter 6 and the appendices.

Chapter 2

Definition and basic properties

We begin by defining our bivariate model.

Definition 2.0.1 *A random vector*

$$(X, N) \stackrel{d}{=} \left(\max_{1 \le i \le N} X_i, N \right), \tag{2.1}$$

where the $\{X_i\}$ *are i.i.d. exponential variables with p.d.f. (1.4), and* N *is a Poisson variable with p.m.f. (1.5), independent of the* $\{X_i\}$*, is said to have a BMEP distribution with parameters* $\beta > 0$ *and* $\lambda > 0$*. This distribution is denoted by* $\mathcal{BMEP}(\beta, \lambda)$*.*

BMEP stands for *bivariate* distribution of the *maximum* of *exponential* variables and *Poisson* marginals.

2.1 The joint probability density function

Let $f(x,n)$ denote the joint p.d.f. of $(X,N) \sim \mathcal{BMEP}(\beta,\lambda)$, while $f_{X|N=n}(x)$ and $f_N(n)$ denote the conditional p.d.f. of X given $N=n$ and the marginal p.d.f. of N, respectively. Since the joint p.d.f. is the product

$$f(x,n) = f_{X|N=n}(x) \cdot f_N(n),$$

it can be derived through a conditioning argument. We have the following two cases.

Case 1: $N=n=0$. If there are 0 events, then $X = \max_{1\le i\le N} X_i = 0$. Therefore,

$$f_{X|N=0}(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}. \tag{2.2}$$

Since $f_N(0) = e^{-\lambda}$, we immediately obtain

$$f(x,0) = f_{X|N=0}(x) \cdot f_N(0) = \begin{cases} e^{-\lambda} & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}. \tag{2.3}$$

Case 2: $N=n \in \mathcal{N}$. Given $N=n\in\mathcal{N}$, where $\mathcal{N}$ is the set of natural numbers, X is the maximum of n i.i.d. $\mathcal{EXP}(\beta)$ variables. Thus, the c.d.f. of X given $N=n$ is

$$\begin{aligned} F_{X|N=n}(x) &= P(X \le x | N=n) \\ &= P\left(\max_{1\le i\le N} X_i \le x \middle| N=n\right) \\ &= P(X_i \le x \text{ for each i} = 1,2,...,n) \end{aligned}$$

$$= [P(X_i \leq x)]^n$$
$$= [1 - e^{-\beta x}]^n, \; x \geq 0. \qquad (2.4)$$

Next, the p.d.f. of X given N is found to be

$$f_{X|N=n}(x) = F'_{X|N=n}(x) = n\beta e^{-\beta x}(1 - e^{-\beta x})^{n-1}, x > 0. \qquad (2.5)$$

Thus, for case 2,

$$\begin{aligned} f(x,n) &= f_{X|N=n}(x) \cdot f_N(n) \\ &= n\beta e^{-\beta x}(1 - e^{-\beta x})^{n-1} \cdot \frac{e^{-\lambda}\lambda^n}{n!} \\ &= \beta e^{-\beta x - \lambda}\lambda^n \frac{(1 - e^{-\beta x})^{n-1}}{(n-1)!}, x > 0. \end{aligned} \qquad (2.6)$$

Combining cases 1 and 2 we now state the joint p.d.f. in the proposition below.

Proposition 2.1.1 *Let $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$. Then the joint p.d.f. of (X, N) is*

$$f(x,n) = \begin{cases} e^{-\lambda} & \text{if } x = n = 0 \\ \beta e^{-\beta x - \lambda}\lambda^n \frac{(1 - e^{-\beta x})^{n-1}}{(n-1)!} & \text{if } x > 0, \; n \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases} . \qquad (2.7)$$

This can be written more compactly as,

$$f(x,n) = e^{-\lambda}\left\{ I_{\{(0,0)\}}(x,n) + \beta e^{-\beta x}\lambda^n \frac{(1-e^{-\beta x})^{n-1}}{(n-1)!} I_{(\mathcal{R}_+ \times \mathcal{N})}(x,n) \right\}, \tag{2.8}$$

where $I_A(\cdot)$ denotes the indicator function of the set A:

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

2.2 The joint cumulative distribution function

Let us denote the joint c.d.f. of (X, N) by $F(x, y)$, and proceed in finding $F(x, y)$ using two cases.

Case 1: $x < 0$ or $y < 0$. Clearly $F(x, y) = 0$.

Case 2: $x \geq 0$ and $y \geq 0$. In this case,

$$\begin{aligned} F(x,y) &\overset{(1)}{=} P(X \leq x, N \leq y) \\ &\overset{(2)}{=} \sum_{k=0}^{n} P(X \leq x, N = k) \\ &\overset{(3)}{=} \sum_{k=0}^{n} P(X \leq x | N = k) \cdot P(N = k) \\ &\overset{(4)}{=} \sum_{k=0}^{n} [1 - e^{-\beta x}]^k \cdot \frac{e^{-\lambda}\lambda^k}{k!} \\ &\overset{(5)}{=} e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!} [1 - e^{-\beta x}]^k, \end{aligned} \tag{2.9}$$

where n in line (2) is the greatest integer less than or equal to y, and in line (4) we use equation (2.4). Note that in the special case $x = 0$ the quantities $[1 - e^{-\beta \cdot 0}]^k$ in line (4) are zero for $k \geq 1$ and 1 for $k = 0$.

Combining all cases we now state the joint c.d.f. in the proposition below.

Proposition 2.2.1 *Let $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$. Then the joint c.d.f. of (X, N) is*

$$F(x, y) = \begin{cases} e^{-\lambda} & \text{if } x = 0 \text{ and } y \geq 0, \text{ or if } x > 0 \text{ and } y = 0 \\ e^{-\lambda} \sum_{k=0}^{[[y]]} \frac{\lambda^k}{k!} [1 - e^{-\beta x}]^k & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{2.10}$$

where $[[\cdot]]$ denotes the greatest integer function.

Chapter 3

Representations

In this chapter we shall derive useful representations of BMEP random vectors.

3.1 Mixture representation

By rewriting the joint p.d.f. of (X, N) found in equation (2.8) as

$$\begin{aligned} f(x,n) &= [e^{-\lambda}]I_{\{(0,0)\}}(x,n) + \\ &\quad [1-e^{-\lambda}]\frac{e^{-\lambda}}{1-e^{-\lambda}}\frac{(1-e^{-\beta x})^{n-1}}{(n-1)!}\beta e^{-\beta x}\lambda^n I_{(\mathcal{R}_+\times\mathcal{N})}(x,n), \end{aligned} \tag{3.1}$$

we see that $f(x, n)$ can be represented as a mixed distribution, which with probability $e^{-\lambda}$ is a point mass at $\{0, 0\}$, or with probability $1-e^{-\lambda}$ is a random vector $(\tilde{X}, \tilde{N}) \in \mathcal{R}_+ \times \mathcal{N}$ given by the p.d.f.

$$g(x,n) = \begin{cases} \dfrac{e^{-\lambda}}{1-e^{-\lambda}} \dfrac{(1-e^{-\beta x})^{n-1}}{(n-1)!} \beta e^{-\beta x} \lambda^n & \text{if } x > 0,\ n \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases} . \tag{3.2}$$

Let us verify that $g(x,n)$ is a valid p.d.f., that is, verify $\sum_{n=1}^{\infty} \int_0^{\infty} g(x,n)\,dx = 1$. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_0^{\infty} g(x,n)\,dx &\overset{(1)}{=} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{(1-e^{-\beta x})^{n-1}}{(n-1)!} \beta e^{-\beta x} \lambda^n\,dx \\
&\overset{(2)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} \int_0^{\infty} (1-e^{-\beta x})^{n-1} \beta e^{-\beta x}\,dx \\
&\overset{(3)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} \lim_{t\to\infty} \int_0^{t} (1-e^{-\beta x})^{n-1} \beta e^{-\beta x}\,dx \\
&\overset{(4)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} \lim_{t\to\infty} \int_0^{1-e^{-\beta t}} u^{n-1}\,du \\
&\overset{(5)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} \lim_{t\to\infty} \frac{(1-e^{-\beta t})^n}{n} \\
&\overset{(6)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} \frac{1}{n} \\
&\overset{(7)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \\
&\overset{(8)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} (e^{\lambda}-1) = \frac{1-e^{-\lambda}}{1-e^{-\lambda}} = 1,
\end{aligned}$$

where in line (4) we use the substitution $u = 1 - e^{-\beta x}$ and in line (8) we note $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Further, rewriting equation (3.1) as

$$f(x,n) = [e^{-\lambda}] I_{\{(0,0)\}}(x,n) + [1-e^{-\lambda}] g(x,n) I_{(\mathcal{R}_+ \times \mathcal{N})}(x,n), \tag{3.3}$$

we see the joint p.d.f. of the BMEP distribution, $f(x,n)$, sums to 1 over its domain as well:

$$\begin{aligned} f(0,0)+\sum_{n=1}^{\infty}\int_0^{\infty} f(x,n)\,dx &= e^{-\lambda}+[1-e^{-\lambda}]\sum_{n=1}^{\infty}\int_0^{\infty} g(x,n)\,dx \\ &= e^{-\lambda}+[1-e^{-\lambda}]\cdot 1 = 1. \end{aligned}$$

We can express the mixture representation of (X,N) as a stochastic identity using the vector $(\tilde{X},\tilde{N})$, along with Bernoulli random variable I, which indicates whether or not $N=0$. The following proposition formalizes this idea.

Proposition 3.1.1 *If* $(X,N)\sim \mathcal{BMEP}(\beta,\lambda)$*, then*

$$(X,N) \stackrel{d}{=} (I\tilde{X}, I\tilde{N}), \tag{3.4}$$

where $(\tilde{X},\tilde{N})$ *is a random vector with p.d.f.* $g(x,y)$ *given by (3.2), and* I *is an indicator random variable, independent of* $(\tilde{X},\tilde{N})$*, taking on the values of 1 and 0 with probabilities* $1-e^{-\lambda}$ *and* $e^{-\lambda}$*, respectively.*

Proof Let $G(x,y)$ represent the joint c.d.f. of $(I\tilde{X}, I\tilde{N})$. We consider the following cases.

Case 1: If $x<0$ or $y<0$, then we have

$$G(x,y)=P(I\tilde{X}\le x, I\tilde{N}\le y)=0,$$

since $I\in\{0,1\}$ and $(\tilde{X},\tilde{N})\in\mathcal{R}_+\times\mathcal{N}$.

Case 2: If $x = 0$ and $y \geq 0$, we have

$$\begin{aligned}
G(x,y) &= P(I\tilde{X} \leq 0, I\tilde{N} \leq y) \\
&= P(I\tilde{X} \leq 0, I\tilde{N} \leq y | I = 0) \cdot P(I = 0) + \\
&\quad P(I\tilde{X} \leq 0, I\tilde{N} \leq y | I = 1) \cdot P(I = 1) \\
&= P(0 \leq 0, 0 \leq y) \cdot e^{-\lambda} + P(\tilde{X} \leq 0, \tilde{N} \leq y)(1 - e^{-\lambda}) \\
&= 1 \cdot e^{-\lambda} + 0 \cdot (1 - e^{-\lambda}) = e^{-\lambda}.
\end{aligned}$$

Case 3: If $x > 0$ and $y = 0$, then

$$\begin{aligned}
G(x,y) &= P(I\tilde{X} \leq x, I\tilde{N} \leq 0) \\
&= P(I\tilde{X} \leq x, I\tilde{N} \leq 0 | I = 0) \cdot P(I = 0) + \\
&\quad P(I\tilde{X} \leq x, I\tilde{N} \leq 0 | I = 1) \cdot P(I = 1) \\
&= P(0 \leq x, 0 \leq 0) \cdot e^{-\lambda} + P(\tilde{X} \leq x, \tilde{N} \leq 0)(1 - e^{-\lambda}) \\
&= 1 \cdot e^{-\lambda} + 0 \cdot (1 - e^{-\lambda}) = e^{-\lambda}.
\end{aligned}$$

Case 4: If $x > 0$ and $y > 0$, then we have

$$\begin{aligned}
G(x,y) &\overset{(1)}{=} P(I\tilde{X} \leq x, I\tilde{N} \leq y) \\
&\overset{(2)}{=} P(I\tilde{X} \leq x, I\tilde{N} \leq y | I = 0) \cdot P(I = 0) + \\
&\quad P(I\tilde{X} \leq x, I\tilde{N} \leq y | I = 1) \cdot P(I = 1) \\
&\overset{(3)}{=} P(0 \leq x, 0 \leq y) \cdot e^{-\lambda} + P(\tilde{X} \leq x, \tilde{N} \leq y)(1 - e^{-\lambda})
\end{aligned}$$

$$\overset{(4)}{=} 1 \cdot e^{-\lambda} + (1 - e^{-\lambda}) \sum_{k=1}^{n} \int_0^x g(t,k)dt$$
$$\overset{(5)}{=} e^{-\lambda} + (1 - e^{-\lambda}) \sum_{k=1}^{n} \int_0^x \frac{e^{-\lambda}}{1 - e^{-\lambda}} \frac{[1 - e^{-\beta t}]^{k-1}}{(k-1)!} \beta e^{-\beta t} \lambda^k dt$$
$$\overset{(6)}{=} e^{-\lambda} + e^{-\lambda} \sum_{k=1}^{n} \frac{\lambda^k}{k!} \int_0^x k\beta[1 - e^{-\beta t}]^{k-1} e^{-\beta t} dt$$
$$\overset{(7)}{=} e^{-\lambda} + e^{-\lambda} \sum_{k=1}^{n} \frac{\lambda^k}{k!} [1 - e^{-\beta x}]^k$$
$$\overset{(8)}{=} e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!} [1 - e^{-\beta x}]^k,$$

where in line (4), n is the greatest integer less than or equal to y. Combining cases 1-4 we see that the joint c.d.f. of $(I\tilde{X}, I\tilde{N})$ is equivalent to $F(x,y)$, the joint c.d.f. of (X,N), given in (2.10). Therefore we have proven that $(X,N) \overset{d}{=} (I\tilde{X}, I\tilde{N})$.

3.2 An alternative representation

Here we develop an alternative representation of BMEP random vectors, where instead of the maximum we have the sum of exponential components. This representation, which is closely connected with properties of order statistics in exponential samples, can be explained by conditioning on N. Indeed, let (X,N) be a BMEP random vector with parameters β and λ. Then, according to Definition 2.0.1, given $N = k$, the random variable X is the maximum of k i.i.d. exponential variables with parameter β, so its c.d.f. takes on the form

$$P(X \leq x | N = k) = (1 - e^{-\beta x})^k, \; x \geq 0.$$

The above is the c.d.f. of the generalized exponential distribution, studied in Gupta and Kundu [3], which is known to have the stochastic representation

$$(X|N{=}k) \stackrel{d}{=} \sum_{j=1}^{k} \frac{E_j}{j}, \tag{3.5}$$

where the E_j are i.i.d. exponential variables with parameter β. Replacing the deterministic number of terms, k, with a random number of terms, N, we obtain a similar representation for X, and more generally, (X, N), which is formally stated in the result below.

Proposition 3.2.1 *If $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$, then*

$$(X, N) \stackrel{d}{=} \left(\sum_{i=1}^{N} \frac{X_i}{i}, N \right). \tag{3.6}$$

Proof Let $H(x, y)$ represent the joint c.d.f. of $\left(\sum_{i=1}^{N} \frac{X_i}{i}, N \right)$. We consider the following cases.

Case 1: If $x < 0$ or $y < 0$, then since $N \geq 0$ and $X_i \geq 0$ for each i, $H(x, y) = P(\sum_{i=1}^{N} \frac{X_i}{i} \leq x, N \leq y) = 0$.

Case 2: $x = 0$ and $y \geq 0$. Here we have

$$\begin{aligned} H(x, y) &\stackrel{(1)}{=} P\left(\sum_{i=1}^{N} \frac{X_i}{i} \leq 0, N \leq y \right) \\ &\stackrel{(2)}{=} P\left(\sum_{i=1}^{N} \frac{X_i}{i} < 0, N \leq y \right) + P\left(\sum_{i=1}^{N} \frac{X_i}{i} = 0, N \leq y \right) \\ &\stackrel{(3)}{=} 0 + P\left(\sum_{i=1}^{N} \frac{X_i}{i} = 0, N \leq y \right) \end{aligned}$$

$$
\begin{aligned}
&\overset{(4)}{=} \quad P\left(\sum_{i=1}^{N}\frac{X_i}{i}=0\middle|N=0\right)\cdot P(N=0)+\\
&\qquad P\left(\sum_{i=1}^{N}\frac{X_i}{i}=0\middle|N=1\right)\cdot P(N=1)+\\
&\qquad P\left(\sum_{i=1}^{N}\frac{X_i}{i}=0\middle|N=2\right)\cdot P(N=2)+...+\\
&\qquad P\left(\sum_{i=1}^{N}\frac{X_i}{i}=0\middle|N=n\right)\cdot P(N=n)\\
&\overset{(5)}{=} \quad 1\cdot P\left(N=0\right)+0+0+...+0=e^{-\lambda},
\end{aligned}
$$

where n in line (4) is the greatest integer less than or equal to y.

Case 3: $x>0$ and $y=0$. In this case, we have

$$
\begin{aligned}
H(x,y) &= P\left(\sum_{i=1}^{N}\frac{X_i}{i}\leq x, N\leq 0\right)\\
&= P\left(\sum_{i=1}^{N}\frac{X_i}{i}\leq x, N<0\right)+P\left(\sum_{i=1}^{N}\frac{X_i}{i}\leq x, N=0\right)\\
&= 0+P\left(\sum_{i=1}^{N}\frac{X_i}{i}\leq x, N=0\right)\\
&= P\left(\sum_{i=1}^{N}\frac{X_i}{i}\leq 0\middle|N=0\right)\cdot P(N=0)\\
&= 1\cdot P\left(N=0\right)=e^{-\lambda}.
\end{aligned}
$$

Case 4: If $x>0$ and $y>0$, we have

$$
\begin{aligned}
H(x,y) &\overset{(1)}{=} P\left(\sum_{i=1}^{N}\frac{X_i}{i}\leq x, N\leq y\right)\\
&\overset{(2)}{=} \sum_{k=0}^{n} P\left(\sum_{i=1}^{N}\frac{X_i}{i}\leq x, N=k\right)\\
&\overset{(3)}{=} \sum_{k=0}^{n} P\left(\sum_{i=1}^{k}\frac{X_i}{i}\leq x, N=k\right)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(4)}{=} \sum_{k=0}^{n} P(\sum_{i=1}^{k} \frac{X_i}{i} \leq x) \cdot P(N = k) \\
&\overset{(5)}{=} \sum_{k=0}^{n} [1 - e^{-\beta x}]^k \cdot \frac{e^{-\lambda}\lambda^k}{k!} \\
&\overset{(6)}{=} e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!} [1 - e^{-\beta x}]^k,
\end{aligned}
$$

where in line (2), n is the greatest integer less than or equal to y. In line (4) we use the fact that the X_i's are independent of N, and in line (5) we use the stochastic representation (3.5) of the generalized exponential distribution of Gupta and Kundu [3].

Combining cases 1-4 we see that $H(x, y)$ coincides with the c.d.f. (2.10). Therefore we have proven that $(X, N) \overset{d}{=} \left(\sum_{i=1}^{N} \frac{X_i}{i}, N \right)$.

Chapter 4

Marginal and conditional distributions

In addition to the joint probability distribution of the vector (X, N) it is useful to know univariate distributions for each variable. Having the marginal probability distribution of one variable we can see how this particular variable behaves irrespective to the second variable. Another important distribution is the conditional distribution, the probability distribution of one variable when the value of the other variable is known. In this chapter we will derive the marginal and conditional probability distributions connected with the BMEP model.

4.1 Marginal distributions of $\tilde{X}$ and $\tilde{N}$

We will begin by finding the marginal distributions of $\tilde{X}$ and $\tilde{N}$ of the Mixture Representation (3.4) given in Proposition 3.1.1. We start with the marginal p.d.f. of $\tilde{X}$, denoted by $f_{\tilde{X}}(x)$:

$$\begin{aligned} f_{\tilde{X}}(x) &\overset{(1)}{=} \sum_{n=1}^{\infty} g(x,n) \\ &\overset{(2)}{=} \sum_{n=1}^{\infty} \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{(1-e^{-\beta x})^{n-1}}{(n-1)!} \beta e^{-\beta x} \lambda^n \\ &\overset{(3)}{=} \frac{\lambda \beta e^{-\beta x} e^{-\lambda}}{1-e^{-\lambda}} \sum_{n=1}^{\infty} \frac{(\lambda[1-e^{-\beta x}])^{n-1}}{(n-1)!} \\ &\overset{(4)}{=} \frac{\lambda \beta e^{-\beta x} e^{-\lambda}}{1-e^{-\lambda}} e^{\lambda[1-e^{-\beta x}]} \\ &\overset{(5)}{=} \frac{\lambda \beta e^{-\beta x}}{1-e^{-\lambda}} e^{-\lambda e^{-\beta x}}, \ x > 0, \end{aligned} \tag{4.1}$$

where in line (4) we note that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. It is easy to see that $f_{\tilde{X}}(x)$ integrates to 1. Indeed, we have

$$\int_0^{\infty} f_{\tilde{X}}(x)\,dx = \int_0^{\infty} \frac{\lambda \beta e^{-\beta x}}{1-e^{-\lambda}} e^{-\lambda e^{-\beta x}}\,dx \overset{(1)}{=} \frac{\lambda\beta}{1-e^{-\lambda}} \int_{-\lambda}^{0} \frac{1}{\lambda\beta} e^y\,dy = \left.\frac{e^y}{1-e^{-\lambda}}\right|_{-\lambda}^{0} = 1,$$

where in (1) we used the substitution $y = -\lambda e^{-\beta x}$.

We can integrate the marginal p.d.f. of $\tilde{X}$ to obtain the marginal c.d.f. of $\tilde{X}$.

$$\begin{aligned} F_{\tilde{X}}(x) &\overset{(1)}{=} \int_0^x f_{\tilde{X}}(x)\,dx \\ &\overset{(2)}{=} \int_0^x \frac{\lambda \beta e^{-\beta x}}{1-e^{-\lambda}} e^{-\lambda e^{-\beta x}}\,dx \end{aligned}$$

$$\overset{(3)}{=} \frac{1}{1-e^{-\lambda}} \int_{-\lambda}^{-\lambda e^{-\beta x}} e^{y}\, dy$$
$$\overset{(4)}{=} \frac{1}{1-e^{-\lambda}} \left(e^{-\lambda e^{-\beta x}} - e^{-\lambda} \right),\ x > 0, \tag{4.2}$$

where in line (3) we use the substitution $y = -\lambda e^{-\beta x}$. We note that the function $F_{\tilde{X}}$ is non-decreasing, $\lim_{x\to\infty} F_{\tilde{X}}(x) = 1$, and $\lim_{x\to 0} F_{\tilde{X}}(x) = 0$, as desired.

On the other hand, $\tilde{N}$ has marginal p.m.f.

$$\begin{aligned} f_{\tilde{N}}(n) &\overset{(1)}{=} \int_0^\infty g(x,n)\, dx \\ &\overset{(2)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^n}{(n-1)!} \int_0^\infty (1-e^{-\beta x})^{n-1} \beta e^{-\beta x}\, dx \\ &\overset{(3)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^n}{(n-1)!} \int_0^1 y^{n-1}\, dy \\ &\overset{(4)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^n}{(n-1)!} \frac{y^n}{n}\bigg|_0^1 \\ &\overset{(5)}{=} \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^n}{n!} \overset{(6)}{=} \frac{\lambda^n}{n!(e^{-\lambda}-1)}, \end{aligned} \tag{4.3}$$

where in (3) we used the substitution $y = 1 - e^{-\beta x}$.

We use the fact that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ in step (2) below, to see that $f_{\tilde{N}}(n)$ summed over all possible values of $\tilde{N}$ equals 1, as desired. We have

$$\sum_{n=1}^{\infty} f_{\tilde{N}}(n) \overset{(1)}{=} \frac{1}{e^\lambda - 1} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \overset{(2)}{=} \frac{1}{e^\lambda - 1}(e^\lambda - 1) \overset{(3)}{=} 1.$$

The marginal c.d.f. of $\tilde{N}$ is

$$F_{\tilde{N}}(n) = \sum_{i=1}^{n} f_{\tilde{N}}(i) = \frac{1}{e^\lambda - 1} \sum_{i=1}^{n} \frac{\lambda^i}{i!}. \tag{4.4}$$

4.2 The marginal probability distributions of X and N

By Definition 2.0.1, the marginal p.m.f. of N is the Poisson distribution,

$$f_N(n) \quad = \quad \frac{e^{-\lambda}\lambda^n}{n!} \text{ for } n = 0, 1, 2, \dots . \tag{4.5}$$

Remark In view of representation (3.4) of Proposition 3.1.1, we have the relation $N \overset{d}{=} I\tilde{N}$, where $\tilde{N}$ has the p.m.f. (4.3) while I, independent of $\tilde{N}$ takes on the values of 1 and 0 with probabilities $1 - e^{-\lambda}$ and $e^{-\lambda}$, respectively. Thus, it is clear that $N = 0$ if and only if $I = 0$, the probability of which is $e^{-\lambda}$, while for $n \in \mathcal{N}$ we have

$$\begin{aligned} P(N = n) &= P(N = n | I = 1) \cdot P(I = 1) = P(\tilde{N} = n) \cdot P(I = 1) \\ &= \frac{e^{-\lambda}\lambda^n}{n!(1 - e^{-\lambda})} \cdot (1 - e^{-\lambda}) = \frac{e^{-\lambda}\lambda^n}{n!}, \end{aligned}$$

which coincides with the Poisson probability (4.5). Incidentally, this also shows the relation

$$f_{\tilde{N}}(n) = P(\tilde{N} = n) = \frac{P(N = n)}{P(N > 0)}, \; n \in \mathcal{N}.$$

Next, we find the marginal c.d.f. and the marginal p.d.f. of X. Recall from equation (3.4) that $(X, N) \overset{(d)}{=} (I\tilde{X}, I\tilde{N})$. This implies $X \overset{d}{=} I\tilde{X}$, where $\tilde{X}$ is a continuous random variable with c.d.f. found in equation (4.2) to be

$$F_{\tilde{X}}(x) = \frac{1}{1-e^{-\lambda}}\left(e^{-\lambda e^{-\beta x}} - e^{-\lambda}\right), x > 0.$$

Thus, we can calculate $F_X(x)$ as follows:

$$\begin{aligned} F_X(x) &= P(X \le x) = P(I\tilde{X} \le x) \\ &= P(I\tilde{X} \le x | I = 0) \cdot P(I = 0) + P(I\tilde{X} \le x | I = 1) \cdot P(I = 1) \\ &= P(0 \le x) \cdot e^{-\lambda} + P(\tilde{X} \le x) \cdot (1 - e^{-\lambda}) \\ &= e^{-\lambda} I_{[0,\infty)}(x) + (1 - e^{-\lambda}) \cdot \frac{1}{1-e^{-\lambda}}\left(e^{-\lambda e^{-\beta x}} - e^{-\lambda}\right) I_{(0,\infty)} \\ &= e^{-\lambda e^{-\beta x}} I_{\{[0,\infty)\}}. \end{aligned} \tag{4.6}$$

This leads to the following result.

Proposition 4.2.1 *Let* $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$. *Then the marginal cumulative distribution function of* X *is*

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-\lambda e^{-\beta x}} & \text{if } x \ge 0 \end{cases}. \tag{4.7}$$

We note that the function F_X is non-decreasing, $\lim_{x\to-\infty} F_X(x) = 0$, and $\lim_{x\to\infty} F_X(x) = 1$, as desired. Observe that $F_X(0) = e^{-\lambda}$ while $\lim_{x\to 0^-} F_X(x) = 0$, showing that the distribution of X has a point mass of $e^{-\lambda}$ at $X = 0$, which agrees with the representation $X \stackrel{d}{=} I\tilde{X}$. In other words, $X = 0$ with probability $e^{-\lambda}$ and $X = \tilde{X}$ with probability

$1 - e^{-\lambda}$, where $\tilde{X}$ is a continuous random variable with the p.d.f.

$$f_{\tilde{X}}(x) = \frac{\lambda\beta e^{-\beta x}}{1 - e^{-\lambda}} e^{-\lambda e^{-\beta x}},\ x > 0.$$

To summarize this, we can express the marginal p.d.f. of X as follows:

$$\begin{aligned} f_X(x) &= e^{-\lambda} I_{\{0\}}(x) + (1 - e^{-\lambda}) f_{\tilde{X}}(x) I_{\{(0,\infty)\}}(x) \\ &= e^{-\lambda} I_{\{0\}}(x) + \lambda\beta e^{-\lambda e^{-\beta x} - \beta x} I_{\{(0,\infty)\}}(x) \\ &= e^{-\lambda} I_{\{0\}}(x) + (1 - e^{-\lambda}) f_{\tilde{X}}(x). \end{aligned} \tag{4.8}$$

Since we verified that $f_{\tilde{X}}(x)$ integrates to 1 in equation (4.2), it follows that $f_X(x)$ integrates to 1 as well. We state the marginal p.d.f. of X formally below.

Proposition 4.2.2 *Let* $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$. *Then the marginal probability distribution of* X *is given by* $f_X(X)$, *where*

$$f_X(x) = \begin{cases} e^{-\lambda} & \text{if } x = 0 \\ \lambda\beta e^{-\lambda e^{-\beta x} - \beta x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} . \tag{4.9}$$

4.3 The conditional probability distribution of X given N

We saw in equation (2.2) that the conditional p.d.f. of X for the case $N = 0$ is as follows:

$$f_{X|N=0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Combining this with equation (2.5), we obtain the following result.

Proposition 4.3.1 *Let $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$. Then the conditional probability distribution of X given $N = n$ is given by $f_{X|N=n}(x)$, where*

$$f_{X|N=n}(x) = \begin{cases} n\beta e^{-\beta x}(1 - e^{-\beta x})^{n-1} & \text{if } n \in \mathcal{N} \text{ and } x > 0 \\ 1 & \text{if } n = 0 \text{ and } x = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{4.10}$$

We can then note that the conditional c.d.f. of X for case $N = 0$ is,

$$F_{X|N=0}(x) = P(X \leq x | N = 0) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with equation (2.4), we obtain the following result.

Proposition 4.3.2 *Let $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$. Then the cumulative condi-*

tional probability distribution function of X given $N = n$ is $F_{X|N=n}(x)$, where

$$F_{X|N=n}(x) = \begin{cases} [1 - e^{-\beta x}]^n & \text{if } n \in \mathcal{N} \text{ and } x \geq 0 \\ 1 & \text{if } n = 0 \text{ and } x \geq 0 \\ 0 & \text{otherwise} \end{cases} . \quad (4.11)$$

4.4 The conditional probability distribution of N given X

Next we find the conditional p.m.f. of N given $X = x$. We proceed using two cases.

Case 1: For $x = 0$ we will use the equation $f_{N|X=0}(n) = \frac{f(0,n)}{f_X(0)}$. First, note that by the marginal p.d.f. of X found in Proposition 4.2.2, we have $f_X(0) = e^{-\lambda}$. Thus, in view of Proposition 2.1.1, for $n = 0$ we have

$$f_{N|X=0}(0) = \frac{f(0,0)}{f_X(0)} = \frac{e^{-\lambda}}{e^{-\lambda}} = 1.$$

In addition, for $n \in \mathcal{N}$, we have

$$f_{N|X=0}(n) = \frac{f(0,n)}{f_X(0)} = \frac{0}{e^{-\lambda}} = 0.$$

In summary,

$$f_{N|X=0}(n) \quad = \quad \begin{cases} 1 & \text{if } n = 0 \\ \\ 0 & \text{otherwise} \end{cases} \quad . \tag{4.12}$$

We note that $\sum_{n=1}^{\infty} f_{N|X=0}(n) = 1$ as desired.

Case 2: If $x > 0$, then by Propositions 2.1.1 and 4.2.2, we arrive at

$$\begin{aligned} f_{N|X=x}(n) &= \frac{f(x,n)}{f_X(x)} \\ &= \frac{\beta e^{-\beta x-\lambda}\lambda^n(1-e^{-\beta x})^{n-1}}{(n-1)!} \cdot \frac{1}{\lambda\beta e^{-\lambda e^{-\beta x}-\beta x}} \\ &= \frac{e^{-\lambda(1-e^{-\beta x})}[\lambda(1-e^{-\beta x})]^{n-1}}{(n-1)!}, n \in \mathcal{N}. \end{aligned} \tag{4.13}$$

Noting that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we see below that $f_{N|X=x}(n)$ summed over all possible values of n equals 1, as desired. We have

$$\sum_{n=1}^{\infty} f_{N|X=x}(n) = \sum_{n=1}^{\infty} \frac{e^{-\lambda(1-e^{-\beta x})}[\lambda(1-e^{-\beta x})]^{n-1}}{(n-1)!} = e^{-\lambda(1-e^{-\beta x})}e^{\lambda(1-e^{-\beta x})} = 1.$$

Having completed cases 1 and 2, we make the following proposition.

Proposition 4.4.1 *Let $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$. Then the conditional probabil-*

ity distribution of N given $X = x$ is given by $f_{N|X=x}(n)$ where,

$$f_{N|X=x}(n) \quad = \quad \begin{cases} 1 & \textit{if } n = x = 0 \\ \dfrac{e^{-\lambda(1-e^{-\beta x})}[\lambda(1-e^{-\beta x})]^{n-1}}{(n-1)!} & \textit{if } n \in \mathcal{N} \textit{ and } x > 0 \\ 0 & \textit{otherwise} \end{cases} \tag{4.14}$$

Remark Note that $N|X = x$ is a shifted Poisson random variable. More precisely, for $x > 0$, $N|X = x$ has the same distribution as $Z + 1$, where Z is a Poisson with parameter $\lambda(1 - e^{\beta x})$.

Chapter 5

Moments

In this chapter we find various representations for bivariate and univariate moments connected with the BMEP model. To begin, we obtain a general expression for $E[X^\eta N^\gamma]$, and then proceed to obtain various special cases for particular values of η and γ.

5.1 The general case $E[X^\eta N^\gamma]$

We recall in equation (3.4) that $(X, N) \stackrel{d}{=} (I\tilde{X}, I\tilde{N})$. This allows us to proceed as follows:

$$E[X^\eta N^\gamma] \quad = \quad E[(I\tilde{X})^\eta (I\tilde{N})^\gamma] = E[I^{\eta+\gamma}] \cdot E[\tilde{X}^\eta \tilde{N}^\gamma].$$

We shall assume throughout that $\eta, \gamma \geq 0$. Noting that for $\eta + \gamma > 0$ we have $E[I^{\eta+\gamma}] = 1 \cdot P(I = 0) + 0 \cdot P(I = 0) = 1 - e^{-\lambda}$, we obtain

$$E[X^\eta N^\gamma] \begin{cases} E[\tilde{X}^\eta \tilde{N}^\gamma] & \text{if } \eta = \gamma = 0 \\ (1 - e^{-\lambda}) \cdot E[\tilde{X}^\eta \tilde{N}^\gamma] & \text{otherwise} \end{cases} . \tag{5.1}$$

We next find $E[\tilde{X}^\eta \tilde{N}^\gamma]$.

$$\begin{aligned}
E[\tilde{X}^\eta \tilde{N}^\gamma] &\overset{(1)}{=} \sum_{n=1}^{\infty} \int_0^\infty x^\eta n^\gamma g(x, n)\, dx \\
&\overset{(2)}{=} \sum_{n=1}^{\infty} \int_0^\infty x^\eta n^\gamma \frac{e^{-\lambda}}{1 - e^{-\lambda}} \frac{(1 - e^{-\beta x})^{n-1}}{(n-1)!} \beta e^{-\beta x} \lambda^n \, dx \\
&\overset{(3)}{=} \frac{\beta e^{-\lambda}}{1 - e^{-\lambda}} \sum_{n=1}^{\infty} \frac{n^\gamma \lambda^n}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \int_0^\infty x^\eta e^{-\beta x(k+1)} \, dx \\
&\overset{(4)}{=} \frac{\beta e^{-\lambda}}{1 - e^{-\lambda}} \sum_{n=1}^{\infty} \frac{n^\gamma \lambda^n}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{1}{[\beta(k+1)]^\eta} \\
&\quad \cdot \int_0^\infty [\beta(k+1)x]^{(\eta+1)-1} e^{-\beta x(k+1)} \, dx \\
&\overset{(5)}{=} \frac{\beta e^{-\lambda}}{1 - e^{-\lambda}} \sum_{n=1}^{\infty} \frac{n^\gamma \lambda^n}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{\Gamma(\eta+1)}{[\beta(k+1)]^{\eta+1}} \\
&\overset{(6)}{=} \frac{\beta e^{-\lambda}}{1 - e^{-\lambda}} \frac{\Gamma(\eta+1)}{\beta^{\eta+1}} \sum_{n=1}^{\infty} \frac{n^\gamma \lambda^n}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \frac{1}{(k+1)^{\eta+1}} \\
&\overset{(7)}{=} \frac{e^{-\lambda}}{1 - e^{-\lambda}} \frac{\Gamma(\eta+1)}{\beta^\eta} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{\eta+1}} \sum_{n=k+1}^{\infty} \frac{n^\gamma \lambda^n}{(n-1)!} \binom{n-1}{k} \\
&\overset{(8)}{=} \frac{e^{-\lambda}}{1 - e^{-\lambda}} \frac{\Gamma(\eta+1)}{\beta^\eta} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{\eta+1}} \frac{1}{k!} \sum_{n=k+1}^{\infty} \frac{n^\gamma \lambda^n}{(n-1)!} \frac{(n-1)!}{(n-k-1)!} \\
&\overset{(9)}{=} \frac{e^{-\lambda}}{1 - e^{-\lambda}} \frac{\Gamma(\eta+1)}{\beta^\eta} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^{\eta+1}} \sum_{n=0}^{\infty} \frac{(n+k+1)^\gamma \lambda^{n+k+1}}{n!},
\end{aligned} \tag{5.2}$$

where in line (5) we introduce the gamma function,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{5.3}$$

It will be useful to note that for a positive integer n, $\Gamma(n) = (n-1)!$.

In particular, when $\eta = \gamma = 0$, we expect $E[\tilde{X}^\eta \tilde{N}^\gamma]$ to be 1. Indeed, we have

$$\begin{aligned} E[\tilde{X}^0 \tilde{N}^0] &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\Gamma(1)}{\beta^0} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^{0+1}} \sum_{n=0}^{\infty} \frac{(n+k+1)^0 \lambda^{n+k+1}}{n!} \\ &= \frac{e^{-\lambda}}{1-e^{-\lambda}} (-1) \sum_{k=0}^{\infty} \frac{(-\lambda)^{k+1}}{(k+1)!} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \\ &= \frac{e^{-\lambda}}{1-e^{-\lambda}} (-1)(e^{-\lambda} - 1) e^{\lambda} = 1. \end{aligned} \tag{5.4}$$

Using equations (5.1), (5.2), and (5.4), we now state an expression for $E[X^\eta N^\gamma]$ in the proposition below.

Proposition 5.1.1 *Let* $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$*. Then for any non-negative* η *and* γ *we have,*

$$E[X^\eta N^\gamma] = \begin{cases} 1 & \text{if } \eta = \gamma = 0 \\ \dfrac{\Gamma(\eta+1)}{e^{\lambda}\beta^\eta} \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^{\eta+1}} \sum_{n=0}^{\infty} \frac{(n+k+1)^\gamma \lambda^{n+k+1}}{n!} & \text{otherwise.} \end{cases} \tag{5.5}$$

5.2 Special cases for moments of (X, N)

We can derive expressions for univariate moments by letting γ or η equal 0. The first moment in the univariate case represents the mean. This, in addition to the second moment, will reveal the variance. In this section we will find the first and second moments for X and N. Additionally, we will find the special case of $E[XN]$, which will be useful in deriving covariance of X and N.

A special case of $E[X^\eta N^\gamma]$ where $\gamma = 0$ gives us $E[X^\eta]$. By equation (5.5), for $\eta \neq 0$,

$$\begin{aligned} E[X^\eta] &= e^{-\lambda}\frac{\Gamma(\eta+1)}{\beta^\eta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)^{\eta+1}}\sum_{n=0}^{\infty}\frac{(n+k+1)^0\lambda^{n+k+1}}{n!} \\ &= e^{-\lambda}\frac{\Gamma(\eta+1)}{\beta^\eta}\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{k+1}}{k!(k+1)^{\eta+1}}e^{\lambda} \\ &= \frac{\Gamma(\eta+1)}{\beta^\eta}\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{k+1}}{k!(k+1)^{\eta+1}}. \end{aligned}$$

Therefore,

$$E[X^\eta] = \begin{cases} 1 & \text{if } \eta = 0 \\ \dfrac{\Gamma(\eta+1)}{\beta^\eta}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{k+1}}{k!(k+1)^{\eta+1}} & \text{otherwise} \end{cases}. \tag{5.6}$$

Further, when we let $\gamma = 0$ and $\eta = 1$ we have $E[X]$. First we consider the function

$$L(\lambda) = \sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n\cdot n!}. \tag{5.7}$$

Then,

$$L'(\lambda) = \sum_{n=1}^{\infty}\frac{n(-\lambda)^{n-1}(-1)}{n\cdot n!} = \frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!} = \frac{e^{-\lambda}-1}{\lambda}.$$

Thus, we have

$$L(\lambda) = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n \cdot n!} = \int_0^{\lambda} \frac{e^{-t} - 1}{t}\, dt. \tag{5.8}$$

Next, by equation (5.6),

$$\begin{aligned} E[X] &= \frac{\Gamma(2)}{\beta} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{k+1}}{k!(k+1)^2} \\ &= \frac{1}{\beta} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{k+1}}{(k+1)!(k+1)} \\ &= -\frac{1}{\beta} \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n \cdot n!} \\ &= -\frac{1}{\beta} L(\lambda) \\ &= \frac{1}{\beta} \int_0^{\lambda} \frac{1 - e^{-t}}{t}\, dt. \end{aligned} \tag{5.9}$$

Also via equation (5.6), this time with $\eta = 2$, we find $E[X^2]$ as follows:

$$\begin{aligned} E[X^2] &= \frac{\Gamma(3)}{\beta^2} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{k+1}}{k!(k+1)^3} \\ &= \frac{2}{\beta^2} \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{k+1}}{k!(k+1)^3}. \end{aligned} \tag{5.10}$$

This can be written in terms of certain definite integrals. Indeed, observe that $E[X^2] = -\frac{2}{\beta^2} \cdot H(\lambda)$, where

$$H(\lambda) = \sum_{k=1}^{\infty} \frac{(-1)^k \lambda^k}{k!\, k^2}.$$

Note that

$$H'(\lambda) = \sum_{k=1}^{\infty} \frac{(-1)^k k \lambda^{k-1}}{k!\, k^2} = \frac{1}{\lambda} L(\lambda),$$

so that

$$H(\lambda) = \int_0^{\lambda} \frac{1}{x} L(x)\, dx.$$

In view of (5.8), we have

$$H(\lambda) = \int_0^{\lambda} \frac{1}{x} \int_0^{x} \frac{e^{-t} - 1}{t}\, dt\, dx,$$

or, by changing the order of integration, we get

$$H(\lambda) = \int_0^{\lambda} \frac{e^{-t} - 1}{t} \int_t^{\lambda} \frac{1}{x}\, dx\, dt,$$

which becomes

$$H(\lambda) = \ln \lambda \int_0^{\lambda} \frac{e^{-t} - 1}{t}\, dt - \int_0^{\lambda} \frac{e^{-t} - 1}{t} \ln t\, dt.$$

We finally obtain

$$E[X^2] = -\frac{2}{\beta^2} \{L(\lambda) \ln \lambda - M(\lambda)\}, \tag{5.11}$$

where $L(\lambda)$ is given by (5.8) while

$$M(\lambda) = \int_0^\lambda \frac{e^{-t}-1}{t} \ln t \, dt. \tag{5.12}$$

Another special case of $E[X^\eta N^\gamma]$ with $\gamma = 1, \eta = 1$ gives us $E[XN]$. First, it will be useful to note,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n+k+1)\lambda^{n+k+1}}{n!} &\overset{(1)}{=} \sum_{n=0}^{\infty} \frac{(k+1)\lambda^{n+k+1}}{n!} + \sum_{n=0}^{\infty} \frac{n \cdot \lambda^{n+k+1}}{n!} \\
&\overset{(2)}{=} (k+1)\lambda^{k+1}e^{\lambda} + \lambda^{k+1}e^{\lambda} \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} n \\
&\overset{(3)}{=} (k+1)\lambda^{k+1}e^{\lambda} + \lambda^{k+1}e^{\lambda}E[N] \\
&\overset{(4)}{=} (k+1)\lambda^{k+1}e^{\lambda} + \lambda^{k+1}e^{\lambda}\lambda \\
&\overset{(5)}{=} e^{\lambda}\lambda^{k+1}(k+1+\lambda),
\end{aligned} \tag{5.13}$$

where N in line (3) had Poisson distribution with mean λ. Thus, by equation (5.5), we have

$$\begin{aligned}
E[XN] &= e^{-\lambda}\frac{\Gamma(2)}{\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^2} \sum_{n=0}^{\infty} \frac{(n+k+1)\lambda^{n+k+1}}{n!} \\
&= e^{-\lambda}\frac{1}{\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^2} e^{\lambda}\lambda^{k+1}(k+1+\lambda) \\
&= \frac{1}{\beta}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^2}\lambda^{k+1}(k+1) + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^2}\lambda^{k+1}(\lambda) \right) \\
&= \frac{1}{\beta}\left(-\sum_{k=0}^{\infty} \frac{(-\lambda)^{k+1}}{(k+1)!} - \lambda \sum_{k=0}^{\infty} \frac{(-\lambda)^{k+1}}{(k+1)(k+1)!} \right) \\
&= \frac{1}{\beta}\left(-(e^{-\lambda}-1) - \lambda L(\lambda) \right)
\end{aligned}$$

$$= \frac{1}{\beta}\left(1 - e^{-\lambda} + \lambda \int_0^{\lambda} \frac{1-e^{-t}}{t}\,dt\right). \tag{5.14}$$

Another special case of $E[X^\eta N^\gamma]$ is when $\eta = 0$ and we are left with $E[N^\gamma]$. By equation (5.5), for $\gamma \neq 0$,

$$E[N^\gamma] = e^{-\lambda}\frac{\Gamma(1)}{\beta^0}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\sum_{n=0}^{\infty}\frac{(n+k+1)^\gamma \lambda^{n+k+1}}{n!}.$$

Thus,

$$E[N^\gamma] = \begin{cases} 1 & \text{if } \gamma = 0 \\ e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\sum_{n=0}^{\infty}\frac{(n+k+1)^\gamma \lambda^{n+k+1}}{n!} & \text{otherwise} \end{cases}. \tag{5.15}$$

The special case of $E[N^\gamma]$ when $\gamma = 2$ yields

$$\begin{aligned}
E[N^2] &= e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\sum_{n=0}^{\infty}\frac{(n+k+1)^2\lambda^{n+k+1}}{n!} \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\lambda^{k+1}\left[2(k+1)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}n + (k+1)^2\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} + \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}n^2\right] \\
&= -e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\cdot \\
&\quad \left[2(k+1)\left(0 + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}n\right) + (k+1)^2 e^{\lambda} + \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(n(n-1)+n)\right] \\
&= -e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\cdot \\
&\quad \left[2(k+1)\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}n + (k+1)^2 e^{\lambda} + \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}n(n-1) + \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}n\right] \\
&= -e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\cdot
\end{aligned}$$

$$
\begin{aligned}
& \left[2(k+1)\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}n + (k+1)^2e^{\lambda} + \sum_{n=2}^{\infty}\frac{\lambda^n}{n!}n(n-1) + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}n\right] \\
= \;& -e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\cdot \\
& \left[2(k+1)\lambda\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!} + (k+1)^2e^{\lambda} + \lambda^2\sum_{n=2}^{\infty}\frac{\lambda^{n-2}}{(n-2)!} + \lambda\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}\right] \\
= \;& -e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\left[2(k+1)\lambda e^{\lambda} + (k+1)^2e^{\lambda} + \lambda^2e^{\lambda} + \lambda e^{\lambda}\right] \\
= \;& -\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\left[2(k+1)\lambda + (k+1)^2 + \lambda^2 + \lambda\right] \\
= \;& 2\lambda^2\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} - \sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{k!}(k+1) - (\lambda^2+\lambda)\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!} \\
= \;& 2\lambda^2\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} - \sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{k!}k - \sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{k!} - (\lambda^2+\lambda)\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!} \\
= \;& 2\lambda^2e^{-\lambda} - \sum_{k=1}^{\infty}\frac{(-\lambda)^{k+1}}{k!}k + \lambda\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} - (\lambda^2+\lambda)(e^{-\lambda}-1) \\
= \;& 2\lambda^2e^{-\lambda} - \lambda^2\sum_{k=1}^{\infty}\frac{(-\lambda)^{k-1}}{(k-1)!} + \lambda e^{-\lambda} - \lambda^2e^{-\lambda} + \lambda^2 - \lambda e^{-\lambda} + \lambda \\
= \;& 2\lambda^2e^{-\lambda} - \lambda^2e^{-\lambda} + \lambda e^{-\lambda} - \lambda^2e^{-\lambda} + \lambda^2 - \lambda e^{-\lambda} + \lambda \\
= \;& \lambda^2 + \lambda, \qquad (5.16)
\end{aligned}
$$

which is what we expect, since $N \sim \mathcal{POISSON}(\lambda) \Rightarrow E[N^2] = \lambda + \lambda^2$.

When $\gamma = 1$, equation (5.15) gives us

$$
\begin{aligned}
E[N] \;&=\; e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}\sum_{n=0}^{\infty}\frac{(n+k+1)\lambda^{n+k+1}}{n!} \\
&=\; -e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\left((k+1)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} + \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}n\right) \\
&=\; -e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\left((k+1)e^{\lambda} + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}n\right) \\
&=\; -e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\left((k+1)e^{\lambda} + \lambda\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\left((k+1)e^{\lambda}+\lambda e^{\lambda}\right)\\
&= -\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}(k+1+\lambda)\\
&= -\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}(k+1)-\lambda\sum_{k=0}^{\infty}\frac{(-\lambda)^{k+1}}{(k+1)!}\\
&= \lambda\sum_{k=0}^{\infty}\frac{(-\lambda)^{k}}{k!}-\lambda(e^{-\lambda}-1)\\
&= \lambda e^{-\lambda}-\lambda e^{-\lambda}+\lambda\\
&= \lambda, \qquad (5.17)
\end{aligned}
$$

as expected since $N \sim \mathcal{POISSON}(\lambda)$. Having demonstrated that $E[N^2] = \lambda^2 + \lambda$ and $E[N] = \lambda$, we have given validity to equation (5.15). However, since $N \sim \mathcal{POISSON}(\lambda)$, the following equation can be used to determine $E[N^\gamma]$, and is much more concise.

$$E[N^\gamma] = \sum_{n=0}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!}n^\gamma. \qquad (5.18)$$

We now have completed several special cases and are able to provide the covariance matrix in the proposition below.

$$L(\lambda) = \sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n\cdot n!}.$$

Proposition 5.2.1 *If* $(X, N) \sim \mathcal{BMEP}(\beta, \lambda)$*, then* $E[X] = \dfrac{-L(\lambda)}{\beta}$*,* $E[N] = \lambda$

and the covariance matrix of (X, N) *is*

$$\begin{pmatrix} \dfrac{2M(\lambda) - 2L(\lambda)\ln\lambda - L^2(\lambda)}{\beta^2} & \dfrac{1 - e^{-\lambda}}{\beta} \\ \dfrac{1 - e^{-\lambda}}{\beta} & \lambda \end{pmatrix}, \tag{5.19}$$

where $L(\lambda) = \int_0^\lambda \frac{e^{-t} - 1}{t}\,dt,$ *and* $M(\lambda) = \int_0^\lambda \frac{e^{-t} - 1}{t}\ln t\,dt.$

Chapter 6

Simulation and Estimation

In practice the values of the parameters of the BMEP distribution are almost never certain and have to be estimated based on the sample data. That is why it is valuable to develop methods of estimation of our parameters, β and λ. In order to test the success of our estimation methods it is useful to simulate data with known β and λ, and then estimate β and λ using the simulated data. In this chapter we develop methods of estimation as well as a technique to simulate BMEP data.

6.1 Simulation

One method to simulate the random variable X of the BMEP distribution is to strictly follow the definition of X by generating N random $\mathcal{EXP}(\beta)$ variables and then set X to the maximum of these exponential variables. A second way to simulate X is to use the alternative representation discussed in Chapter 3 by generating N random exponential variables and summing the $\frac{E_j}{j}$ terms.

A third, more efficient, way to simulate X is to generate one random variable rather than N random variables. We note that a continuous random variable with c.d.f. F can be generated by first generating U, a random variable from the standard uniform distribution, and next computing $F^{-1}(U)$. This is because $F^{-1}(U)$ has distribution F (See [10]). We can use this to simulate the random variable X, however X is dependent on N. Therefore N needs to be generated first. Random discrete variables can be simulated by generating U and setting the discrete variable to N where, if $P(N = n_j) = p_j$ then,

$$N = \begin{cases} n_0 & \text{if } U < p_0 \\ n_1 & \text{if } p_0 \leq U < p_0 + p_1 \\ \vdots & \\ n_j & \text{if } \sum_{i=1}^{j-1} p_i \leq U < \sum_{i=1}^{j} p_i \\ \vdots & \end{cases} . \tag{6.1}$$

Recall from Proposition 4.3.2 that $F_{X|N=n}(x) = [1 - e^{-\beta x}]^n$ if $n \in \mathcal{N}$ and $x \geq 0$. So

$$F^{-1}_{X|N=n}(U) = -\frac{\ln(1 - U^{1/n})}{\beta} \text{ if } n \in \mathcal{N} \text{ and } U \in (0, 1). \tag{6.2}$$

Thus, to simulate random data from the BMEP distribution with parameters β and λ we follow the steps below.

Step 1: Generate a standard uniform random variable U and determine N according to equation (6.1) and the p.m.f. of N, $P(N = n) = \frac{e^{-\lambda}\lambda^n}{n!}$, $n \in \mathcal{N}$.

Step 2: If $N = 0$ then set X to zero since the maximum of zero exponentials is zero. Otherwise, proceed to steps 3 and 4.

Step 3: Regenerate the standard uniform random variable U.

Step 4: Set X equal to $F^{-1}(U)$ where $F^{-1}(U)$ is defined in equation (6.2).

Appendix I contains a program written in C++ which generates random data from the BMEP distribution.

6.2 Moment Estimators

Moment estimators are found by setting sample moments equal to the equation for the population moments and solving for the parameters which are to be estimated. Below are the equations for the first moments of X and N we will use to estimate the two parameters of the BMEP distribution, β and λ. By Proposition 5.2.1,

$$\bar{X} = \frac{1}{\beta}\int_0^\lambda \frac{1 - e^{-t}}{t}\, dt \text{ and } \bar{N} \;=\; \lambda,$$

where $\bar{X} = \frac{1}{m}\sum_{j=1}^{m} X_j$ and $\bar{N} = \frac{1}{m}\sum_{j=1}^{m} N_j$. Solving this system for β and λ leads to the proposition below.

Proposition 6.2.1 *Suppose* (X_1, N_1), (X_2, N_2), ..., (X_m, N_m) *form a random sample where* $(X_i, N_i) \sim \mathcal{BMEP}(\beta, \lambda)$ *and there exists at least one* N_i *such that*

$N_i > 0$. *Then the moment estimators,* $\check{\beta}$ *and* $\check{\lambda}$ *of* β *and* λ*, respectively, are*

$$\check{\beta} = \frac{1}{\bar{X}} \int_0^{\bar{N}} \frac{1 - e^{-t}}{t} dt \text{ and } \check{\lambda} \quad = \quad \bar{N}. \tag{6.3}$$

We note that by the method of moments that estimators for β and λ can be explicitly found.

6.3 Maximum Likelihood Estimators

Suppose (X_1, N_1), (X_2, N_2), ..., (X_m, N_m) form a random sample from the BMEP distribution for which the parameters β and λ are unknown, and that there exists at least one N_i such that $N_i > 0$. In this section we find the maximum likelihood estimators (M.L.E.s) for β and λ.

Recall from Proposition 2.1.1 that,

$$f(x, n) = \begin{cases} e^{-\lambda} & \text{if } x = n = 0 \\ \beta e^{-\beta x - \lambda} \lambda^n \dfrac{(1 - e^{-\beta x})^{n-1}}{(n-1)!} & x > 0, \ n \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases} .$$

Since $f(x, n)$ varies when $x = 0$ versus when $x > 0$ we define

$$k(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z = 0 \end{cases} . \tag{6.4}$$

We also let

$$A = \sum_{j=1}^{m} k(N_j), \tag{6.5}$$

which represents the number of the values in the random sample that have the N component greater than 0. Note that $0 \leq A \leq m$. Finally we let $C =$ the set of all j's in the range $1, 2, 3, ..., m$ such that $N_j > 0$. Therefore $A = |C| =$ the number of elements in C.

We begin by finding, $f_n((\vec{x}, \vec{n})|(\beta, \lambda))$, the p.d.f. associated with our random sample, and then proceed to find the λ and β that maximize this p.d.f.. By independence, we have

$$\begin{aligned} f_n\left((\vec{x}, \vec{n})|(\beta, \lambda)\right) &= \prod_{j=1}^{m} f(X_j, N_j) \\ &= \prod_{i=1}^{m-A} e^{-\lambda} \prod_{j \in C} \beta e^{-\beta X_j - \lambda} \lambda^{N_j} \frac{(1 - e^{-\beta X_j})^{N_j - 1}}{(N_j - 1)!} \\ &= \frac{e^{-\lambda m} \beta^A \lambda^{\sum_{j \in C} N_j} e^{-\beta \sum_{j \in C} X_j} \prod_{j \in C} (1 - e^{-\beta X_j})^{N_j - 1}}{\prod_{j \in C} (N_j - 1)!}. \end{aligned} \tag{6.6}$$

Notice that $\sum_{j \in C} N_j = m\bar{N}$, since for $j \notin C$, $N_j = 0$. Similarly, $\sum_{j \in C} X_j = m\bar{X}$. We note also there are no λ or β in the denominator in (6.6). So, to maximize $f_n((\vec{x}, \vec{n})|(\beta, \lambda))$, we need to find the $\lambda > 0$ and $\beta > 0$ that maximize the numerator in (6.6). We let $k(\lambda, \beta)$ represent the numerator. Therefore,

$$k(\lambda, \beta) = e^{-\lambda m} \lambda^{m\bar{N}} \cdot \beta^A e^{-\beta m \bar{X}} \prod_{j \in C} (1 - e^{-\beta X_j})^{N_j - 1}$$

$$= \quad v(\lambda) \cdot u(\beta),$$

where

$$v(\lambda) \quad = \quad e^{-\lambda m} \lambda^{m\bar{N}} \tag{6.7}$$

and

$$u(\beta) \quad = \quad \beta^A e^{-\beta m \bar{X}} \prod_{j \in C} (1 - e^{-\beta X_j})^{N_j - 1}. \tag{6.8}$$

Having separated $k(\lambda, \beta)$ into the product of a function of λ and β will be useful in finding each M.L.E. of our parameters.

6.3.1 Maximum Likelihood Estimator of λ

To find the M.L.E. for λ we maximize $\ln v(\lambda)$. First we see there is an inflection point at $\lambda = \bar{N}$:

$$\ln v(\lambda) = -\lambda m + m\bar{N} \ln \lambda \Longrightarrow \frac{d}{d\lambda} \ln v(\lambda) = -m + \frac{m\bar{N}}{\lambda} = 0 \Longrightarrow \lambda = \bar{N} \tag{6.9}$$

Next we see $\lambda = \bar{N}$ is indeed the maximum of $\ln v(\lambda)$, since

$$\frac{d}{d\lambda} \ln v(\lambda) > 0 \text{ for } \lambda < \bar{N} \text{ and } \frac{d}{d\lambda} \ln v(\lambda) < 0 \text{ for } \lambda > \bar{N}.$$

We conclude the M.L.E. for λ is $\bar{N}$, and denote this as

$$\hat{\lambda} = \bar{N}. \tag{6.10}$$

6.3.2 Maximum Likelihood Estimator of β

To find the M.L.E. of β we wish to maximize $w(\beta) = \ln u(\beta)$. However, this cannot be done explicitly. First we prove $\hat{\beta}$ exists by showing that $w(\beta)$ has a maximum. We begin by showing the limit to the left and right of $w(\beta)$ are both $-\infty$.

$$\begin{aligned}
\lim_{\beta\to 0} w(\beta) &= \lim_{\beta\to 0}\left\{A\ln\beta - \beta m\bar{X} + \sum_{j\in C}\ln[(1-e^{-\beta X_j})^{N_j-1}]\right\} \\
&= -\infty - 0 - \infty = -\infty.
\end{aligned}$$

$$\begin{aligned}
\lim_{\beta\to\infty} w(\beta) &\overset{(1)}{=} \lim_{\beta\to\infty}\left\{A\ln\beta - \beta m\bar{X} + \sum_{j\in C}\ln[(1-e^{-\beta X_j})^{N_j-1}]\right\} \\
&\overset{(2)}{=} \lim_{\beta\to\infty}\left\{A\ln\beta - \beta m\bar{X}\right\} + 0 \\
&\overset{(3)}{=} \lim_{\beta\to\infty}\beta\left(\frac{A\ln\beta}{\beta} - m\bar{X}\right) \\
&\overset{(4)}{=} \lim_{\beta\to\infty}\beta\cdot\lim_{\beta\to\infty}\left(\frac{A\ln\beta}{\beta} - m\bar{X}\right) \\
&\overset{(5)}{=} \lim_{\beta\to\infty}\beta\cdot\lim_{\beta\to\infty}\left(\frac{A}{\beta} - m\bar{X}\right) \\
&\overset{(6)}{=} \infty\cdot(0 - m\bar{X}) = -\infty,
\end{aligned}$$

where in step (5) L'Hospital's Rule was used. Next we show that $w(\beta)$ is concave down everywhere. Indeed, the first derivative of w is found to be

$$\begin{aligned} \frac{d}{d\beta} w(\beta) &= \frac{A}{\beta} - m\bar{X} + \sum_{j \in C} (N_j - 1) X_j \frac{e^{-\beta X_j}}{1 - e^{-\beta X_j}} \\ &= \frac{A}{\beta} - m\bar{X} + \sum_{j \in C} (N_j - 1) X_j \frac{1}{e^{\beta X_j} - 1}, \end{aligned} \quad (6.11)$$

so that the second derivative becomes

$$\begin{aligned} \frac{d^2}{d\beta^2} w(\beta) &= -\frac{A}{\beta^2} + \sum_{j \in C} (1 - N_j) X_j{}^2 \frac{e^{\beta X_j}}{(e^{\beta X_j} - 1)^2} \quad (6.12) \\ &< 0, \end{aligned}$$

since $(1 - N_j) \geq 0$ for $j \in C$. Therefore, since $w(\beta)$ has limits to the left and right approaching $-\infty$ and it is concave down everywhere, $w(\beta)$ has a unique maximum. We summarize our findings in the following result.

Approximating $\hat{\beta}$ can be accomplished through the Newton Method. Completing iterations of

$$\beta_{n+1} = \beta_n - \frac{\frac{d}{d\beta} w(\beta_n)}{\frac{d^2}{d\beta^2} w(\beta_n)} \quad (6.13)$$

will approximate the solution of $\frac{d}{d\beta} w(\beta) = 0$, which represents the M.L.E. of β. Appendix II contains a computer program written in C++ which calculates iterations of (6.13) given m sample points (X_j, N_j) to find $\hat{\beta}$.

We now state the M.L.E.s of β and λ in the preposition below.

Proposition 6.3.1 *Suppose* (X_1, N_1), (X_2, N_2), ..., (X_m, N_m) *form a random sample where* $(X_i, N_i) \sim \mathcal{BMEP}(\beta, \lambda)$ *and there exists at least one* N_i *such that* $N_i > 0$. *Then the maximum likelihood estimators,* $\hat{\beta}$ *and* $\hat{\lambda}$ *of* β *and* λ, *respectively, are*

$$\hat{\beta} = \beta \text{ such that } \tfrac{A}{\beta} - m\bar{X} + \textstyle\sum_{j \in C}(N_j - 1)X_j \tfrac{1}{e^{\beta X_j} - 1} = 0 \tag{6.14}$$

and

$$\hat{\lambda} = \bar{N}. \tag{6.15}$$

Where C = *the set of all* j*s in the range* $1, 2, 3, ..., m$ *such that* $N_j > 0$, *and* $A = |C|$ = *the number of elements in* C.

6.4 Results

As mentioned above, the program in Appendix II uses the Newton Method to find $\hat{\beta}$, which will approximate the solution to (6.14). The program also calculates $\check{\beta}$ and $\hat{\lambda} = \check{\lambda}$ using equations (6.3) and (6.15). To provide data for, as well as test, the program in Appendix II, a program was written (see Appendix I) which generates n random vectors of the BMEP distribution according to the steps outlined in Section 6.1. That is, we choose values for β and λ, generate sample data from the BMEP distribution, and then use this sample data to estimate the values of β and λ to test our derivations for $\check{\beta}$, $\hat{\beta}$, and $\hat{\lambda} = \check{\lambda}$. Some results of the two programs are listed in Table 6.1 below. The sample sizes used for the illustration were 50, 100, 500, 1,000, 5,000, and 10,000. For each sample size parameter pairs were chosen to be $\beta = 0.5$ with $\lambda = 0.5$, $\beta = 0.5$ with $\lambda = 1$, $\beta = 0.5$ with $\lambda = 5$, $\beta = 1$ with $\lambda = 0.5$, $\beta = 1$ with $\lambda = 1$, $\beta = 1$ with $\lambda = 5$, $\beta = 5$ with $\lambda = 0.5$, $\beta = 5$ with $\lambda = 1$, and $\beta = 5$ with $\lambda = 5$.

In Table 6.1, we see as n increases the variance of the estimators decreases as expected, and the mean estimators become closer to the true parameters. Since the method of moment estimator for λ, $(\check{\lambda})$ is equal to the M.L.E. for λ, $(\hat{\lambda})$, the estimators are equally successful. In regard to β, the method of moments estimators means and M.L.E. means are consistently approximately equal, and thus equally close to the theoretical means. However, an advantage of the M.L.E. can been seen in the mean squared error entries. We see that the maximum likelihood estimator for β, $(\hat{\beta})$ is less volatile than the method of moments estimator for β, $(\check{\beta})$. This is true in every case presented in the table, which is interesting since the bias of an M.L.E. can be large in smaller samples. On the other hand, a small variance in the M.L.E. is expected in the larger samples since the M.L.E. is an asymptotically unbiased estimator.

n	$\breve{\beta}$	$\hat{\beta}$	$\breve{\lambda} = \hat{\lambda}$	$\breve{\beta}$	$\hat{\beta}$	$\breve{\lambda} = \hat{\lambda}$	$\breve{\beta}$	$\hat{\beta}$	$\breve{\lambda} = \hat{\lambda}$
	$\beta = 0.5, \lambda = 0.5$			$\beta = 0.5, \lambda = 1$			$\beta = 0.5, \lambda = 5$		
50	0.521690	0.521479	0.500344	0.512084	0.512013	0.999960	0.502601	0.502317	5.001254
	0.014204	0.013469	0.010150	0.006731	0.006162	0.020346	0.001540	0.001439	0.099671
100	0.509753	0.509923	0.500292	0.505476	0.505592	1.001680	0.501585	0.501378	4.999193
	0.006087	0.005808	0.005098	0.003044	0.002831	0.009811	0.000758	0.000713	0.049830
500	0.501668	0.501749	0.500135	0.501358	0.501367	0.999407	0.500261	0.500199	4.999949
	0.001133	0.001070	0.001028	0.000585	0.000535	0.001961	0.000152	0.000142	0.010150
1,000	0.501115	0.501137	0.499894	0.500768	0.500755	1.000142	0.500138	0.500107	4.999665
	0.000548	0.000521	0.000506	0.000304	0.000278	0.000988	0.000074	0.000070	0.005052
5,000	0.500115	0.500157	0.500061	0.500187	0.500209	0.999954	0.500064	0.500072	5.000253
	0.000112	0.000105	0.000098	0.000060	0.000057	0.000213	0.000015	0.000014	0.000987
10,000	0.500076	0.500081	0.499962	0.500095	0.500109	0.999905	0.499984	0.499984	5.000152
	0.000056	0.000053	0.000049	0.000029	0.000027	0.000096	0.000008	0.000007	0.000508
	$\beta = 1, \lambda = 0.5$			$\beta = 1, \lambda = 1$			$\beta = 1, \lambda = 5$		
50	1.046488	1.046851	0.498360	1.020425	1.020306	0.999354	1.005790	1.004589	5.004230
	0.059819	0.057032	0.009793	0.026836	0.024751	0.019722	0.006269	0.005868	0.100198
100	1.019168	1.020045	0.500768	1.009600	1.009731	1.000323	1.003609	1.003207	4.995724
	0.024563	0.023495	0.004917	0.012580	0.011619	0.009871	0.003043	0.002855	0.049148
500	1.002693	1.002715	0.499665	1.002381	1.002448	0.999700	1.001000	1.000953	4.999688
	0.004543	0.004325	0.001013	0.002377	0.002175	0.002006	0.000599	0.000563	0.010118
1,000	1.002555	1.002521	0.499720	1.001225	1.001334	0.999651	1.000500	1.000460	4.999166
	0.002214	0.002119	0.000509	0.001189	0.001109	0.000994	0.000305	0.000286	0.005079
5,000	1.000486	1.000432	0.499952	1.000481	1.000473	0.999839	1.000064	1.000060	5.000242
	0.000439	0.000420	0.000100	0.000233	0.000216	0.000201	0.000059	0.000056	0.001015
10,000	1.000247	1.000232	0.500003	1.000188	1.000180	0.999817	0.999932	0.999947	5.000028
	0.000219	0.000205	0.000048	0.000122	0.000114	0.000106	0.000031	0.000029	0.000499
	$\beta = 5, \lambda = 0.5$			$\beta = 5, \lambda = 1$			$\beta = 5, \lambda = 5$		
50	5.193465	5.198325	0.501436	5.106850	5.105734	1.001156	5.034111	5.027455	5.003208
	1.385641	1.318273	0.010144	0.675965	0.624088	0.019881	0.154776	0.143213	0.101965
100	5.108695	5.108118	0.499246	5.055750	5.055452	1.000886	5.015890	5.013152	4.999424
	0.621071	0.591093	0.005029	0.309984	0.285638	0.009925	0.073144	0.068861	0.049726
500	5.023684	5.024920	0.499865	5.011401	5.011646	1.000413	5.002543	5.002090	5.000172
	0.110548	0.104959	0.000972	0.060590	0.055929	0.001962	0.015408	0.014262	0.009912
1,000	5.007123	5.007701	0.499998	5.005585	5.005750	1.000064	5.001554	5.000902	4.999356
	0.055109	0.051851	0.000499	0.029858	0.027593	0.000988	0.007592	0.007043	0.005028
5,000	5.003261	5.003096	0.499991	5.001090	5.000851	0.999950	5.000554	5.000481	5.000314
	0.011142	0.010479	0.000099	0.005839	0.005387	0.000198	0.001483	0.001394	0.000980
10,000	5.001876	5.001891	0.499887	5.001415	5.001433	0.999969	5.000500	5.000490	4.999453
	0.005461	0.005148	0.000049	0.003047	0.002872	0.000105	0.000720	0.000671	0.000477

Table 6.1: Mean (top entry) and mean squared error (bottom entry) of the method of moments and the maximum likelihood estimators of the BMEP parameters β and λ based on 10,000 simulations.

Appendix I

```
/*
This program simulates random data from the BMEP distribution.
The inputs the program will prompt for are beta, lambda, and
the number of data points desired.

The program needs to be compiled with a C++ compiler and the
results will be written to a text file named "random data.txt."

Example:

After compiling the C++ code and following the prompts
with 5 as the input for beta, 10 as the input for lambda, and
100 as the input for the number of data points, a sample
output results file shows,

100 data points.
          X                 N
0.34747472536598017       3
0.33569883513526738       15
. . .
0.54027051818993688       9,

which is a list of 100 sample vectors from the BMEP
distribution with beta = 5, lambda = 10.
*/
```

```
#include <iostream>
#include <fstream>
#include <cstdlib> // for rand()
#include <ctime>
#include <cmath>
using namespace std;

void main()
    {
    ofstream fout("random data.txt");
    srand( time(0) ); // seed random number generator

    double num_points;
    double lambda;
    double beta;
    cout << "Enter beta ( beta>0 ) : ";
    cin  >> beta;
    cout << "Enter lambda ( lambda>0 ): ";
    cin  >> lambda;
    cout << "How many data points? (1 - 10,000) ";
    cin  >> num_points;

    int N;
    double X;
    fout << num_points << " data points." << endl;
    fout << "\t X \t\t N "<< endl;
```

```
for (int i=0; i<num_points; i++)
   {


   // begin Step 1

   // initialize standard uniform R.V.
   double U = 0;

   // generate 20 decimal places for continuous random
   // variable by choosing a # 0-9 for each decimal place
   double power_of_10 = 1;
   for (int j=0; j<20; j++)
      {
      power_of_10 *= 10;
      U += (double) ( rand() % 10 ) / power_of_10;
      }

   // apply equation (6.1)
   double cumulative = 0;
   int done = 0;
   for (int n = 0; done!=1; n++)
      {
      // p = e^(-lambda) lambda^n / n!
      double p = exp(-lambda);
      for (double denom = n; denom>=1; denom--)
         p = p * lambda / denom;
      cumulative += p;
      if (U < cumulative)
         {
         N = n;
         done = 1; // ends loop
         }
      }
```

```
    // begin Step 2

    if ( N == 0 )
       {
       X = 0;
       }

    else

       // begin Step 3

       {

       // initialize standard uniform R.V.
       U = 0;

       // generate 20 decimal places for continuous
       // random variable by choosing a number 0-9
       // for each decimal place
       power_of_10 = 1;
       for (j = 0; j<20; j++)
          {
          power_of_10 *= 10;
          U += (double) ( rand() % 10 ) / power_of_10;
          }


       // begin Step 4

       // apply equation (6.2)
       X = -log(1 - pow(U,(double)1/N)) / beta;

       } // end of else

    fout.precision(20);
    fout << X << "\t" << N << endl;
    } // for each point

 cout << "View results in random data.txt file." << endl;
 fout.close();
 while(1); // keeps window open
 }
```

Appendix II

```
/* This program estimates the BMEP parameters beta and lambda
using sample data from the BMEP distribution.  The program's
inputs are sample data points from the BMEP distribution and
the outputs are the Method of Moments Estimators and the
Maximum Likelihood Estimators for beta and lambda.  The
program can handle up to 10,000 data points as input.

The program needs to be compiled with a C++ compiler and the
results will be written to a text file named "results.txt,"
which can be found in the same folder as the program code is
saved.  The sample data points for the program should be
saved to a file named "random data.txt," which should be
saved to the same folder as the program code is saved.  The
first line of the input file should state the number of data
points in the format "XXX data points."  The second line of
the input file should contain the headers "X" and "N"
separated by a tab.  The following lines of the input file
should contain the random BMEP data with X listed first and
N listed second on each line, where X and N are separated by
a tab.

For example, the input file should look like this:
100 data points.
        X               N
0.34747472536598017     3
0.33569883513526738     15
. . .
0.54027051818993688     9

After compiling the C++ code typical output will look like
this:
moment estimator of beta = 4.705048429
MLE of beta = 4.677619166
moment estimator of lambda = MLE of lambda = 9.29  */
```

```
#include <iostream>
#include <fstream>
#include <cmath>
using namespace std;

void main()
   {
   double X[10000];
   int N[10000];
   ofstream fout("results.txt");

   // Open the file generated by the generate data program
   ifstream inData;
   inData.open ("random data.txt");
   if ( !inData )
      {
      cout << "File could not be opened." << endl;
      while(1); // keeps window open
      }

   // determine number of data points
   int num_points;
   inData >> num_points;

   // skip text information
   char junk[10];
   inData >> junk >> junk >> junk >> junk;

   // read in (X , N) points
   for (int i = 0; i < num_points; i++)
      inData >> X[i] >> N[i];
```

```
// find sum of X and sum of N
double sum_of_X=0, sum_of_N=0;
for (i=0; i < num_points; i++)
   {
   sum_of_X += X[i];
   sum_of_N += N[i];
   }

// find X bar and N bar
double X_bar = sum_of_X/num_points;
double N_bar = sum_of_N/num_points;

// find A of equation (6.5)
double A = 0;
for (i=0; i < num_points; i++)
   {
   if (N[i] >0)
      A++;
   }


// find moment estimator of beta by applying equation (6.3)

double MOM_of_beta;
if ( A > 0 ) // otherwise, if A = 0 then X_bar = 0
             // and moment estimator of beta is undefined
   {

   // compute Riemann integral with 100000 partitions
   double width = N_bar/100000;
   double integral = 0;
   for (i = 0; i<100000; i++)
      {
      double t = i*width + width/2;
      integral += width * ( 1 - exp(-t) ) / t;
      }

   MOM_of_beta = integral / X_bar;
   }
```

```
// find MLE estimator of beta
// by applying iterations of (6.13)

double MLE_of_beta;
if (A > 0) // otherwise, if A = 0 then 2nd derivative
           // of w(beta) will equal zero and MLE of
           // beta is undefined
   {
   double beta_to_try = MOM_of_beta, last_beta_tried;
   double sum;  // sum found in equation (6.11)
   double sum2; // sum found in equation (6.12)

   for (i = 0; i < 100; i++) // set max iterations to 100
      {
      sum = 0, sum2=0;

      for (int j=0; j < num_points; j++)
         {
         if (N[j] > 0) // for j in C
            {
            sum  += (N[j]-1) * X[j]
                    / (exp( beta_to_try * X[j] ) - 1);
            sum2 += (1-N[j]) * pow(X[j],2)
                    * exp( beta_to_try * X[j])
                    / pow((exp( beta_to_try*X[j] ) - 1), 2);
            }
         }

      // equation (6.13)
      beta_to_try = beta_to_try
                    - ( A / beta_to_try
                    - num_points * X_bar + sum )
                    / (-A / pow(beta_to_try,2) + sum2);

      // if converges already, quit loop
      if (beta_to_try/last_beta_tried == 1.00000 && i>1)
         i=123;
      last_beta_tried = beta_to_try;
      }

   MLE_of_beta = beta_to_try;
   }
```

```
fout.precision(10);

if (A > 0)
   {
   fout << "moment estimator of beta = "
        << MOM_of_beta << endl;
   fout << "MLE of beta = " << MLE_of_beta << endl;

   fout << "moment estimator of lambda = MLE of lambda = "
        << N_bar << endl;
   }
else
   {
   fout << "All points are (0,0).  No estimate made.";
   }


fout.close();
cout << "View results in results.txt file." << endl;
while(1); // keeps window open
}
```

Notation

$\mathcal{R}_+$ The set of positive real numbers

$\mathcal{N}$ The set of natural numbers {1,2,3,...}

$\overset{d}{=}$ Equality in distribution

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